Can. J. Math., Vol. XXXI, No. 6, 1979, pp. 1247–1268

UNION OF REALCOMPACT SPACES AND LINDELÖF SPACES

AKIO KATO

0. Introduction. All spaces in this paper are completely regular Hausdorff and all maps are continuous onto, unless otherwise stated. The purpose of this paper is to investigate the realcompactness of a space X which contains a Lindelöf space L such that every zero-set Z (in X) disjoint from L is realcompact. We show in § 2 that such a space X is very close to being realcompact (Theorems I, II and III). But in general such a space fails to be realcompact. Indeed, in §§ 3 and 4 the following questions of Mrówka [**18**, **19**] are answered, both in the negative:

(Q. 1) If $X = L \cup G$ where L is Lindelöf closed and G is E-compact, then is X E-compact?

(Q. 2) Suppose $f: X \to Y$ is a perfect map such that the set $M(f) = \{y \in Y | \, |f^{-1}(y)| > 1\}$ of multiple points of f is Lindelöf (especially, countable) closed. If X is E-compact, is Y also E-compact?

Here, E is chosen to be the real line $\mathbf{R}$ or the countable discrete space N, and in case $E = N$ the above spaces X, Y are supposed to be 0-dimensional, i.e., the small inductive dimension $= 0$. (A space X is called *E-compact* if it is embeddable as a closed subset into the product E^m for some cardinal m [**17**]. Clearly "N-compact" implies "$\mathbf{R}$-compact", and "$\mathbf{R}$-compact" is identical with "realcompact". A map f is called *perfect* if it is a closed map with compact fibers.) In order to answer these questions we construct various examples each of which is an almost realcompact, non-realcompact space. Hence, finally we will come to appreciate the gap between realcompact spaces and almost realcompact spaces. It should be noted that the space of Mrówka [**18**] was the only example of an almost realcompact, non-realcompact space known hitherto.

Throughout this paper we adopt the notation and terminology of Gillman & Jerison [**10**]. βX and υX denote respectively the Stone-Čech compactification and the Hewitt realcompactification of X. $Z(X)$ (resp. $\mathrm{Coz}(X)$) denotes the family of all zero-sets (resp. cozero-sets) in X. The remainder $\beta X \backslash X$ of Stone-Čech compactification is always denoted by X^*. The symbol $\oplus$ means the topological sum. N is the discrete space of positive integers. For a locally compact space X, ωX denotes its one-point compactification, except in § 4 Example D. "dim" means the covering dimension. For the notion of E-compactness we refer to [**17**].

Received June 9, 1977 and in revised form October 13, 1978.

https://doi.org/10.4153/CJM-1979-104-8 Published online by Cambridge University Press

Remark 0. Most results for realcompact (= R-compact) spaces in this paper can be generalized to E-compact spaces, so that, especially their "N-compact" versions remain true. To obtain an "N-compact" version of the given statement for realcompact spaces, in most cases it suffices to replace the left terms in the following table by the corresponding right terms.

TABLE 1.

"Realcompact" version	"N-compact" version
completely regular spaces, i.e., subspaces of $\mathbf{R}^m$	0-dimensional spaces, i.e., subspaces of N^m
realcompact spaces, i.e., closed subspaces of $\mathbf{R}^m$	N-compact spaces, i.e., closed subspaces of N^m
compact spaces, i.e., closed subspaces of I^m or $(\omega\mathbf{R})^m$	0-dimensional compact spaces, i.e., closed subspaces of D^m or $(\omega N)^m$
$\beta X = \beta_I X = \beta_{\omega\mathbf{R}} X$	$\beta_D X = \beta_{\omega N} X$
$\upsilon X = \beta_{\mathbf{R}} X$	$\beta_N X$
real-valued function	N-valued function

In this table m ranges over all cardinal numbers, and D denotes the two-point discrete space $\{0, 1\}$. $\beta_E X$ denotes the E-compact extension of X uniquely determined by the property that every continuous function $f:X \to E$ admits a continuous extension $f^*:\beta_E X \to E$. See [**17**, Theorem 4.14].

