CORE-CONSISTENCY AND TOTAL INCLUSION FOR METHODS OF SUMMABILITY

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1. Introduction. We shall consider methods of summation $A, B, \ldots$ defined by matrices of real elements (a_{mn}), (b_{mn}), $(m, n = 1, 2, \ldots)$ which are regular, that is, have the three well-known properties of Toeplitz (**4**, p. 43). A method A is said to be *core-consistent with the method B for bounded sequences* if the A-core (**3**, p. 137; and **4**, p. 55) of each real bounded sequence is contained in its B-core. B is *totally included* in A, $B \ll A$, if each real sequence which is B-summable to a definite limit (this limit may be finite or infinite of a definite sign) is also A-summable to the same limit. It will be shown in the present paper that if the matrix A is core-consistent with the positive matrix B, then A is "almost" divisible by B on the right. This statement is made precise in Theorem 1 below. The proof (§2) involves some elementary properties of convex sets in Banach spaces. In §3, the same method is used to prove a similar result for the relation $B \ll A$ (Theorem 2). Some simple corollaries are given in §4.

Let l_1 be the Banach space of elements $\mathbf{x} = (x_n)$, with norm

$$||\mathbf{x}|| = \sum_{n=1}^{\infty} |x_n|,$$

so that the rows of the matrices A, B are elements $\mathbf{a}_m, \mathbf{b}_m$ of l_1. Elements $\mathbf{x}, \mathbf{y} \in l_1$ are called disjoint if $x_n y_n = 0$ $(n = 1, 2, \ldots)$; an element $\mathbf{x} \in l_1$ is positive, $\mathbf{x} \geqslant 0$, if $x_n \geqslant 0$ $(n = 1, 2, \ldots)$. If $\mathbf{x} = (x_1, x_2, \ldots, x_n, \ldots) \in l_1$, we shall write

$$\mathbf{x}^q = (x_1, \ldots, x_q, 0, 0, \ldots), \qquad \mathbf{x}_p = (0, \ldots, 0, x_p, x_{p+1}, \ldots),$$

$$\mathbf{x}_p{}^q = (0, \ldots, 0, x_p, \ldots, x_q, 0, \ldots), \qquad p \leqslant q.$$

We also use the same notation for sets $E \subset l_1$, for instance $E_p{}^q$ is the set of all $\mathbf{x}_p{}^q$ with $\mathbf{x} \in E$. A *cone* $K \subset l_1$ is a set such that

$$\sum_{1}^{n} c_k \mathbf{x}_k \in K$$

whenever $c_k \geqslant 0$, $\mathbf{x}_k \in K$. For instance, the set of all positive elements is a cone in l_1.

We shall prove the following theorems:

THEOREM 1. *Let A, B be regular matrices and let A be core-consistent with B. If B is positive, that is if $\mathbf{b}_m \geqslant 0$ $(m = 1, 2, \ldots)$, there is a positive regular matrix C such that the norm of the mth row of $CB - A$ tends to zero for $m \to \infty$.*

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The case where the elements of the sequences, or of the matrices, are complex is not essentially different as will be shown in §2.

If $A = (a_{mn})$, we shall write A_p for the matrix obtained from A by replacing all a_{mn} with $n < p$ by zeros.

THEOREM 2. *If A, B are regular row-finite matrices, B positive and*

(i) $$B \ll A,$$

there is an integer p and a regular positive row-finite matrix C such that

(1) $$CB_p = A_p;$$

this remains true if (i) *is replaced by the* (*formally weaker*) *hypothesis that*

(ii) *$\tau_n \to +\infty$ always implies $|\sigma_n| \to +\infty$, where σ_n and τ_n are the A- and the B- transforms of a sequence s_n, respectively.*

If B is the unit matrix I, these results were known before; for the case of Theorem 1 see Agnew **(1)**, also **(3**, p. 149**)**; for Theorem 2, Hurwitz **(5)** or **(4**, p. 53**)**.

