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ON THE INTERSECTION OF A CLASS OF MAXIMAL SUBGROUPS OF A FINITE GROUP

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1. Introduction. Of late there has been considerable interest in the study of analogs of the Frattini subgroup of a finite group and the investigation of their properties, particularly their influence on the structure of the group, see [**2-11**], [**14-16**] and [**18**]. Gaschütz [**11**] and more recently Bechtell [**2**] and Rose [**18**] have considered extensively the intersection of the family of all non-normal, maximal subgroups of a finite group. Deskins [**8**] has discussed the intersection of the family of all maximal subgroups of a finite group whose indices are not divisible by a given prime. Bhatia [**7**] considered the intersection of the class of all maximal subgroups of a given group whose indices are composites. In this paper we investigate the intersection of another class of maximal subgroups and its relationship with the structure of the group. The subgroup we consider here contains the Frattini subgroup and also the two subgroups introduced in [**8**] and [**7**].

Let p be any given prime. Let $F(G)$ denote the family of all maximal subgroups of a given group G whose indices are both composite and also co-prime to p. Let $S(G)$ denote the intersection of the members of $F(G)$. If $F(G)$ is empty, then we define $S(G) = G$. We consider the structure of $S(G)$ and its relationship with the properties of G. A number of characterisations of $S(G)$ and its relationship with the properties of G are given. It is shown that $S(G)$ is solvable if p is the largest prime dividing the order of G. In general, $S(G)$ is not solvable and may turn out to be a simple group. For example, consider the group $G = PSL(2, 7)$. It is well known that the maximal subgroups of $PSL(2, 7)$ can have indices only 7 and 8. Therefore, the family

$$\{M:[G:M]_2 = 1 \text{ and } [G:M] \text{ is composite}\}$$

is empty. Hence we have that $S(G) = G$ itself which is a simple group. However, we prove that $S(G)$ is solvable if G is p-solvable. We use standard group theoretic notation as in [**12**] and [**13**]. If $M \leqq G$, then $[G:M]_p$ denotes the p-part of $[G:M]$.

Definition. Let G be a group. Let p be a given prime. Define

$$\Phi_p(G) = \cap\{M:M \text{ is a maximal subgroup of } G, [G:M]_p = 1\}$$

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$L(G) = \cap\{M{:}M$ is a maximal subgroup of G, $[G{:}M]$ is composite$\}$

$S(G) = \cap\{M{:}M$ is a maximal subgroup of G, $[G{:}M]_p = 1$ and $[G{:}M]$ is composite$\}$.

In case G has no maximal subgroup M such that $[G{:}M]_p = 1$, we define $\Phi_p(G) = G$. Similarly we set $L(G) = G$ if every maximal subgroup of G is of prime index. Likewise, $S(G)$ is defined to be G if there does not exist any maximal subgroup of composite index which is not co-prime to p.

In [**8**], $\Phi_p(G)$ is defined and several important properties are mentioned. The subgroup $L(G)$ is considered in Bhatia [**7**] who shows that it is supersolvable and further that if $G/L(G)$ has the Sylow tower property, then G has the same property; both $\Phi_p(G)$ and $L(G)$ are contained in $S(G)$ and these three subgroups are characteristic subgroups of G. Moreover, if p does not divide the order of G, then we have that $S(G) = L(G)$ and also that G is supersolvable if and only if $G = S(G)$. Unlike $L(G)$, the subgroup $S(G)$ has neither the Sylow tower property nor is in general supersolvable. For example, consider when $G = \text{Sym}(4)$ and $p = 2$. Since the intersection of all maximal subgroups in the family $F(G)$ is empty in this particular case, we have that $S(G) = \text{Sym}(4)$ which is neither supersolvable nor is Sylow towered.

We consider further the subgroup $S(G)$ corresponding to two distinct primes. In this context, the subgroup $S(G)$ corresponding to the prime p will be denoted by $S_p(G)$ and similarly $S_q(G)$ denotes the subgroup $S(G)$ corresponding to the prime q. If H denotes the intersection of $S_p(G)$ and $S_q(G)$, we show that H is supersolvable if either G is p-solvable or q-solvable. If either p or q happens to be the largest prime dividing the order of G, then H is supersolvable without any conditions on G. Hence, G is supersolvable if and only if $G = H$, that is $G = S_p(G) = S_q(G)$. This fact illustrates vividly how some purely set theoretic conditions for a group may control the structure of the group and force it to be supersolvable.

