EXTREMUM PROPERTIES OF THE REGULAR POLYHEDRA

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1. Historical remarks. In this paper we extend some well-known extremum properties of the regular polygons to the regular polyhedra. We start by mentioning some known results in this direction.

First, let us briefly consider the problem which has received the greatest attention among all the extremum problems for polyhedra. It is the determination of the polyhedron of greatest volume V of a class of polyhedra of equal surface areas F, i.e., the isepiphan problem.

The simple fact that the regular tetrahedron is the best among the tetrahedra was already known to Lhuilier.[1] But let us at once note that, among the 8- and 20-cornered polyhedra, the cube and the regular dodecahedron are not the best ones, and similarly, the regular octahedron and icosahedron are not the best polyhedra among the 8- and 20-faced polyhedra.

Steiner,[2] who was certainly in possession of this fact, announced only the conjecture that any regular polyhedron is the best one among the topologically isomorphic polyhedra. In proving this conjecture he succeeded, apart from the tetrahedron, only for the octahedron. The case of the icosahedron is, up to the present day, unsettled.

In 1935, M. Goldberg[3] made an attempt to prove the inequality

$$F^3/V^2 \geqslant 54(f-2)\tan\omega_f(4\sin^2\omega_f - 1); \quad \omega_f = \frac{f}{f-2}\frac{\pi}{6}$$

concerning a convex f-faced polyhedron. This inequality (for which I subsequently gave a complete proof[4]) is exact for $f = 4$, 6 and 12 and gives an exact asymptotical estimate for large values of f. Equality holds only for a regular tetrahedron, hexahedron, and dodecahedron.

According to this the regular hexahedron and dodecahedron are proved to be the best not only among the polyhedra of their type but also among all convex polyhedra with 6 and 12 faces, respectively.

Received September 24, 1948. The earlier publications of the author appeared under the name "Fejes". In order to explain this fact the author communicates the following at the request of the editors: Kolozsvár (Roumanian Cluj, the capital of Transsylvania, the native town of J. Bólyai, and where L. Fejér, F. Riesz and A. Haar began their career as young professors) was ceded to Roumania by the Treaty of 1920. From 1940 to 1944 it belonged temporarily to Hungary. The author generally worked in Kolozsvár during the time 1941-1944. Returning to Budapest he took the name "Fejes Tóth" (to be found in old family documents, and already used by some other members of his family), partly in order to avoid confusion with the name of Professor L. Fejér.

[1]S. Lhuilier, *De relatione mutua capacitatis et terminorum figurarum*, etc. (Varsaviae, 1782).

[2]J. Steiner, *Gesammelte Werke* II, 117-308.

[3]M. Goldberg, "The Isoperimetric Problem for Polyhedra," *Tôhoku Math. J.*, vol. 40 (1935), 226-236.

[4]L. Fejes Tóth, "The Isepiphan Problem for n-hedra," *Amer. J. Math.*, vol. 70 (1948), 174-180.

Also, the following inequality

$$F^3/V^2 \geqslant \frac{27\sqrt{3}}{2}(v-2)(3\tan^2\omega_v - 1); \ \omega_v = \frac{v}{v-2}\frac{\pi}{6}$$

probably holds for any convex polyhedron with v vertices. It is exact for $v = 4$, 6 and 12 and gives an exact asymptotical estimate for large values of v. This would mean that the regular octahedron and icosahedron are—again far beyond Steiner's conjecture—the best polyhedra among all 6- and 12-cornered polyhedra.

The state of affairs in the isepiphan problem is characteristic of a number of other problems.[5] Therefore in order to give a general orientation in the possibilities of transferring different extremum properties of the regular n-gon to space we can say:

When f is given it is the trihedral-cornered regular polyhedra, and when v is given the triangular-faced, that generally play a prominent part in the solutions of the extremum problems. It is inherent in the problem that—contrary to the problems in the plane—we cannot expect to determine the extremal polyhedra for all values of f or v. We must rather be content with inequalities exact for 4, 6 and 12 and asymptotically exact for large values of f or v.

Let us note that—taking into account the great number of researches dealing with various extremum properties of the regular polygons—it is surprising that, for instance, no extremum property of the regular icosahedron or dodecahedron occurs, as far as I know, in earlier literature. Still less do we find a systematic treatment of such extremum properties. Therefore, much remains to be done in the extremum problems for polyhedra to bring our knowledge, in this respect, to a level with that of the polygons. These attractive questions offer ample scope for work.