2. Core-consistency. If Theorem 1 is true for a given pair of matrices A, B, it is also true for any two matrices A', B' with rows $\mathbf{a}'_m$, $\mathbf{b}'_m$ satisfying

$$||\mathbf{a}_m - \mathbf{a}'_m|| \to 0, \ \ ||\mathbf{b}_m - \mathbf{b}'_m|| \to 0.$$

This and the regularity of A, B imply that we may assume A, B to be row-finite, and such that there is a sequence $n(m)$ increasing to $+\infty$ with $a_{mn} = b_{mn} = 0$ for $n < n(m)$.

LEMMA. *In the above conditions there exist two sequences $p = p(m) < q(m)$ such that $p(m) \to \infty$ for $m \to \infty$ and that*

(2) $$\rho(\mathbf{a}_m, K) = \rho(\mathbf{a}_m, K_p{}^q);$$

here $\rho(\mathbf{a}_m, K)$ is the distance from $\mathbf{a}_m$ to the cone K generated by the $\mathbf{b}_\lambda$ ($\lambda = 1, 2, \ldots$).

Proof. For a given m, let $m_1 \leqslant m_2$ be such that $\mathbf{b}_\mu$ is disjoint with $\mathbf{a}_m$ if μ does not satisfy $m_1 \leqslant \mu \leqslant m_2$; we may assume that $m_1 \to \infty$ for $m \to \infty$. Let K' be the cone generated by the $\mathbf{b}_\mu$, $m_1 \leqslant \mu \leqslant m_2$, let $p(m) = n(m_1)$ and let q be so large that $b_{\mu n} = 0$, $m_1 \leqslant \mu \leqslant m_2$, $a_{mn} = 0$ for $n > q$. Then $\mathbf{a}_{mp}{}^q = \mathbf{a}_m$, $K'_p{}^q = K'$, and therefore

(3) $$\rho(\mathbf{a}_m, K) \leqslant \rho(\mathbf{a}_m, K') = \rho(\mathbf{a}_m, K'_p{}^q).$$

On the other hand, let $\mathbf{x} \in K$, then $\mathbf{x}$ is a linear combination, with positive coefficients, of some of the $\mathbf{b}_\lambda$. If we omit from it all those $\mathbf{b}_\lambda$ which are not $\mathbf{b}_\mu$, we shall obtain another element $\mathbf{x}' \in K'$. The omitted $\mathbf{b}_\lambda$ are disjoint with $\mathbf{a}_m$ and all $b_{\lambda n}$ satisfy $b_{\lambda n} \geqslant 0$. This implies

$$||\mathbf{a}_m - \mathbf{x}_p{}^q|| \geqslant ||\mathbf{a}_m - \mathbf{x}'_p{}^q||.$$

Since $K'_p{}^q \subset K_p{}^q$, it follows that $\rho(\mathbf{a}_m, K_p{}^q) = \rho(\mathbf{a}_m, K'_p{}^q)$ and using (3) we obtain $\rho(\mathbf{a}_m, K) \leqslant \rho(\mathbf{a}_m, K_p{}^q)$. The inverse inequality is obvious, and (2) follows.

Proof of Theorem 1. We shall show that

$$\rho(\mathbf{a}_m, K) \to 0. \tag{4}$$

If this is not true, there exists by the Lemma an $\epsilon > 0$, a sequence of disjoint $\mathbf{a}_{m_i}$ and a sequence of disjoint intervals $[p_i, q_i]$ with

$$\rho(\mathbf{a}_{m_i}, K_{p_i}^{q_i}) > \epsilon.$$

If

$$\mathbf{y} = \sum c_i \mathbf{a}_{m_i}$$

is a linear combination of the $\mathbf{a}_{m_i}$ with $c_i > 0$, $\sum c_i = 1$ and if $\mathbf{x} \in K$, we can put

$$\mathbf{z}_i = c_i^{-1} \mathbf{x}_{p_i}^{q_i} \in K_{p_i}^{q_i}$$

and have

$$||\mathbf{y} - \mathbf{x}|| = ||\sum c_i \mathbf{a}_{m_i} - \sum c_i \mathbf{z}_i|| = \sum c_i ||\mathbf{a}_{m_i} - \mathbf{z}_i|| > \epsilon.$$