1. Fundamental theorems. The main theorem in this section is Theorem 1.4 which shows how the operation υ is distributed over subspaces, and from which many corollaries are derived. Our concern is for a space X that is the union of a Lindelöf space L and a realcompact space, or more generally a space X that contains a Lindelöf space L such that every zero-set (or equivalently, closed set) in X disjoint from L is realcompact. Recall that a subset A is called *z-embedded* in X if every zero-set in A is the restriction to A of a zero-set in X. The *G_δ-closure* of A in X, denoted by G_δ-$\mathrm{cl}_X A$, consists of all points $x \in X$ for which each G_δ-set (or equivalently, zero-set) about x meets A. The following fundamental facts are needed.

Fact 1.1. [**12**] [**1**, 7.8] If L is a Lindelöf subset of a space X, then L is z-embedded in X.

Fact 1.2. Let A be a realcompact subset of X.
(1) [**10**, 8.10] If A is C-embedded in X, then $\mathrm{cl}_{\upsilon X} A = A$.
(2) [**4**, 2.6] If A is z-embedded in X, then G_δ-$\mathrm{cl}_{\upsilon X} A = A$.

Fact 1.3. [**10**, 8.8] [**2**, § 5] For any function $f \in C(X)$ and its extension $f^\upsilon \in C(\upsilon X)$, we have $Z(f^\upsilon) = \mathrm{cl}_{\upsilon X} Z(f)$ and $\mathrm{Coz}(f^\upsilon) = \upsilon(\mathrm{Coz}(f))$, i.e., $\mathrm{Coz}(f)$ is C-embedded in the realcompact space $\mathrm{Coz}(f^\upsilon)$.

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THEOREM 1.4. *Let A be a subset of a space X. Then*

(1) $\upsilon X = \mathrm{cl}_{\upsilon X} A \cup \bigcup \{\upsilon U | U \in \mathrm{Coz}(X)$ *and* U *is completely separated (in* X*) from* $A\}$ *where* υU *is understood as in* 1.3.

(2) $\upsilon X = G_\delta\text{-}\mathrm{cl}_{\upsilon X} A \cup \bigcup \{\mathrm{cl}_{\upsilon X} Z | Z \in Z(X)$ *and* $Z \cap A = \emptyset\}$.

Proof. (1) Let $p \in \upsilon X \backslash \mathrm{cl}_{\upsilon X} A$. Then, since p does not belong to the compact set $\mathrm{cl}_{\beta X} A$, there exists $\tilde{U} \in \mathrm{Coz}(\beta X)$ which contains p and is completely separated from $\mathrm{cl}_{\beta X} A$ in βX. Put $U = \tilde{U} \cap X$. Then by 1.3, $p \in \upsilon U$. Clearly U is completely separated from A in X. This proves (1).

(2) Let $p \in \upsilon X \backslash G_\delta\text{-}\mathrm{cl}_{\upsilon X} A$. Then there exists a zero-set $\tilde{Z} \in Z(\upsilon X)$ that contains p and misses A. Put $Z = \tilde{Z} \cap X$. By 1.3 we have $p \in \tilde{Z} = \mathrm{cl}_{\upsilon X} Z$ and this proves (2).

COROLLARY 1.5. *Let A be a realcompact closed subset of X such that every zero-set $Z \in Z(X)$ completely separated from A is realcompact. Then X is realcompact if and only if $\mathrm{cl}_{\upsilon X} A = A$. Especially, X is realcompact if A is C-embedded in X.*

Proof. The non-obvious part that requires a proof is that $\mathrm{cl}_{\upsilon X} A = A$ implies that X is realcompact. By 1.4(1) we need to show that every $U \in \mathrm{Coz}(X)$ completely separated from A is realcompact. Let U be such a cozero-set; then we can choose $Z \in Z(X)$ which contains U and is completely separated from A. By our hypothesis Z is realcompact, and hence U is also realcompact.

The last assertion of 1.5 was proved by Mrówka [**18**].

COROLLARY 1.6. *Let A be a realcompact z-embedded subset of X. Then X is realcompact if and only if $\mathrm{cl}_{\upsilon X} Z = Z$ for every $Z \in Z(X)$ disjoint from A. Especially, X is realcompact if every $Z \in Z(X)$ disjoint from A is realcompact and C-embedded (equivalently, z-embedded) in X.*

This corollary follows from 1.2(2) and 1.4(2). Note here that for any zero-set Z the concepts of C-, C^*-, and z-embedding are all equivalent [**1**, 7.5]. The last part of 1.6, in case A Lindelöf, was pointed out by A. Okuyama. As special cases of 1.5 and 1.6 we obtain the next useful result. Recall that a space is a *P-space* if every zero-set is open.