2. Aim and results. As we have seen, the researches made hitherto related to polyhedra of a given type or to polyhedra of a given number of faces or vertices.

But the consideration of a type of polyhedra is too special to obtain general results. On the other hand, the class of polyhedra of a given number of faces or vertices is too large to obtain all the five regular polyhedra as solutions of the same extremum problem. Therefore, in the following, we are going to compare polyhedra having a given number of faces f *and* a given number of vertices v. In this way we shall obtain inequalities in which equality holds for *all the five regular solids*.

In this paper we shall prove the following

THEOREM. *If V denotes the volume, r the radius of the insphere and R the radius of the circumsphere of a convex polyhedron having f faces, v vertices and e edges, then*

[5]See, for instance, the paper L. Fejes Tóth, "An Inequality Concerning Polyhedra," *Bull. Amer. Math. Soc.*, vol. 54 (1948), 139-146, where further bibliographical data can be found.

$$V \geqslant \frac{e}{3} \sin \frac{\pi f}{e} \left(\tan^2 \frac{\pi f}{2e} \tan^2 \frac{\pi v}{2e} - 1 \right) r^3 \tag{1}$$

$$V \leqslant \frac{2e}{3} \cos^2 \frac{\pi f}{2e} \cot \frac{\pi v}{2e} \left(1 - \cot^2 \frac{\pi f}{2e} \cot^2 \frac{\pi v}{2e} \right) R^3. \tag{2}$$

Equality holds in both inequalities only for the regular polyhedra.

Letting $p = 2e/f$ and $q = 2e/v$, we obtain by combining (1) and (2) the following

COROLLARY. *If r and R denote the radii of the in- and circumsphere of a convex polyhedron for which the average number of the sides of the faces and the average number of the edges of the vertices is p and q, respectively, then*[6]

$$\frac{R}{r} \geqslant \tan \frac{\pi}{p} \tan \frac{\pi}{q}. \tag{3}$$

Professor H. S. M. Coxeter wrote to me calling my attention to the equality $R/r = \tan \pi/p \tan \pi/q$ which holds for any regular polyhedron having p-gonal faces, q at each vertex. By this remark I was impelled to prove the nice inequality (3) which was the point of departure of the present paper.

3. Proofs. In order to prove (1) we may obviously suppose—without loss of generality—that the insphere of centre O has the radius $r = 1$. Denote the faces of the polyhedron Π, and their area as well by F_i ($i = 1, 2, \ldots, f$), the solid angle under which F_i appears from O by σ_i, and the number of the sides of the polygon F_i by p_i.

It is easy to see that for given values of p_i and σ_i the area F_i takes its minimum if F_i is a regular p_i-gon touching the insphere at its own centre. This minimum property is expressed—as a simple computation shows—by the inequality

$$F_i \geqslant \Phi(\sigma_i, p_i); \quad \Phi(\sigma, p) = \frac{p}{2} \sin \frac{2\pi}{p} \left(\tan^2 \frac{\pi}{p} \cot^2 \frac{2\pi - \sigma}{2p} - 1 \right).$$

Now we make use of the fact that the function of two variables $\Phi(\sigma, p)$ is convex from below for $0 \leqslant \sigma \leqslant 2\pi$, $3 \leqslant p$. Hence by Jensen's inequality[7]

$$3V \geqslant \sum_{i=1}^{f} F_i \geqslant \sum_{i=1}^{f} \Phi(\sigma_i, p_i) \geqslant f \Phi(4\pi/f, 2e/f); \qquad \text{q.e.d.}$$

The only difficulty of this very simple proof—which is properly Goldberg's proof mentioned above—is the unfortunate circumstance that the function

[6]The inequality (3) is a generalization of the inequality $R \geqslant 3r$ concerning tetrahedra—found in 1943 by a young Hungarian mathematician I. Ádám at the suggestion of Professor L. Fejér—and of certain results of the author (see the paper referred to in footnote 5).

[7]J. L. W. V. Jensen, "Sur les fonctions convexes et les inégalités entre les valeurs moyennes," *Acta Math.*, vol. 30 (1906), 175-193.