This shows that the convex set E generated by the $\mathbf{a}_{m_i}$ is at a distance $\geqslant \epsilon$ from E, hence the ϵ-neighbourhood E_ϵ of E is disjoint with K. If K_ϵ is the cone generated by E_ϵ, K and K_ϵ are disjoint except for the origin. By a well-known theorem **(7**, Theorem 1.2**)**, there is in l_1 a bounded linear functional $f(\mathbf{x})$ of norm one which is positive on K and negative on K_ϵ. Hence $f(\mathbf{y}) \leqslant -\epsilon$ on E **(7**, Lemma 1.2**)**. This means that there is a bounded sequence s_ν with

$$\sum b_{m\nu} s_\nu \geqslant 0 \qquad (m = 1, 2, \ldots),$$

$$\sum a_{m_i,\nu} s_\nu \leqslant -\epsilon \qquad (i = 1, 2, \ldots),$$

and contradicts the hypothesis of Theorem 1.

From (4) it follows that for some row-finite positive matrix $C = (c_{mn})$,

$$||\mathbf{a}_m - \sum_n c_{mn} \mathbf{b}_n|| \to 0, \qquad m \to \infty.$$

Finally, this C will be necessarily regular, provided we agree to take $c_{mn} = 0$ whenever $\mathbf{b}_n = 0$. For

$$c_{mn} b_{n\nu} \leqslant \sum_{n=1}^{\infty} c_{mn} b_{n\nu} = a_{m\nu} + o(1) = o(1), \qquad m \to \infty$$

implies that $c_{mn} \to 0$ for $m \to \infty$ and each n. On the other hand,

$$\sum_{\nu=1}^{\infty} a_{m\nu} = \sum_\nu \sum_n c_{mn} b_{n\nu} + o(1)$$

$$= \sum_n c_{mn} \sum_\nu b_{n\nu} + o(1)$$

together with

$$\sum_\nu a_{m\nu} = 1 + o(1), \quad \sum_\nu b_{n\nu} = 1 + o(1)$$

imply that $\sum_n c_{mn} \to 1$ for $m \to \infty$. This completes the proof.

The concept of the core is defined also for sequences of complex numbers **(3**, p. 137**)**. Accordingly, we may introduce the concept of core-consistency

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as well for matrices and sequences with complex elements. With this new definition, Theorem 1 holds literally as before.

For the proof assume that A is a regular matrix with complex elements, B is positive and that A is core-consistent with B for bounded sequences. Hence, by Knopp's core theorem (**6**, p. 115) or (**4**, p. 55), the core of the B-transform of any bounded sequence s_n is included in the core of s_n. But A is core-consistent with B, and so the core of the A-transform of s_n also is included in the core of s_n. This implies (**3**, p. 149) that $A = A' + V$ where A is a positive regular matrix, and the norm of the mth row of the matrix V tends to zero as $m \to \infty$. Clearly, A' also is core-consistent with B, for complex or more particularly, for real sequences. It then follows from the original Theorem 1 that there exists a positive regular matrix C such that the norm of the mth row of $CB - A'$ tends to zero from $m \to \infty$. Consequently, the norm of the mth row of

$$CB - A = CB - A' - V$$

also tends to zero for $m \to \infty$. This proves our assertion.