THEOREM 1.7. *Let $X = L \cup G$ where L is a Lindelöf subspace such that every $Z \in Z(X)$ disjoint from L is realcompact. Then X is realcompact if one of the following conditions is satisfied.*

(1) *L is a zero-set in X.*

(2) *G is a P-space and open in X.*

(3) *G is normal and z-embedded in X.*

Proof. (1) and (2): In order to apply 1.5 it suffices to note that the z-embedded L is C-embedded if and only if every $Z \in Z(X)$ disjoint from L is completely separated from L. In case (2) such a Z is clopen (= closed and open) in X.

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(3): Let Z be a zero-set disjoint from L. Since G is normal, Z is z-embedded in G. Since G is z-embedded in X, it follows that Z is z-embedded in X. Hence 1.6 applies.

When G (in 1.7) itself is realcompact, we know

THEOREM 1.8. *Let $X = L \cup G$ be the union of a Lindelöf space L and a realcompact space G. If G is z-embedded in X, then X is realcompact.*

This theorem is a direct consequence of the next lemma [**3**, 3.9], which follows from 1.2(2).

LEMMA 1.9. (Blair) *If $X = \bigcup_{n \in N} X_n$ is the union of a countable family of z-embedded realcompact subspaces X_n, then X is realcompact.*

It is interesting to compare 1.8 with 1.7(3). We see later in § 3 Example B that the condition of realcompactness of G in 1.8 can not be replaced by the weaker condition that every $Z \in Z(X)$ disjoint from L is realcompact. As a special case of 1.8 or 1.5 the following theorem due to Mrówka [**18**] is known. As Mrowka did not give its proof in explicit form, we present the proof.

THEOREM 1.10. (Mrówka) *Let $X = L \cup F$ be the union of a Lindelöf space L and a realcompact space F. If both L and F are closed in X, then X is realcompact.*

To prove this, we need only see, by 1.8 or 1.5, that F is C-embedded in X, which follows immediately from

LEMMA 1.11. *Let F be a closed subset of a space X. If the boundary $\partial_X F$ of F is C-embedded in $X \backslash \mathrm{int}_X F$, then F is C-embedded in the whole space X.*

Proof. Let $f \in C(F)$ and put $E = X \backslash \mathrm{int}_X F$. Then, by the hypothesis there exists an extension $g \in C(E)$ of $f \mid \partial_X F$. Define $h: X \rightarrow \mathbf{R}$ by $h|F = f$ and $h|E = g$. Then h is continuous because E and F are both closed in X. Hence h is an extension of f.

Remark 1.12. In view of 1.4 as well as 1.5 and 1.6, it seems an important task to obtain an inner characterization of the equality $\mathrm{cl}_{vX} F = F$ for a realcompact closed subset F of X. Such a characterization is not yet known in the complete form (cf. [**10**, 8.10]). Here we point out some sufficient conditions.

Let F be a realcompact closed subset of X. Then the implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) *hold*:

(1) *F is C-embedded in X.*

(2) *Given any decreasing sequence $\{Z_n'\}_{n \in N}$ of zero-sets in F with empty intersection, there exists a sequence $\{Z_n\}_{n \in N}$ of zero-sets in X with empty intersection such that $Z_n' \subset Z_n$ for each $n \in N$.*

(3) $\mathrm{cl}_{vX} F = F$.

Proof. (1) $\Rightarrow$ (2): For each Z_n' choose $Z_n'' \in Z(X)$ such that $Z_n' = Z_n'' \cap F$. Put $Z'' = \bigcap_{n \in N} Z_n''$. Then, since F is C-embedded and disjoint from

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Z'', there exists $Z \in Z(X)$ such that $F \subset Z$ and $Z \cap Z'' = \emptyset$. Put $Z_n = Z \cap Z_n''$. Then the sequence $\{Z_n\}_{n \in N}$ satisfies (2).