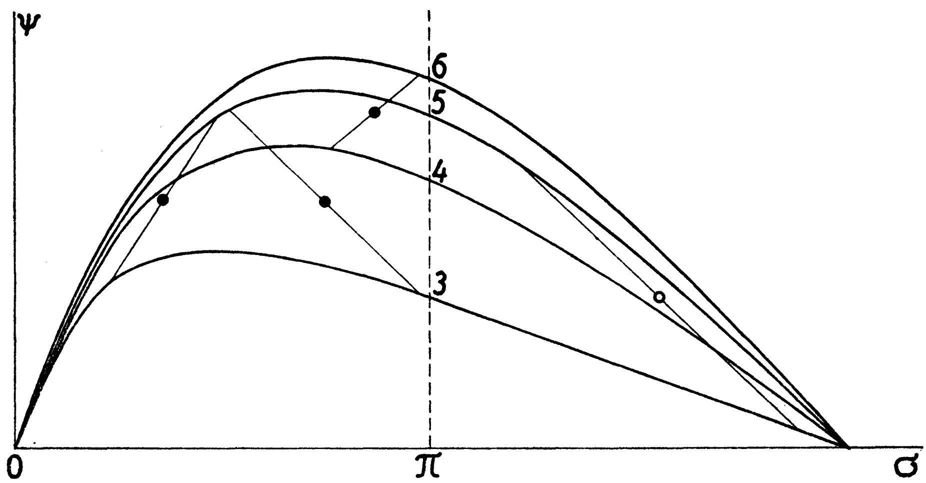
$\Phi(\sigma, p)$ is too complicated to arrange clearly the computations necessary for the proof of convexity.[8] On the other hand, it is easy to give a graphical representation of $\Phi(\sigma, p)$ from which the convexity can be seen empirically.[9]

Let us now turn to the inequality (2), the proof of which is analogous to the foregoing. Decompose Π into f pyramids of volumes $V_1, V_2, \ldots V_f$, having the centre O of the circumsphere of radius $R = 1$ as a common vertex, with bases formed by the respective faces $F_1, F_2, \ldots F_f$ of Π.

We have now the inequality

$$V_i \leqslant \Psi(\sigma_i, p_i); \quad \Psi(\sigma, p) = \frac{p}{3} \cos^2 \frac{\pi}{p} \tan \frac{2\pi - \sigma}{2p} \left(1 - \cot^2 \frac{\pi}{p} \tan^2 \frac{2\pi - \sigma}{2p}\right)$$

which means that, for given values of p_i and for given values of the area σ_i of the projection of F_i from O upon the circumsphere, the volume V_i takes its maximum if F_i is a regular p_i-gon the vertices of which lie on the circumsphere.



But now in addition to the difficulty indicated in the above proof a further one arises, namely: the function $\Psi(\sigma, p)$, as a function of two variables, is not convex from above in the whole strip $0 \leqslant \sigma \leqslant 2\pi$, $p \geqslant 3$. But it will be sufficient to make use of the convexity, say, for $0 \leqslant \sigma \leqslant \pi$, which can be surmised with great confidence from the above graphical representation of a few functions $\Psi(\sigma, \text{const.})$. The convexity is expressed by the fact that, for instance, the midpoint of any segment joining a point of the curve $\Psi = \Psi(\sigma, p_1)$ with a point of $\Psi = \Psi(\sigma, p_2)$ lies always below the curve $\Psi = \Psi\left(\sigma, \frac{p_1 + p_2}{2}\right)$.

[8]On this occasion I take the liberty to cite from the letter of M. Goldberg written to me in connection with my paper referred to in footnote 4: "Your rigorous proof . . . has removed a difficulty which I have tried to overcome without success."

[9]See my paper: "Über einige Extremaleigenschaften der regulären Polyeder und des gleichseitigen Dreiecksgitters," *Annali della Scuola Norm. Sup. di Pisa* (2) 13 (1948), 51-58.