The converse of this (as well as the converse of Theorem 1) is a direct consequence of Knopp's core theorem. Thus, let A, B, C, be three regular matrices, C positive, such that the norm of the mth row of $CB - A$ tends to zero for $m \to \infty$. Then the core of the transform of any bounded complex sequence s_n by CB coincides with the core of the transform of s_n by A. The transform of s_n by CB is the transform by C of the transform of s_n by B. Hence the core of the transform of s_n by CB is included in the core of the transform of s_n by B, by virtue of Knopp's core theorem. In other words, CB, and hence A, are core-consistent with B for bounded sequences.

3. Total inclusion. We shall now prove Theorem 2, deducing (1) from the hypothesis (ii). Let $\rho_{mp} = \rho(\mathbf{a}_{mp}, K_p)$; we first show that

$$\rho_{mp} = 0 \text{ for all } p \text{ sufficiently large and } m = 1, 2, \ldots . \tag{5}$$

Let (5) be false. Since the ρ_{mp} decrease for m fixed and increasing p and finally become zero, we deduce that for each p, $\rho_{mp} > 0$ for an infinity of m. Now $\rho_{mp} > 0$ implies the existence of $\delta > 0$, $\epsilon > 0$ such that the sphere S in l_1 with center $\mathbf{a}_{mp}$ and radius δ does not have common points with the cone K' generated by the points $\mathbf{b}_{\lambda p}$ ($\lambda = 1, 2, \ldots$), and by the spheres with radii ϵ around those of the $\mathbf{b}_{\mu p}$ ($\mu = 1, 2, \ldots, m$) which are not zero. Hence, there is a functional

$$f(\mathbf{x}) = \sum x_n s_n, \; ||f|| = 1 \text{ in } l_1,$$

generated by a bounded sequence s_n with $s_n = 0$ for $n < p$, such that the hyperplane $f(\mathbf{x}) = 0$ separates S and K' and supports S (by Eidelheit's theorem, (**7**, Theorem 1.6)). If $f(\mathbf{x}) \geqslant 0$ on K', we have

$$\tau_\mu = f(\mathbf{b}_{\mu p}) \geqslant \epsilon \quad \text{for } \mathbf{b}_{\mu p} \neq 0 \qquad (\mu = 1, 2, \ldots, m)$$

(**7**, Lemma 1.2) and

$$0 > \sigma_m = f(\mathbf{a}_{mp}) \geqslant -||f||\delta = -\delta.$$

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By fixing $\epsilon > 0$, taking $\delta > 0$ sufficiently small, and then multiplying the s_n with a sufficiently large positive number, we obtain the following statement:

(*) For each m, p with $\rho_{mp} > 0$ and for any two positive numbers M, η, there is a bounded sequence s_n with $s_n = 0$ for $n < p$ such that

$$\sigma_m = \sum_n a_{mn} s_n = -\eta, \qquad \tau_\lambda = \sum_n b_{\lambda n} s_n \geqslant 0 \qquad (\lambda = 1, 2, \ldots),$$

$$\tau_\mu > M \text{ if } 1 \leqslant \mu \leqslant m \text{ and } \mathbf{b}_{\mu p} \neq 0.$$

We now define inductively increasing sequences of integers $p_1, p_2, \ldots, m_1, m_2, \ldots$ and bounded sequences $\mathbf{s}^{(i)}$ satisfying $s_n^{(i)} = 0$ for $n < p_i$. If

$$p_1, \ldots, p_{i-i}; m_1, \ldots, m_{i-i}; \mathbf{s}^{(1)}, \ldots, \mathbf{s}^{(i-1)}$$

are already defined, take p_i so large that $a_{\mu n} = 0$ for $n \geqslant p_i$, $\mu = m_1, \ldots, m_{i-1}$, then find an $m_i > m_{i-1}$ with

$$\left|\sum a_{m_i,n}(s_n^{(1)} + \ldots + s_n^{(i-1)})\right| < \tfrac{1}{2}, \; \rho(\mathbf{a}_{m_i p_i}, K_{p_i}) > 0.$$