2. Preliminary results.

LEMMA 1. [**1**, p. 118]. *If N is a minimal normal subgroup of a group G then either N is elementary abelian, or N is the direct product of isomorphic copies of a simple group. Further if N is abelian and M is any maximal subgroup of G such that $N \not\subseteq M$, then $G = MN$ and $M \cap N = \langle 1 \rangle$.*

We shall use the following generalisation of the "Frattini Argument" to solvable groups whose proof which we omit is a direct consequence of P. Hall's extended Sylow theorems.

LEMMA 2. *Let N be a normal, solvable subgroup of G, let π be a set of primes. If H is a Hall π-subgroup of N, then we have $G = N_G(H)N$.*

The following result will be useful in some proofs using induction arguments.

PROPOSITION 3. *Let* $K \lhd G$. *Then*

(i) $S(G)K/K \subseteq S(G/K)$

(ii) $L(G)K/K \subseteq L(G/K)$

(iii) $\Phi(G)K/K \subseteq \Phi(G/K)$

(iv) $\Phi_p(G)K/K \subseteq \Phi_p(G/K)$.

Proof. (i) Write $\bar{G} = G/K$. Let $\bar{M}$ be a maximal subgroup of $\bar{G}$ of composite index such that $[\bar{G}:\bar{M}]_p = 1$. Now $\bar{M} = M/K$ where M is a maximal subgroup of G containing K. Therefore

$$[\bar{G}:\bar{M}]_p = [G:M]_p = 1.$$

Let J be the intersection of all M corresponding to each $\bar{M}$ of composite index such that $[\bar{G}:\bar{M}]_p = 1$. Then $S(G) \subseteq J$ and $K \subseteq J$ and therefore $S(G)/K \subseteq J/K$. It is easy to show that $J/K \subseteq S(G/K)$ and so (i) follows. The proofs of (ii) and (iii) are similar to the above argument.

COROLLARY 4. *Let* $K \lhd G$.
(i) *If* $K \subseteq S(G)$ *then* $S(G/K) = S(G)/K$.
(ii) *If* $K \subseteq L(G)$ *then* $L(G/K) = L(G)/K$
(iii) *If* $K \subseteq \Phi_p(G)$ *then* $\Phi_p(G/K) = \Phi_p(G)/K$.

For the subgroup $S(G)$ we prove the following result about the structure of its Sylow subgroups.

PROPOSITION 5. *Let p be the prime taken in the definition of* $S(G)$. *Then* (i) *if p divides the order of* $S(G)$ *and p is moreover the largest prime dividing the order of* $S(G)$ *then any Sylow p-subgroup P of* $S(G)$ *is normal in G.* (ii) *If p does not divide the order of* $S(G)$ *then if* $Q \in \mathrm{Syl}_q(S(G))$ *where q is the largest prime dividing the order of* $S(G)$ *we have* $Q \lhd G$.

Proof. We denote $S(G)$ by S for convenience. (i) Suppose if possible that P is not normal in G. Then $N_G(P) \neq G$ and by the Frattini argument, $G = SN_G(P)$. Let M be a maximal subgroup of G containing $N_G(P)$. We have that $G = SM$. Now $N_G(P) \subseteq M$ and by the Sylow theory, $[G:N_G(P)] = 1 + kp$ for some integer k and consequently $[G:M] = 1 + sp$ for some non-zero integer s. We observe that $[G:M]$ cannot be composite. For, if so then as $[G:M]_p = 1$ it would imply that $S \subseteq M$ and therefore $G = M$, a contradiction. So, $[G:M] = 1 + sp$ is a prime dividing the order of S which is a contradiction to the fact p is the largest prime dividing $|S(G)|$. Hence $P \lhd G$. (ii) Suppose that Q is not normal in G. By the Frattini argument, $G = SN_G(Q)$. Let M be a maximal subgroup of G containing $N_G(Q)$. Then $G = SM$ and $[G:M] = 1 + kq = r$, say. Arguing as in (i) we have that r must be composite and also that p divides r. Since r divides the order of S, we then have that p divides the order of S, a contradiction to the hypothesis. Hence $Q \lhd G$.

We recall the definition that a group G has the *Sylow tower property* if every homomorphic image of G has a normal Sylow subgroup.