First of all, we are going to prove the inequality (2) for $f \geqslant 8$. Let us note for this purpose that for any value of $p \geqslant 3$ we have the inequalities

$$\begin{aligned} &\Psi(\sigma, p) \leqslant \Psi(\pi, p) && \text{for} \quad \sigma \geqslant \pi, \\ &\Psi(\sigma_1, p) \leqslant \Psi(\sigma_2, p) && \text{for} \quad 0 \leqslant \sigma_1 \leqslant \sigma_2 \leqslant \pi/2. \end{aligned}$$

Let us replace any value $\sigma_i > \pi$ by π. Let us denote the new values by $\sigma'_1, \sigma'_2, \ldots, \sigma'_f$ and their sum by $\sigma' (\leqslant 4\pi)$. Owing to the above inequalities, we have for $f \geqslant 8$

$$V = \sum_{i=1}^{f} V_i \leqslant \sum_{i=1}^{f} \Psi(\sigma_i, p_i) \leqslant \sum_{i=1}^{f} \Psi(\sigma'_i, p_i) \leqslant f\Psi(\sigma'/f, p) \leqslant f\Psi(4\pi/f, p).$$

This is just the inequality (2).

The detailed discussion of the several types of polyhedra for which $f < 8$ contains no interesting new ideas. Instead of such a discussion let us consider, for example, only the type of a 5-sided prism ($f = 7$, $v = 10$), or more generally the case $f \geqslant 6$, $p \geqslant 4$. Since, for a fixed value of p ($p \geqslant 4$), the function $\Psi(\sigma, p)$ is an increasing function of σ up to a constant $c_p \geqslant 2\pi/3$, the proof runs word for word as above.

The cases of equality are evident, by the above proofs, in both inequalities (1) and (2).

Now we are going to give two further rigorous proofs of (1). On the other hand, we must admit that an attempt at a similar proof of the inequality (2) did not succeed.

Again let O be the centre of the insphere and put $r = 1$. Let us consider a face F_i of Π and denote the foot of the perpendicular from O to the face-plane by A. Further, let BD be an edge of F_i and C the foot of the perpendicular from A on it.

Suppose that C lies on the segment BD, just as A lies within F_i, and that this proves to be right for all faces and edges of Π. The surface of Π can in this case be decomposed into $4e$ right triangles one of which is ABC.

Consider the right spherical triangle $A'B'C'$ arising by central projection of ABC from O upon the insphere. Denote the angle at A' by α, the angle at B' by β and the hypotenuse $A'B'$ by c. Since $AB \geqslant \tan c$ and $\cos c = \cot \alpha \cot \beta$, the area t of the triangle ABC is given by

$$t \geqslant \tfrac{1}{4} \sin 2\alpha \tan^2 c = \tfrac{1}{4} \sin 2\alpha \,(\tan^2 \alpha \tan^2 \beta - 1) \equiv \Theta(\alpha, \beta).$$

Furthermore, since

$$\Theta_{\alpha\alpha}\Theta_{\beta\beta} - \Theta_{\alpha\beta}^{\;2} = \frac{2 \tan^4\alpha}{\cos^6\beta} [1 - (\sin^2\alpha + \sin^2\beta)]^2 \geqslant 0,$$

the function $\Theta(\alpha, \beta)$ in the domain determined by the inequalities $0 \leqslant \alpha \leqslant \pi/2$, $0 \leqslant \beta < \pi/2$, $\alpha + \beta \geqslant \pi/2$, is convex from below[10] and we have

[10]For the transformation of the Jacobian $\Theta_{\alpha\alpha}\Theta_{\beta\beta} - \Theta_{\alpha\beta}^2$ into the above simple form I am obliged to Mr. J. Molnár.

$$3V \geqslant \sum t \geqslant 4e\Theta(2\pi f/4e, 2\pi v/4e).$$

This is just the inequality (1) to be proved.

In order to get rid of the above restriction concerning the feet of the perpendiculars we can use the inequalities $F_i \geqslant \Phi(\sigma_i, p_i)$ of the first proof. In other words, we can replace any face by an admissible polygon of smaller area of which the number of sides and the area of the projection remain invariant.