By (*), there is a bounded sequence $\mathbf{s}^{(i)}$ with $s_n^{(i)} = 0$ for $n < p_i$ such that

$$\sum_{n=1}^{\infty} a_{m_i,n}(s_n^{(1)} + \ldots + s_n^{(i)}) = -1, \tag{6}$$

$$\sum_n b_{\lambda n} s_n^{(i)} \geqslant 0 \qquad (\lambda = 1, 2, \ldots), \tag{7}$$

$$\sum_n b_{\mu n} s_n^{(i)} > i \text{ if } b_{\mu p_i} \neq 0 \qquad (\mu = 1, \ldots, m). \tag{8}$$

Let $\mathbf{s} = (s_n)$ be the sequence defined by $s_n = \sum_i s_n^{(i)}$; for each n this sum has only a finite number of terms. Since $\mathbf{a}_{m_i}$ and $\mathbf{s}^{(j)}$ are disjoint for $j > i$, we have by (6), $\sigma_{m_i} = -1$, and by (7) and (8), $\tau_\lambda \to \infty$, which contradicts the hypothesis and proves (5).

Fixing a p for which (5) holds, we consider an arbitrary m. For each $\epsilon > 0$ there is an $\mathbf{x}$ in K such that

$$||\mathbf{a}_{mp} - \mathbf{x}_p|| < \epsilon, \qquad \mathbf{x} = \sum c_\mu \mathbf{b}_\mu,\; c_\mu \geqslant 0. \tag{9}$$

Let q be the last index n with $a_{mn} \neq 0$. If we omit from the last sum all $\mathbf{b}_\mu$ for which

$$\sum_{n>q} b_{\mu n} > \sum_{n\leqslant q} b_{\mu n},$$

we shall obtain an element $\mathbf{x}' \in K$ with

$$||\mathbf{a}_{mp} - \mathbf{x}'_p|| \leqslant ||\mathbf{a}_{mp} - \mathbf{x}_p||.$$

It follows that μ in (9) may be assumed bounded for all ϵ. Then we must have

$$\mathbf{a}_{mp} = \sum_{n=1}^{N} c_{mn}\mathbf{b}_{np}, \qquad c_{mn} \geqslant 0. \tag{10}$$

This proves the theorem, for the argument used in the proof of Theorem 1 shows that $C = (c_{mn})$ is regular, provided in (10) we take $c_{mn} = 0$ whenever $\mathbf{b}_{np} = 0$.

We give some corollaries to Theorem 2, assuming that the matrices A, B are regular and row-finite and that B is positive. We compare the following relations (for the definition of the core of a possibly unbounded sequence see (**4**, p. 55)):

(i) $B \ll A$.

(ii) For each sequence s_n, $\tau_n \to +\infty$ implies that $|\sigma_n| \to +\infty$.

(iii) $A_p = CB_p$ for some p with $C \geqslant 0$.

(iv) A is core-consistent with B for all real sequences.

(v) A is core-consistent with B for all complex sequences.

Then we have:

THEOREM 3. *Conditions* (i)-(v) *are equivalent.*

Proof. Clearly, (i) $\to$ (ii). Theorem 2 shows that (ii) implies (iii) and it is easy to see that (iii) $\to$ (i). From the definitions of the properties concerned we have (v) $\to$ (iv) $\to$ (ii). Finally, Knopp's core theorem states that (iii) $\to$ (v). This completes the proof.

4. Applications. For further illustration of Theorems 1 and 2 we shall give some applications to totally equivalent and core equivalent methods. Two methods A, B are *totally equivalent*, if $A \ll B$ and $B \ll A$; they are *core-equivalent for bounded sequences* if the A-core of each bounded sequence coincides with its B-core. In what follows, V is a matrix such that the norm of the mth row tend to zero for $m \to \infty$, and I is the unit matrix.