(2) $\Rightarrow$ (3): Since $\mathrm{cl}_{vX}F = (\mathrm{cl}_{\beta X}F) \cap vX$, we need to see that

$$\mathrm{cl}_{\beta X}F \backslash F \subset \beta X \backslash vX.$$

Let $\Phi: \beta F \to \mathrm{cl}_{\beta X}F$ be the extension of the identity map 1_F of F, and let $p \in \mathrm{cl}_{\beta X}F \backslash F$. Choose a point $q \in \Phi^{-1}(p)$. Then, since F is realcompact, there exists a decreasing sequence $\{Z_n'\}_{n \in N}$ of zero-sets in F such that

$$q \in \bigcap_{n \in N} \mathrm{cl}_{\beta F} Z_n' \text{ and } \bigcap_{n \in N} Z_n' = \emptyset.$$

For this sequence choose a sequence $\{Z_n\}_{n \in N}$ of zero-sets in X as in (2). Then

$$p = \Phi(q) \in \Phi(\mathrm{cl}_{\beta F} Z_n') = \mathrm{cl}_{\beta X} Z_n' \subset \mathrm{cl}_{\beta X} Z_n$$

for each $n \in N$. Thus we have $p \in \bigcap_{n \in N} \mathrm{cl}_{\beta X} Z_n$ and $\bigcap_{n \in N} Z_n = \emptyset$, which means $p \in \beta X \backslash vX$. The proof is complete.

Remark 1.13. The most concrete case of 1.7(2) is when G is a discrete space. In this case 1.7 reads as follows:

The union $L \cup D$ of a Lindelöf closed space L and a discrete space D is realcompact if and only if every clopen subset disjoint from L is of nonmeasurable cardinal.

If one considers topological completeness instead of realcompactness, the cardinality condition can be deleted in the above statement. In fact the following stronger assertion holds:

The union $L \cup D$ of a Lindelöf closed space L and a discrete space D is always paracompact.

The proof is an easy exercise.

2. Nearly realcompact properties. The main results in this section are Theorems I, II and III. Let $X = L \cup G$ be the union of a Lindelöf space L and a realcompact space G. To sum up roughly, Theorem I shows that a little change of L makes X realcompact, while Theorem II shows that a little change of G makes X realcompact; Theorem III finds a dense realcompact subspace of X. Let us first recall that a space X is called *almost realcompact* if it is the perfect image of some realcompact space [**21**, § 6 Theorem 3].

THEOREM I. *Let $X = L \cup G$ where L is a Lindelöf space such that every $Z \in Z(X)$ disjoint from L is realcompact. Suppose G is either realcompact or normal. If G is dense or open in X, then there exist a realcompact space $\tilde{X} = \tilde{L} \cup G$ such that $\tilde{L}$ is Lindelöf and a perfect irreducible map $\varphi: \tilde{X} \to X$ which keeps G pointwise fixed and maps $\tilde{L}$ onto L. Consequently, X is almost realcompact.*

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Proof. (In case G is dense in X): In this case βX is a compactification of G, and hence there is the Stone extension $\Phi:\beta G \rightarrow \beta X$ of the identity map $1_G:G \rightarrow G$. Put $\tilde{X} = \Phi^{-1}(X)$, $\tilde{L} = \Phi^{-1}(L)$ and $\varphi = \Phi|\tilde{X}:\tilde{X} \rightarrow X$. Then $\tilde{L}$ is Lindelöf and φ is perfect and irreducible. Since G is C^*-embedded hence z-embedded in $\tilde{X}$, it follows from 1.8 and 1.7(3) that $\tilde{X}$ is realcompact.

(In case G is open in X): Put $X_0 = \mathrm{cl}_X G$, $L_0 = L \cap X_0$ and $L' = L\backslash G$. Then $X_0 = L_0 \cup G$ where L_0 is Lindelöf and G is dense in X_0. So, by the above argument there exist a realcompact space $\tilde{X}_0 = \tilde{L}_0 \cup G$ and a perfect irreducible map $\varphi_0:\tilde{X}_0 \rightarrow X_0$ which is an extension of $1_G:G \rightarrow G$. Let $f:L' \oplus X_0 \rightarrow X = L' \cup X_0$ be the canonical two-to-one map. Then, since L' and X_0 are both closed in X, f is perfect and irreducible. Put $\tilde{X} = L' \oplus \tilde{X}_0$ and $\tilde{L} = L' \oplus \tilde{L}_0$. Then $\tilde{X} = \tilde{L} \cup G$, and $\tilde{L}$ is Lindelöf. Define a map φ by

$$\varphi = f \circ (1_{L'} \oplus \varphi_0):\tilde{X} = L' \oplus \tilde{X}_0 \rightarrow L' \oplus X_0 \rightarrow X.$$

It is easy to see that $\tilde{X}$ and φ satisfy the conditions in Theorem I. This completes the proof.