COROLLARY 6. (i) *If G satisfies the hypothesis of Proposition* 5 (i), *then $S(G)/P$ has the Sylow tower property*; (ii) *If G satisfies the hypothesis of Proposition* 5 (ii) *then $S(G)$ has the Sylow tower property.*

THEOREM 7. (i) *$\Phi_p(G)$ is solvable.*

(ii) *If p does not divide the order of $\Phi_p(G)$, then $\Phi_p(G) = \Phi(G)$.*

(iii) *$\Phi_p(G) = PT$ where P is a normal Sylow p-subgroup of $\Phi_p(G)$ and T is a nilpotent complement of P.*

(iv) *$\Phi_p(G)/P$ is nilpotent. (Thus $\Phi_p(G)$ is meta-nilpotent.)*

Proof. We distinguish two cases.

Case 1. p does not divide the order of $\Phi_p(G)$. Let M be a maximal subgroup of G. If $\Phi_p(G)$ is not contained in M, then $G = M\Phi_p(G)$. Then clearly, $[G:M]_p = 1$. So $\Phi_p(G) \subseteq M$, a contradiction. Therefore $\Phi_p(G)$ is contained in every maximal subgroup of G and consequently $\Phi_p(G) = \Phi(G)$ proving (ii). Now (i), (iii) and (iv) follow immediately in this case from the properties of $\Phi(G)$.

Case 2. p divides the order of $\Phi_p(G)$. Let P be a Sylow p-subgroup of $\Phi_p(G)$. We claim that $P \lhd G$. For, suppose that P is not normal in G. Then by the Frattini argument

$$G = N_G(P)\Phi_p(G).$$

Let M be a maximal subgroup of G containing $N_G(P)$. It follows that $G = M\Phi_p(G)$. Since P is a Sylow p-subgroup of $\Phi_p(G)$ and P is contained in M, we now get that $[G:M]_p = 1$. This implies that $\Phi_p(G) \subseteq M$ and consequently

$$G = M\Phi_p(G) = M,$$

a contradiction. Hence we conclude that P is normal in G. So by the Schur-Zassenhaus theorem we have that $\Phi_p(G) = PT$ where T is a p-complement. By Corollary 4 (iii) we have that

$$\Phi_p(G/P) = \Phi_p(G)/P = PT/P$$

which is isomorphic to T. Since p does not divide the order of T, we have by using Case 1 that

$$\Phi_p(G/P) = \Phi(G/P).$$

Since a Frattini subgroup is always nilpotent, it follows that $\Phi(G/P)$ is nilpotent and so T is nilpotent, proving (iii) and (iv). Further, it follows that $\Phi_p(G)/P$ is nilpotent (and so solvable). Therefore $\Phi_p(G)$ is solvable since P is solvable, proving (i).

3. Conditions implying solvability of $S(G)$.

THEOREM 8. *Let p be the prime taken in the definition of $S(G)$. Then* (i) *if p is the largest prime dividing the order of G then $S(G)$ is solvable;* (ii) *if G is p-solvable, then $S(G)$ is solvable.*

Proof. (i) We use induction on the order of G. We distinguish two cases:

Case 1. $S(G) \neq G$. Let N be a minimal normal subgroup of G contained in $S(G)$. By induction on G/N, we have that $S(G)/N$ is solvable. For convenience we denote $S(G)$ by S. If W is another minimal normal subgroup of G contained in S, then again S/W is solvable and so $S(W \cap N) \cong S$ is solvable. So, we may suppose that N is the unique minimal normal subgroup of G which is contained in S. Further, let B be another minimal normal subgroup of G. Then T/B, the intersection of all maximal subgroups of G/B, which have composite indices and which are also prime to p, is solvable by applying the induction hypothesis. But $SB/B \subseteq T/B$. Therefore

$$SB/B \cong S/(S \cap B) \cong S$$

is solvable; we may now assume that N is the unique minimal normal subgroup of G.