THEOREM 4. (i) *A method A is core-equivalent with I for bounded sequences if and only if A has a representation*

$$A = A' + V \tag{11}$$

with positive A', where A' contains a sequence of rows of the form

$$\mathbf{a}'_{m_n} = (0, \ldots, 0, a_{m_n,n}, 0, \ldots), \qquad n = 1, 2, \ldots. \tag{12}$$

(then necessarily $m_n \to \infty$, $a_{m_n,n} \to 1$ for $n \to \infty$.)

(ii) *A regular row-finite method A is totally equivalent with I if and only if for some p, A_p is positive and contains a sequence of rows of the form* (12).

Proof. (i) The conditions are clearly sufficient. It follows from Theorem 1 that (11) with a positive A' is necessary. Again by Theorem 1, there is a positive regular matrix C and a V' with $CA' = I + V'$. For each n we have

$$\sum_{m=1}^{\infty} c_{nm}\mathbf{a}'_m = \mathbf{e}_n \tag{13}$$

with $e_{nl} \geqslant 0$, $\mathbf{e}_n = (e_{nl})$, $e_{nn} \to 1$ and

$$\sum_{l \neq n} e_{nl} \to 0$$

for $n \to \infty$. Let

$$\epsilon_n = \sum_{l \neq n} \frac{e_{nl}}{e_{nn}}.$$

Then $\epsilon_n \to 0$ for $n \to \infty$. Since the c_{nm} are all positive, it follows from (13) that there is at least one $m = m_n$ such that

$$\sum_{l \neq n} \frac{a'_{ml}}{a'_{mn}} \leqslant \epsilon_n.$$

For otherwise, multiplying the relations

$$\sum_{l \neq n} a'_{ml} > \epsilon_n a'_{mn} \qquad (m = 1, 2, \ldots)$$

with c_{nm} and adding we would obtain by means of (13) that

$$\sum_{l \neq n} e_{nl} > \epsilon_n e_{nn},$$

which contradicts the definition of ϵ_n.

We now replace by zero the elements a_{ml} of the rows of A' with $m = m_n$, $l \neq n (n = 1, 2, \ldots)$. Denoting the matrix thus obtained again by A', we see that (11) and (12) are satisfied. This proves (i); the proof of (ii) is similar.

Theorem 4(i) may serve to show, for instance, that if a regular Hausdorff method H_g is core-equivalent with I for bounded sequences, then H_g is identical with I.

A method A is *normal* if $a_{mn} = 0$ for $n > m$ and $a_{nn} \neq 0$ $(n = 1, 2, \ldots)$. In this case A has an inverse A^{-1}. If A, B are normal, there is a triangular matrix C with $A = CB$.

THEOREM 5. *Let the regular normal methods A, B be totally equivalent. Then there exists a sequence $c_m \to 1$ such that for some p,*

$$(14) \qquad a_{mn} = c_m b_{mn}, \qquad m = 1, 2, \ldots; n = p, p+1, \ldots.$$

Proof. Let $A = CB$, $B = DA$, then the matrices C, D are triangular, regular and totally equivalent with I. We have

$$a_{mm} = c_{mm} b_{mm}, \quad b_{mm} = d_{mm} a_{mm},$$

hence

$$c_{mm} d_{mm} = 1,$$

and we obtain $c_{mm} \to 1$. From Theorem 4 (ii) it follows that for all sufficiently large n, $c_{mn} = 0$ if $n \neq m$. Putting $c_m = c_{mm}$, we obtain (14).

It should be added that sometimes it is even possible to prove that A, B are identical if they are totally equivalent. Let $A = H_g$, $B = H_{g_1}$ be two regular and normal Hausdorff methods. Then

$$\sum_{n=0}^{m} |a_{mn}|$$

converges for $m \to \infty$ to the "essential" total variation of $g(x)$. From (14) it follows that

$$\sum_{n=0}^{m} |a_{mn} - b_{mn}| \to 0,$$

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hence g and g_1 are essentially identical. Thus we obtain a remark of Bosanquet (**2**, p. 452) that H_g, H_{g_1} are identical if they are totally equivalent.

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