The final assertion of Theorem I, in case G is realcompact, can be generalized as follows.

THEOREM 2.1. *Let $X = L \cup \bigcup_{n \in N} G_n$ be the union of a Lindelöf space L and almost realcompact spaces G_n for $n \in N$. If each G_n is dense or open in X, X is almost realcompact.*

Proof. Consider the absolute, or projective cover, $E(X)$ of X and the associated perfect irreducible map $\varphi:E(X) \rightarrow X$ [**23**]. $E(X)$ is known to be extremally disconnected. Put $\tilde{L} = \varphi^{-1}(L)$ and $\tilde{G}_n = \varphi^{-1}(G_n)$ for $n \in \mathbf{N}$. Since φ is perfect and irreducible, $\tilde{L}$ is Lindelöf and $\tilde{G}_n$ is almost realcompact [**8**, Theorem 8] and dense or open in $E(X)$. (Irreducible maps pull dense subsets back to dense subsets.) Note that any dense or open subset of an extremally disconnected space is extremally disconnected and C^*-embedded [**10**, 1H, 6M], and that any extremally disconnected, almost realcompact space is realcompact [**5**, Theorem 1.2]. Hence each $\tilde{G}_n$ is realcompact and C^*-embedded in $E(X)$. Therefore by 1.9, $E(X) = \tilde{L} \cup \bigcup_{n \in N} \tilde{G}_n$ is realcompact; consequently, X is almost realcompact.

Remark 2.2. Note that, in Theorem I, the multiple points of φ are contained in L. So Theorem I shows in general that in case $E = \mathbf{R}$ a negative answer to the question (Q.1) in §0 provides a negative answer to (Q.2). This is true also for the case $E = N$. In fact, the "N-compact" version of Theorem I holds, i.e., in Theorem I we can replace the term "realcompact" by "N-compact" if the space X is supposed to be 0-dimensional (see Remark 0 in the introduction). It should be noted that the proof of the "N-compact" version of Theorem I coincides with that of Theorem I if βG and βX are 0-dimensional, i.e., $\beta G = \beta_D G$ and $\beta X = \beta_D X$ where $D = \{0, 1\}$.

Remark 2.3. The hypothesis in Theorem I (resp. Theorem 2.1) that G

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(resp. G_n) is dense or open is essential. Indeed the space $\Psi = N \cup D$ in [**10**, 5I] is the union of a Lindelöf space N and a realcompact space D, but it is not almost realcompact, because any pseudocompact, almost realcompact space must be compact.

From Theorem I we can derive a general condition for the space X there to be realcompact. A locally finite family $\{F_n\}_{n\in N}$ of subsets in X is called *expandable* (resp. *z-expandable*) if each F_n is contained in an open set U_n (resp. a zero-set Z_n) such that $\{U_n\}_{n\in N}$ (resp. $\{Z_n\}_{n\in N}$) is locally finite in X.

THEOREM 2.4. *Let $X = L \cup G$ be as in Theorem I. Then X is realcompact if every expandable sequence $\{F_n\}_{n\geqq 0}$ in X with the property (*) is z-expandable:*

(*) *$\{F_n\}_{n\geqq 0}$ is decreasing, hence $\bigcap_{n\geqq 0} F_n = \emptyset$, and consists of $F_0 \in Z(X)$ disjoint from L and $F_n \in Z(G)$ for $n \geqq 1$.*

In particular, X is realcompact if one of the following conditions is satisfied:

(1) *Every decreasing sequence $\{F_n\}_{n\in N}$ of regular closed sets in X, with $\bigcap_{n\in N} F_n = \emptyset$, is z-expandable.*

(2) *For every $Z \in Z(X)$ disjoint from L, any decreasing sequence $\{F_n\}_{n\in N}$ of zero-sets in Z with $\bigcap_{n\in N} F_n = \emptyset$ is z-expandable.*