Let L be a maximal subgroup of G such that $[G:L]_p = 1$. Now we claim that $N \subseteq L$. Suppose that $N \nsubseteq L$. Then we get that $G = LN$. Also, $[G:L]$ cannot be composite, for if so then $S \subseteq L$ implying that $N \subseteq L$ and so $G = LN = L$, a contradiction. Let $[G:L] = t$, a prime. Now we note that L must be corefree otherwise $N \subseteq L$, contradicting our supposition. Now by representing G on the t cosets of L, it follows that the order of G divides $t!$ and consequently t must be equal to q, the largest prime dividing the order of G. Since t divides $|S|$, $t = p$, a contradiction. Hence, we must have that $N \subseteq L$. Therefore, $N \subseteq \Phi_p(G)$ and so N is solvable since $\Phi_p(G)$ is solvable by Theorem 7 (i). Then, together with the fact that $S(G)/N$ is solvable, it implies that $S(G)$ is solvable proving the result.

Case 2. $S(G) = G$. This implies that if M is a maximal subgroup of G such that $[G:M]_p = 1$ then $[G:M]$ must be a prime; let M be such a maximal subgroup and let $[G:M] = t$, a prime. Now by representing G on the t cosets of M, we get that the core of M is nontrivial as otherwise the order of G must divide $t!$ which is not possible because $t \neq p$. Thus G is not a simple group. Now by the same argument as in Case 1, we have a unique minimal normal subgroup H in G and also $H \subseteq S(G)$. Since the core of H in M is non-trivial, $H \subseteq M$. Hence, we have that $H \subseteq \Phi_p(G)$. So H is solvable since $\Phi_p(G)$ is solvable by Theorem 7 (i). Now by induction hypothesis, $S(G)/H = G/H$ is solvable. Hence, we get that $S(G)$ is solvable proving the result.

(ii) We use induction on the order of G. If N is a minimal normal subgroup of G contained in $S(G)$, then by the same argument as in (i), Case 1, we have that N is the unique minimal normal subgroup of G, otherwise, the result holds. Now since G is p-solvable, N is either a p-group or a p'-group. If N is a p-group, then N is solvable because by induction $S(G)/N$ is solvable. Now suppose that N is a p'-group. If N is contained in every maximal subgroup whose index is prime to p, then $N \subseteq \Phi_p(G)$ and so N is solvable since $\Phi_p(G)$ is solvable by Theorem 7 (i). Hence, it will follow that $S(G)$ is solvable. We may now assume that $N \not\subseteq M$ for some maximal subgroup M such that $[G:M]_p = 1$. Then $G = NM$. We note that M is core free. Again $[G:M]$ must be a prime because the two facts that $[G:M]$ is composite and $[G:M]_p = 1$ would imply that $S(G) \subseteq M$ and so $N \subseteq M$. This gives that $G = MN = M$, a contradiction. Let $[G:M] = r$ where $r \neq p$ is a prime. Now by representing G on the cosets of M in G, it now follows that the order of G divides $r!$. Hence $r = q$ where q is the largest prime dividing the order of G.

Further we have that N is contained in every maximal subgroup K such that $[G:K]_q = 1$. For, if there is such a K which does not contain N, then $G = KN$. Now since N is a p'-group, we have $[G:K]_p = 1$ and moreover if $[G:K]$ is composite then it will follow that $N \subseteq K$, a contradiction. So $[G:K] = s$, a prime and $s \neq q$ since $[G:K]_q = 1$. Now by representing G on the cosets of K in G as before, we get that the order of G divides $s!$ which is not possible since then q divides $s!$, a contradiction. Hence it follows that $N \subseteq K$, that is, $N \subseteq \Phi_q(G)$ and so N is solvable using Theorem 7 (i). As in Case 1 this now implies that $S(G)$ is solvable.

It is a well known result of Huppert (see for example [**17**, 9.45, p. 268]) that a group G is supersolvable if and only if $G/\Phi(G)$ is supersolvable. The following theorem is a generalisation of this and we shall use it later in the proof of Theorem 10.

THEOREM 9. *Let G be a group containing a normal subgroup H which contains $\Phi(G)$. Then H is supersolvable if and only if $H/\Phi(G)$ is supersolvable.*

Proof. If H is supersolvable then obviously $H/\Phi(G)$ is supersolvable. Now, suppose that $H/\Phi(G)$ is supersolvable. We may assume that $\Phi(G) \neq \langle 1 \rangle$ as otherwise the result holds trivially. We use induction on the order of H. We split the proof into three steps.