The following alternative proof makes no use of the discussion of any special function.[11] We shall obtain the inequality in question as a corollary of the following general

THEOREM. *Decompose the surface S of the unit sphere by a net N having v vertices and e edges into a finite number $f \geqslant 4$ of convex spherical polygons $\sigma_1, \sigma_2, \ldots, \sigma_f$. Further let $P_1, P_2, \ldots, P_f$ be f points of S and $\varphi(\rho)$ a strictly increasing function defined for $0 \leqslant \rho < \pi$. Then*

$$\sum_{i=1}^{f} \int_{\sigma_i} \varphi(P_iP)d\omega \geqslant 4e \int_{\Delta} \varphi(AP)d\omega \tag{4}$$

where $d\omega$ denotes the area element of S at the variable point P, and Δ a right spherical triangle ABC the acute angles of which are $\alpha = \pi f/2e$ at A and $\beta = \pi v/2e$ at B. Equality holds only if N is the central projection of the edges of a regular polyhedron circumscribed about S and $P_1, P_2, \ldots, P_f$ the points of contact of the faces of this polyhedron.

Preparatory to the proof we make two remarks, easy to prove, in which a spherical domain and its area are denoted by the same symbol.

REMARK 1. Let s be a segment of a spherical cap c $(< 2\pi)$ with the top point T. Then the function

$$\Omega(s) = \int_s \varphi(TP)d\omega$$

is convex from above for $0 \leqslant s \leqslant c/2$.

REMARK 2. For any convex domain d lying in a "hemicap" of c,

$$\int_d \varphi(TP)d\omega \leqslant \Omega(d).$$

Let us first note that the integral $\int_{\sigma_i} \varphi(P_i P)d\omega$ obviously takes its minimum for a variable P_i at an inner point of σ_i. Therefore we may suppose that P_i lies within σ_i $(i = 1, 2, \ldots, f)$.

Let c_i be the spherical cap with the top point P_i and the radius AB, while $Q_1, Q_2, \ldots Q_{p_i}$ are the vertices of σ_i and $s_1, s_2, \ldots, s_{p_i}$ are the convex partial-domains of c_i lying outside of σ_i, the first bordered by the great circles P_iQ_1, Q_1Q_2, P_iQ_2, the second by P_iQ_2, Q_2Q_3, P_iQ_3, etc. Omitting the common integrand $\varphi(P_i P)d\omega$ under the integral signs we have

[11]Cf. the proof in the paper referred to in footnote 4.

$$\int_{\sigma_i} = \int_{c_i} - \sum_{\nu=1}^{p_i} \int_{s_\nu} + \int_{\sigma'_i}$$

where σ'_i denotes the part of σ_i not covered by c_i.

Consider the corresponding equalities for $i = 1, 2, \ldots, f$. The total number of the domains s_ν being $2e$, we get by the above remarks and Jensen's inequality

$$\sum_{i=1}^{f} \int_{\sigma_i} = f\int_c - \sum_{\nu=1}^{2e} \int_{s_\nu} + \sum_{i=1}^{f} \int_{\sigma'_i} \geqslant f\int_c - \sum_{\nu=1}^{2e} \Omega(s_\nu) + \sum_{i=1}^{f} \int_{\sigma'_i}$$
$$\geqslant f\int_c - 2e\,\Omega\left(\sum_{\nu=1}^{2e} \frac{s_\nu}{2e}\right) + \sum_{i=1}^{f} \int_{\sigma'_i},$$

where c denotes a spherical cap of radius AB and top point A, and

$$\int_c = \int_c \varphi(AP)d\omega.$$

Since, with the notation $\sum_{i=1}^{f} \sigma'_i = S'$, we have

$$S = fc - \sum_{\nu=1}^{2e} s_\nu + S',$$

we may write

$$\sum_{i=1}^{f} \int_{\sigma_i} \geqslant f\int_c - 2e\Omega\left(\frac{fc - S + S'}{2e}\right) + \sum_{i=1}^{f} \int_{\sigma'_i}$$
$$= f\int_c - 2e\Omega\left(\frac{fc - S}{2e}\right) + \sum_{i=1}^{f} \int_{\sigma'_i} - 2e\int_d \varphi(AP)d\omega,$$

denoting by d the partial domain of c which completes the segment of the cap c of area $(fc - S)/2e$ to the segment of area $(fc - S + S')/2e$. Furthermore, since by the monotonicity of $\varphi(\rho)$ the sum of the last two terms in the above inequality is $\geqslant 0$, we have

$$\sum_{i=1}^{f} \int_{\sigma_i} \geqslant f\int_c - 2e\,\Omega\left(\frac{fc - S}{2e}\right) = 4e\left\{\frac{\alpha}{2\pi}\int_c - \tfrac{1}{2}\Omega\left(\frac{fc - S}{2e}\right)\right\}.$$

But $(fc - S)/2e$ equals the area of the segment s of the cap c cut off by BC. This is obvious by

$$\Delta = \alpha + \beta - \frac{\pi}{2} = \frac{\alpha}{2\pi}c - \tfrac{1}{2}s.$$

This completes the proof of (4).