(3) *G is z-embedded in X.*

Proof. Suppose that X is not realcompact. Then we can choose a point $p \in \upsilon X\backslash X$. By 1.4(2) or 1.6 there exists an $F_0 \in Z(X)$ such that $F_0 \cap L = \emptyset$ and $p \in \mathrm{cl}_{\upsilon X} F_0$. Let $\tilde{X}$ and $\varphi:\tilde{X} \to X$ be as in Theorem I. Let $\Phi:\beta\tilde{X} \to \beta X$ be the extension of φ. Since $p \in \mathrm{cl}_{\beta X} F_0$, we have

$$\Phi^{-1}(p) \cap \mathrm{cl}_{\beta\tilde{X}} F_0 \neq \emptyset$$

(F_0 is identified with $\varphi^{-1}(F_0)$); select a point q from this non-empty set. Then q is contained in $\beta\tilde{X}\backslash\tilde{X}$ since φ is perfect. As $\tilde{X}$ is realcompact, there is a function $f \in C(\beta\tilde{X})$ that is positive on $\tilde{X}$ and vanishes at q. Put $U_n = \tilde{X} \cap f^{-1}[0, 1/n)$ and $F_n = F_0 \cap f^{-1}[0, 1/n + 1]$ for $n \geqq 1$. Since the map φ is the identity map on G, we henceforth identify F_n in $\tilde{X}$ with $\varphi(F_n)$ in X. It is clear that $q \in \mathrm{cl}_{\beta\tilde{X}} F_n$ and $\bigcap_{n\in N} F_n = \emptyset$. Put $V_n = X\backslash\varphi(\tilde{X}\backslash U_n)$. Then V_n is open in X and contains F_n. Since $\{U_n\}$ is locally finite in $\tilde{X}$ and φ is perfect, $\{\varphi(U_n)\}$ is also locally finite in X. Since $V_n \subset \varphi(U_n)$, $\{V_n\}$ is locally finite. Now it follows that $\{F_n\}_{n\geqq 0}$ is an expandable sequence in X satisfying the condition (*). By the hypothesis there exists a sequence $\{Z_n\}_{n\geqq 0}$ of zero-sets in X such that $\bigcap_{n\geqq 0} Z_n = \emptyset$ and $F_n \subset Z_n$ for each n. Therefore

$$p \in \Phi(\mathrm{cl}_{\beta\tilde{X}} F_n) = \mathrm{cl}_{\beta X} F_n \subset \mathrm{cl}_{\beta X} Z_n$$

and hence

$$p \in \bigcap_{n\geqq 0} \mathrm{cl}_{\beta X} Z_n,\ \bigcap_{n\geqq 0} Z_n = \emptyset \text{ and } Z_n \in Z(X).$$

This shows $p \notin \upsilon X$ which contradicts our assumption. Thus, X is proved to be realcompact. Next we prove the special cases (1), (2) and (3). Let $\{F_n\}_{n\geqq 0}$ be

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any expandable sequence in X with the property (*). We need to show in each case that this $\{F_n\}$ is z-expandable.

(1) Since $\{F_n\}$ is expandable, there exists a decreasing sequence $\{R_n\}$ of regular closed sets in X with $\bigcap_{n\geqq 0} R_n = \emptyset$ and $F_n \subset R_n$ for each n. By (1) this R_n is z-expandable so that $\{F_n\}$ is also z-expandable.

(2) It suffices to note $F_n \in Z(F_0)$ and $F_0 \in Z(X)$.

(3) By (3) there exists $Z_n \in Z(X)$ such that $F_n = Z_n \cap G$ for each $n \geqq 1$. Put $Z_0 = F_0$. Then

$$\bigcap_{n\geqq 0} Z_n = F_0 \cap \bigcap_{n\geqq 1} Z_n \subset F_0 \cap L = \emptyset;$$

hence $\{F_n\}$ is z-expandable.

Remark 2.5. A space X satisfying the condition (1) in 2.4 is often called a *weak cb-space*. It is well known that any almost realcompact, weak *cb*-space is realcompact [**5**]. Hence 2.4(1) follows also, only from the final assertion of Theorem I that X is almost realcompact. 2.4(2) follows also from 1.6 and 1.12. Of course 2.4(3) follows also from 1.8.

For a space X consider the new topology generated by $Z(X)$. The set X endowed with this new topology is denoted by pX. Clearly pX is a P-space. According to T. Terada [**24**] a space X is called *P-realcompact* if pX is realcompact. The following lemma is known [**9**, Theorem 4] (cf. [**15**, 4.8] [**25**, 5.2]).

