whence

$$\int_a^b \frac{x}{e^x - 1}\, dx = 2\int_{\sqrt{a}}^{\sqrt{\beta}} \frac{\log(1+u)}{u}\, du + 2\left\{\int_a^{\sqrt{a}} - \int_\beta^{\sqrt{\beta}} - \frac{\log(1+u)}{u}\, du\right\}.$$

Let $a \to 0$ and $b \to \infty$, then $\sqrt{a} \to 0$, $\sqrt{\beta} \to 1$ and so (the limit existing by Theorem 3) $\int_{\sqrt{a}}^{\sqrt{\beta}} \frac{\log(1+u)}{u}\, du \to \int_0^1 \frac{\log(1+u)}{u}\, du$; furthermore, by (v) and (vi), $\int_a^{\sqrt{a}} \frac{-\log(1-u)}{u}\, du \to 0$ and $\int_\beta^{\sqrt{\beta}} \frac{-\log(1-u)}{u}\, du \to 0$, whence $\int_0^\infty \frac{x}{e^x - 1}\, dx$ exists and equals $2\int_0^1 \frac{\log(1+u)}{u}\, du$.

Combining theorems 3 and 4 we obtain Planck's integral

$$\int_0^\infty \frac{x}{e^x - 1}\, dx = \tfrac{1}{6}\pi^2.$$

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Some series for π

By C. E. WALSH.

Consider three sequences a_n, D_n, k_n $(n = 1, 2, 3, \ldots)$, such that $D_n a_n \to 0$ and, for $n > 1$,

(1) $$a_n + D_n a_n = D_{n-1} a_{n-1} + k_n a_n$$

Then $\sum_1^m a_n + D_m a_m = a_1(1 + D_1 - k_1) + \sum_1^m k_n a_n$. Hence, writing Σ for $\sum_1^\infty$,

(2) $$\Sigma a_n = a_1(1 + D_1 - k_1) + \Sigma k_n a_n$$

if either series converges. This will be applied to derive various series for π from the two known results[1]

[1] Knopp, *Infinite Series*, p. 269, Ex. 110 (a), and p. 246, Ex. 2.

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(3) $$\pi/2 = \Sigma(n+1)!\,N_n^{-1} = 1 + \Sigma n!\,N_n^{-1},$$
where $N_n = 3.\ 5.\ \ldots\ .(2n+1).$

We have first $\pi = 2\Sigma a_n$, where $a_n = (n+1)!\,N_n^{-1}$. Taking $D_n = (n+2)\,(n+a+1)^{-1}$, we find that the formula for k_n given by (1) is simplest when $a = 0$. Then $k_n = -n^{-1}(n+1)^{-1}$, and (2) yields

(4) $$\pi = 4 - 2\Sigma(n-1)!\,N_n^{-1}.$$

Repeating the procedure on the series in (4), we chose $D_n = n(n+a+1)^{-1}$, $a = 2$, $k_n = 3(n+2)^{-1}(n+3)^{-1}$, and obtain

(5) $$\pi = 3\tfrac{1}{3} - 6\Sigma(n-1)!\,[n+2)\,(n+3)\,N_n]^{-1}.$$

For this series, chose $D_n = n(n+a+1)^{-1}$, $a = 6$,
$$k_n = 3(3n+7)\,[(n+1)\,(n+6)\,(n+7)]^{-1},$$
from which follows

(6) $$\pi = 3\tfrac{4}{21} - 18\Sigma(3n+7)(n-1)!\,[(n+1)(n+2)(n+3)(n+6)\,(n+7)N_n]^{-1}.$$

Again, from (5), in two stages, if we first take $D_n = n\,(n+1)^{-1}$, rearrange[1] the resulting series slightly, then, at the second stage, take $D_n = (n+1)\,(n+a+1)^{-1}$, $a = 9$, there results the series

(7) $$\pi = 3\tfrac{19}{120} + 90\Sigma(n-8)(n-1)!\,[(n+1)(n+2)(n+3)(n+4)(n+9)(n+10)N_n]^{-1},$$

five terms of which give π to six decimal places.

If we proceed similarly from the second of the series (3), taking at the first stage $D_n = (n+1)\,(n+2)^{-1}$, and at the second stage $D_n = (n+1)\,(n+6)^{-1}$, rearranging[1] slightly the final result, we find

(8) $$\pi = 3 + 2\Sigma n!\,[(n+1)\,(n+2)N_n]^{-1}$$

(9) $$= 3\tfrac{1}{7} + 6\Sigma(n-3)n!\,[(n+2)\,(n+3)\,(n+6)(n+7)N_{n+1}]^{-1}.$$

This gives in series form the error[2] in the approximation $\pi = 22/7$. Taking only the two negative terms with which the series begins, we obtain π with an error in the fifth decimal place.

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[1] These rearrangements consist in taking the first term of $\sum$ separately and changing n to $n+1$.

[2] Another such expression was found by D. P. Dalzell, "On 22/7," *Journ. London Math. Soc.*, 19 (1944), pp. 133-4.

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