Step 1. Suppose that $\Phi(G)$ is a subgroup whose order is a composite number. Then, let p_1, p_2 be two distinct primes dividing the order of $\Phi(G)$. Let P_1, P_2 be Sylow p_1 and Sylow p_2 – subgroups respectively of $\Phi(G)$. Since $\Phi(G)$ is nilpotent, P_1 and P_2 are characteristic subgroups of $\Phi(G)$. Then it follows that P_1, P_2 are normal in G. So, we have that

$$(H/P_1)/\Phi(G/P_1) = (H/P_1)/(\Phi(G)/P_1) = H/\Phi(G)$$

is supersolvable by the hypothesis. Therefore by the induction hypothesis we get H/P_1 is supersolvable. Similarly we get that H/P_2 is supersolvable. Therefore,

$$H/(P_1 \cap P_2) \simeq H$$

is supersolvable, proving the theorem.

Step 2. Now, we consider the case when $\Phi(G)$ is a subgroup of order p^s where p is a prime and s is an integer, $s \geqq 1$. For convenience let us write $P = \Phi(G)$. Assume, first, that $\Phi(P) \neq \langle 1 \rangle$. Then we have that

$$(H/\Phi(P))/\Phi(G/\Phi(P)) = (H/\Phi(P))/(\Phi(G)/\Phi(P)) = H/\Phi(G)$$

is supersolvable. So by induction, we get that $H/\Phi(P)$ is supersolvable. Thus we have that

$$(H/\Phi(P))/(\Phi(H)/\Phi(P)) = H/\Phi(H)$$

is supersolvable. Consequently, H is supersolvable using a result of Huppert (see for example, [**17**, 9.45, p. 268]) and the proof is complete.

Now, we consider the possibility that $\Phi(P) = \langle 1 \rangle$. Then we have that $P = \Phi(G)$ is an elementary abelian p-group. Note that H is solvable since $H/\Phi(G)$ is supersolvable and $\Phi(G)$ is nilpotent.

Step 3. $\Phi(G)$ is an elementary abelian p-group. Let $S_{\mathscr{F}}$ be the supersolvable residual of H. Since $S_{\mathscr{F}} \subseteq P$, $S_{\mathscr{F}}$ is abelian. Since H is solvable we have by [**13**, VI, Satz 7.15, p. 703] that $H = S_{\mathscr{F}}T$ where $S_{\mathscr{F}} \cap T = \langle 1 \rangle$ and T is supersolvable. Moreover if $N_G(T) \neq G$ then choose a maximal subgroup M of G containing $N_G(T)$. For $x \in G$, $T^x \subset H$ and by [**13**, VI, Satz 7.15, p. 703] we have that $T^x = T^s$ for some $s \in H$. Now $xs^{-1} \in N_G(T)$ implies that $G = N_G(T)H$. Therefore we get that

$$G = N_G(T)H = N_G(T)S_{\mathscr{F}} = MS_{\mathscr{F}} = M$$

since $S_{\mathscr{F}} \subseteq \Phi(G) \subseteq M$. However, this is a contradiction to the fact that $G \neq M$. Therefore we have that $N_G(T) = G$ and so T is normal in G. Consequently $H = S_{\mathscr{F}} \times T$ and so H is supersolvable, being the direct product of two supersolvable groups.

Hence the proof of the theorem is now complete.

Finally, we consider the subgroup $S(G)$ corresponding to two distinct primes, the subgroup $S(G)$ corresponding to the primes p and q are then denoted by $S_p(G)$ and $S_q(G)$ respectively; when only one prime p is under consideration we write as before $S(G)$ for $S_p(G)$. We have given an example in Section 1 that $S(G)$ is not in general supersolvable, or even solvable. However, we have

THEOREM 10. *Let p, q be two distinct primes dividing the order of a group G. Suppose that G is either p-solvable or q-solvable. Then we have that $S_p(G) \cap S_q(G)$ is supersolvable.*

Proof. Let H denote the intersection of $S_p(G)$ and $S_q(G)$. By Theorem 8 (ii) we have that H is solvable. Using induction on the order of G we now prove that H is supersolvable. Let N be a minimal normal subgroup of G contained in H. By the induction hypothesis and Corollary 4, we get that

$$S_p(G/N) \cap S_q(G/N) = H/N$$

is supersolvable. Now if X is another minimal normal subgroup of G contained in H, then again we have that H/X is supersolvable and consequently $H/(N \cap X) \cong H$ is supersolvable and we are done. So we may suppose that N is the unique minimal normal subgroup of G contained in H. Since H is solvable, N is elementary abelian. (We observe that if N is cyclic then the theorem follows now immediately from the fact that H/N is supersolvable.)