LEMMA 2.6. *Any realcompact space X is P-realcompact, i.e., pX is realcompact.*

THEOREM II. *Suppose X contains a Lindelöf space L such that every $Z \in Z(X)$ disjoint from L is realcompact. Then there exist a realcompact space $\tilde{X} = L \cup P$ and a continuous bijection $\varphi: \tilde{X} \to X$ such that P is a P-space and the topology about $L \subset \tilde{X}$ is identical with the topology about $L \subset X$. Consequently, X is P-realcompact.*

Proof. Let τ be the topology of the space X. Let $\tilde{X}$ be a new space with the same underlying set as X and the topology generated by

$$\tau \cup \{Z \in Z(X) |\ Z \cap L = \emptyset\}.$$

(Observe that $\tilde{X}$ is completely regular.) Let $\varphi: \tilde{X} \to X$ be the identity function and put $P = \varphi^{-1}(X \backslash L)$. Then P is a P-space and open in $\tilde{X}$. To prove $\tilde{X}$ realcompact, we need only see by 1.7(2) that every $Z' \in Z(\tilde{X})$ disjoint from $L = \varphi^{-1}(L)$ is realcompact. Let Z' be such a zero-set in $\tilde{X}$. Then by our definition of the topology of $\tilde{X}$, $\mathrm{cl}_X \varphi(Z')$ misses L. Since L is Lindelöf and disjoint from $\mathrm{cl}_X \varphi(Z')$, there is a $Z \in Z(X)$ missing L and containing $\varphi(Z')$. As Z is realcompact, $\varphi^{-1}(Z) \cong pZ$ is also realcompact by 2.6. Hence the closed subspace Z' of $\varphi^{-1}(Z)$ is realcompact. Thus $\tilde{X}$ is proved to be realcompact. Since pX coincides with $p\tilde{X}$, pX is also realcompact.

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If we pay attention just to P-realcompactness, the very same proof as Theorem II yields the following.

COROLLARY 2.7. *Suppose X contains a Lindelöf space L such that every $Z \in Z(X)$ disjoint from L is P-realcompact. Then X is P-realcompact.*

To prove the coming Theorem III we need a couple of lemmas.

LEMMA 2.8. [**14**, 2.10] *A space X is locally realcompact if and only if $\mathrm{cl}_{\beta X}(\upsilon X \backslash X)$ is disjoint from X in βX.*

LEMMA 2.9. *Suppose X contains a Lindelöf closed subspace L such that every $Z \in Z(X)$ disjoint from L is realcompact. If L is locally compact, X is locally realcompact.*

Proof. Observe that $X \cup \mathrm{cl}_{\beta X} L$ is realcompact. Hence $\upsilon X \backslash X$ is contained in $\mathrm{cl}_{\beta X} L \backslash L$. As L is locally compact, $\mathrm{cl}_{\beta X} L \backslash L$ is compact. Therefore

$$\mathrm{cl}_{\beta X}(\upsilon X \backslash X) \subset \mathrm{cl}_{\beta X} L \backslash L.$$

Since L is closed in X, $\mathrm{cl}_{\beta X} L \backslash L$ is disjoint from X. Thus, by 2.8 X is locally realcompact.

THEOREM III. *Suppose X contains a Lindelöf closed subspace such that every $Z \in Z(X)$ disjoint from L is realcompact and of nonmeasurable cardinal. If X is locally realcompact (especially when L is locally compact), then X contains a realcompact, open and dense subspace X_0 including L.*

Proof. Note that the statement in the parentheses follows from 2.9. Put $K = \mathrm{cl}_{\beta X}(\upsilon X \backslash X)$. Then by 2.8 K is compact and disjoint from X. Since L is Lindelöf, there exists $\tilde{Z} \in Z(\beta X)$ containing K and missing L. Put $Z = \tilde{Z} \cap X$. Select a maximal disjoint collection $\{U_\lambda | \lambda \in \Lambda\}$ in Z such that each U_λ is a realcompact cozero-set in X. Put $U = \bigcup_{\lambda \in \Lambda} U_\lambda$. This $U = \bigoplus_{\lambda \in \Lambda} U_\lambda$ is realcompact because the cardinality of Λ is nonmeasurable by our assumption. Set $X_0 = (X \backslash Z) \cup U \subset X$. Then X_0 is the topological sum of realcompact spaces $X \backslash Z = \upsilon X \backslash \tilde{Z}$ and U; hence X_0 is realcompact. Since U is open and dense in Z, X_0 is open and dense in X.