Equality holds only if S is entirely covered by the caps c_i without any part being covered three times, and the domains s_ν are all congruent segments of a spherical cap. This is just the case indicated above.

The inequality (1) is an immediate consequence of (4) for the function $\varphi(\rho) = \sec^3\rho$, for which $\frac{1}{3}\int_d \varphi(TP)d\omega$ equals the volume of the cone with a vertex at the centre of S cutting out from S the domain d and the base plane of which touches S at the point T.

Let us still note that the consideration of the function

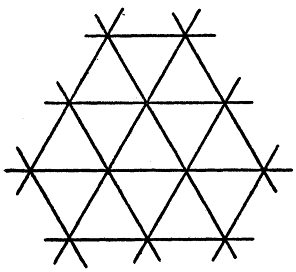
$$\varphi(\rho) = \begin{cases} \sec^3 \rho & \text{for } \quad 0 \leqslant \rho \leqslant AB \\ \sec^3 AB & \text{for } AB \leqslant \rho \leqslant \pi/2 \end{cases}$$

involves a sharpening of (1), according to which the volume V of Π can be replaced by the volume of the part of Π which lies in a sphere of radius $r \tan \frac{\pi}{p} \tan \frac{\pi}{q}$ concentric with the insphere.

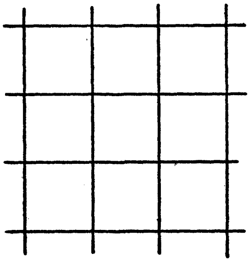
4. The regular degenerate polyhedra.[12] The five Platonic solids can be supplemented in a natural manner by three further "polyhedra" inscribed in or circumscribed to the sphere of infinite radius, or more correctly: tessellations in the plane. If we denote by $\{p, q\}$ the regular polyhedron having p-gonal faces, q at each vertex, then the eight regular polyhedra can be arranged into the following scheme:

$$\begin{array}{llll} \{3,3\} & \{3,4\} & \{3,5\} & \{3,6\} \\ \{4,3\} & \{4,4\} & & \\ \{5,3\} & & & \\ \{6,3\} & & & \end{array}$$

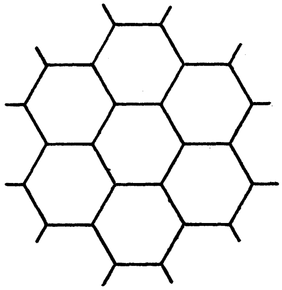
Any of the three regular degenerate polyhedra, represented by the figures below, can be considered as the limiting form of a set of convex polyhedra.



$\{3, 6\}$



$\{4, 4\}$



$\{6, 3\}$

The introduction of this terminology will prove suitable in the investigation of the question when our inequalities give exact asymptotic estimates for large values of e. Let us consider, for instance, a set of polyhedra of increasing values of e for which

$$\lim e\left(\frac{R}{r} - \tan\frac{\pi}{p}\tan\frac{\pi}{q}\right) = 0.$$

[12]Cf. H. S. M. Coxeter, *Regular Polytopes* (New York, 1949), chap. IV.

The "limiting poryhedra" of such sets are the regular ones.

In a certain sense we can briefly say that equality holds in our inequalities for, and only for, the eight regular polyhedra.

5. Further problems. Let us consider the inequalities analogous to (1) and (2) for the surface area F of the polyhedron Π:

$$e \sin \frac{2\pi}{p}\left(\tan^2 \frac{\pi}{p} \tan^2 \frac{\pi}{q} - 1\right) r^2 \leqslant F \leqslant e \sin \frac{2\pi}{p}\left(1 - \cot^2 \frac{\pi}{p} \cot^2 \frac{\pi}{q}\right) R^2.$$

The inequality on the left is equivalent to (1). On the other hand, the inequality on the right seems to involve some difficulties. We are going to prove this inequality only for polyhedra the faces and edges of which contain the foot of the centre of the circumsphere on their plane or line, respectively.