Let M be any maximal subgroup of G. If N is not contained in M, then $G = MN$. Since N is abelian we have from Lemma 1 that $M \cap N = \langle 1 \rangle$. Therefore $[G:M] = |N|$. Suppose if possible that $[G:M]_p = 1$. Then $[G:M]$ must be a prime since if $[G:M]$ is composite, then by the definition of $S_p(G)$ it will follow that $N \subseteq M$ and we then obtain that $G = M$, a contradiction. Thus $|N| = [G:M]$ is a prime implying that N is cyclic and so the theorem follows by the remark made at the end of the last paragraph. Therefore, we may now assume that $[G:M]_p \neq 1$ and so N is an elementary abelian p-group. Now, if $[G:M]_q = 1$ then the result will follow by arguing as in the case when $[G:M]_p = 1$. Thus we must have that $[G:M]_q \neq 1$. Now, this implies q divides the order of N which is not possible since N is an elementary abelian p-group whereas p and q are distinct primes. It now follows that N is contained in every maximal subgroup of G. So $N \subseteq \Phi(G)$. Since clearly $\Phi(G) \subseteq H$, it now follows that

$$H/\Phi(G) \cong (H/N)/(\Phi(G)/N).$$

Since H/N is supersolvable we get now that $H/\Phi(G)$ is supersolvable. Using Theorem 9 it follows that H is supersolvable proving the theorem.

Using Theorem 8 (i) and arguing as in the proof of Theorem 10, we get the following result whose proof we omit.

THEOREM 11. *Let G be a group and p, q be two distinct primes dividing the order of G, one of them being the largest prime divisor of the order of G. Then $H = S_p(G) \cap S_q(G)$ is supersolvable.*

REFERENCES

1. R. Baer, *Classes of finite groups and their properties*, Illinois J. Math. *1* (1957), 115-187.
2. H. Bechtell, *Pseudo-Frattini subgroups*, Pacific J. Math. *14* (1964), 1129-1136.

3. J. Beidleman, *Generalized Frattini subgroups of finite group II*, Can. J. Math. *21* (1969), 418-429.
4. ——— *The influence of generalized Frattini subgroups on the solvability of a finite group*, Can. J. Math. *22* (1970), 41-46.
5. J. Beidleman and T. Seo, *Generalized Frattini subgroups of finite groups*, Pacific J. Math. *23* (1967), 441-450.
6. J. Beidleman and D. Dykes, *θ-Frattini subgroups of finite groups*, J. Algebra *17* (1971), 326–340.
7. H. C. Bhatia, *A generalized Frattini subgroup of a finite group*, Ph.D. thesis, Michigan State Univ., East Lansing (1972).
8. W. E. Deskins, *On maximal subgroups*, in First Sympos. in Pure Math. (Amer. Math. Soc., Providence, 1959).
9. D. Dykes, *Weakly hypercentral subgroups of finite groups*, Pacific J. Math. *31* (1969), 337-346.
10. ——— *Properties of generalized Frattini subgroups*, Boll. Unione Math. Italiana *5* (1970), 337-346.
11. W. Gaschütz, *Über die Φ-Untergruppe endlicher Gruppen*, Math. Zeit. *58* (1953), 160-170.
12. D. Gorenstein, *Finite groups* (Harper & Row, New York, 1968).
13. B. Huppert, *Endliche Gruppen I* (Springer Verlag, New York, 1967).
14. N. Ito, *Über die Φ-Gruppen einer endliche Gruppe*, Proc. Japan Acad. *31* (1955), 123-127.
15. ——— *Über eine Zur Frattini-Gruppe duale Bildung*, Nagyoa Math. Jour. *9* (1955), 123-127.
16. Z. Janko, *Eine Bemerkung Über die Φ-Untergruppe endlicher Gruppen*, Acta Sci. Math (Szeged) *23* (1962), 247-248.
17. D. J. Robinson, *A course in the theory of groups* (Springer Verlag, New York, 1982).
18. J. Rose, *The influence on a group of its abnormal structure*, Lond. Math. Soc. *40* (1964), 348-361.

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