The proof is a dual counterpart of the second proof of (1). Let us keep the notations of this proof surrendering the rôle of the insphere to the circumsphere of radius $R = 1$. We have now $AB \leqslant \sin c$ and hence

$$t \leqslant \tfrac{1}{4} \sin 2\alpha \sin^2 c = \tfrac{1}{4} \sin 2\alpha\, (1 - \cot^2\alpha \cot^2 \beta)$$
$$\equiv \Gamma(\alpha, \beta) = -\Theta\left(\frac{\pi}{2} - \alpha, \frac{\pi}{2} - \beta\right).$$

Since $\Gamma_{\alpha\alpha}\,\Gamma_{\beta\beta} - \Gamma_{\alpha\beta}{}^2 = \dfrac{2 \cot^4 \alpha}{\sin^6 \beta}\,[1 - (\cos^2 \alpha + \cos^2 \beta)\,]^2 \geqslant 0$, the function $\Gamma(\alpha, \beta)$, for $0 \leqslant \alpha < \pi/2,\ 0 \leqslant \beta < \pi/2,\ \alpha + \beta \geqslant \pi/2$, is concave from below and we have

$$F = \sum t \leqslant 4e\, \Gamma(2\pi f/4e, 2\pi v/4e); \qquad \text{q.e.d.}$$

The proof of the general case miscarries for the following reason. Let us change the face F_i within the insphere so that the number p_i of its sides and the area σ_i of its projection from O upon the circumsphere remain invariant. Then the area F_i has only a local maximum for the regular p_i-gon inscribed in the circumsphere and takes its absolute maximum just in the case when A lies outside F_i, provided that σ_i remains below a certain constant which depends only on p_i.

Let us now return to the isepiphan problem. According to a well-known result of L. Lindelöf[13] the f-hedron which minimizes, by a given value of f, the quotient F^3/V^2 has the property of being circumscribed about a sphere. Hence for the best f-hedron $F^3/V^2 = 9F/r^2$. But this holds not only for the best f-hedra, but also, for instance, for the best dipyramids of given number of vertices and for the best polyhedra among many other classes of polyhedra as well. All these induce us to announce the following conjecture concerning any convex polyhedron:

[13] L. Lindelöf, "Propriétés générales des polyèdres etc.," *St. Petersburg Bull. Acad. Sci.*, vol. 14 (1869), 258-269.

$$F^3/V^2 \geqslant 9e \sin \frac{2\pi}{p}\left(\tan^2\frac{\pi}{p}\tan^2\frac{\pi}{q} - 1\right).$$

The proof of this inequality would, in a certain sense, close the range of the isepiphan problems for polyhedra.

Let us now agree upon the notations $A(x; k)$ and $H(x; k)$ for the arithmetic and harmonic means of certain numbers x_i with the weights k_i. Let further Π be a convex polyhedron, O an arbitrary inner point of it, $p_1, p_2, \ldots, p_f$ the numbers of the sides of the faces distant $r_1, r_2, \ldots, r_f$ from O, $q_1, q_2, \ldots, q_v$ the numbers of the edges running into the vertices distant $R_1, R_2, \ldots, R_v$ from O and put, as above, $p = A(p; 1)$, $q = A(q; 1)$. With these notations the following inequality probably holds:

$$A(R; q)/H(r; p) \geqslant \tan\frac{\pi}{p}\tan\frac{\pi}{q}.$$

This may be a generalization of certain previous results[14] suggested by a triangle inequality of L. J. Mordell and P. Erdös.[15] Here $A(R; q)$ cannot be replaced by $A(R; 1)$ just as $H(r; p)$ cannot be replaced by $H(r; 1)$. Similarly, the above inequality cannot be improved by putting $H(R; q)$ instead of $A(R; q)$ or $A(r; p)$ instead of $H(r; p)$.

[14] L. Fejes Tóth, "Inequalities Concerning Polygons and Polyhedra," *Duke Math. J.*, vol. 15 (1948), 817-822.

[15] L. J. Mordell, Problem 3740, proposed by Paul Erdös, *Amer. Math. Monthly*, vol. 44 (1937), 252.

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