3. Dieudonné plank. Now we present two simple examples. Example A provides negative answers to both of the questions (Q.1) and (Q.2) in § 0. Example B relates to the condition on G in 1.7 and 1.8. (Different examples with better topological properties will be constructed in §§ 4 and 5, yet we want to present here Examples A and B because of their simplicity.) Both examples are subspaces of the Tychonoff plank T [**10**, 8.20].

Let $W = [0, \omega_1)$ be the space of ordinals less than the first uncountable ordinal ω_1, with the interval topology. Let $\tilde{W} = [0, \omega_1]$ be its one-point compactification. Let us denote by D the discrete subspace of W consisting of all isolated ordinals, and put $\tilde{D} = D \cup \{\omega_1\} \subset \tilde{W}$. Let $\omega N = N \cup \{\omega\}$ be the

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one-point compactification of N. T denotes the Tychonoff plank, i.e., $T = \tilde{W} \times \omega N \setminus \{p\}$ where $p = (\omega_1, \omega)$. Define $T_D = \tilde{D} \times \omega N \setminus \{p\}$. This subspace T_D of T is called the *Dieudonné plank* [**22**, Example 89].

Example A. The Dieudonné plank T_D is the union of a countable discrete, closed subspace and a realcompact (N-compact) metric space, hence is almost realcompact (by Theorem I), but it is not realcompact.

Put $L = \{\omega_1\} \times N$ and $M = D \times \omega N$. Then $T_D = L \cup M$ and L is a countable discrete, closed space and M is a metric space. Since D and ωN are N-compact, M is also N-compact and hence realcompact. The next assertion can be found also in [**16**].

ASSERTION A.1. *$vT_D = T_D \cup \{p\} = \tilde{D} \times \omega N$ and hence T_D is not realcompact.*

Proof. Since $\tilde{D}$ is Lindelöf, $T_D \cup \{p\} = \tilde{D} \times \omega N$ is also Lindelöf hence realcompact. So, to prove Assertion A.1 we need only see that T_D is C-embedded in $T_D \cup \{p\}$. Let $f \in C(T_D)$. Since each point of L is a P-point in T_D, we can choose a neighborhood U of $\omega_1 \in \tilde{D}$ such that f is constant on $U \times \{n\}$ for each $n \in N$. Thence f is also constant on $(U \setminus \{\omega_1\}) \times \{\omega\}$; let t be the constant value of f. Extend f over $T_D \cup \{p\}$ by setting $f(p) = t$. It is easy to see that this extended f is continuous. Thus, T_D is proved to be C-embedded in $T_D \cup \{p\}$.

ASSERTION A.2. *There exist an N-compact space $\tilde{T}_D$ and a perfect irreducible map $\varphi: \tilde{T}_D \to T_D$ with the countable discrete closed set $L = \{\omega_1\} \times N$ of multiple points.*

Proof. Since $\dim \beta M = \dim M = 0$ and $\dim \beta T_D = \dim vT_D = 0$, our Assertion follows from Theorem I and Remark 2.2.

Thus, Assertions A.1 and A.2 answer (Q.1) and (Q.2) in § 0 respectively, in the negative.

Example B. A space $X = L \cup G$, with the property that L is Lindelöf and every $Z \in Z(X)$ disjoint from L is realcompact, fails to be realcompact even if G is C-embedded in X.

(This shows that the condition on G in 1.7 (3) and 1.8 that G is either normal or realcompact is essential.)

Consider the subspace $X = (\tilde{W} \times N) \cup (D \times \{\omega\})$ of T. Put $L = \{\omega_1\} \times N$ and $G = (W \times N) \cup (D \times \{\omega\})$. We show that $X = L \cup G$ is the required space. First note the well-known property of W that any continuous real-valued function on W is constant on its tail. Thence a similar argument as in the proof of Assertion A.1 shows that G is C-embedded in X and that $vX = X \cup \{p\}$, hence X is not realcompact. Next, put $Z_\alpha = ([0, \alpha] \times N) \cup (D \times \{\omega\})$ for each $\alpha \in W$. It is easy to see that Z_α is a metric space (of non-

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measurable cardinal) and hence is realcompact. Therefore it follows that every $Z \in Z(X)$ disjoint from L is realcompact, because such a Z is contained in some Z_α. Thus, X is proved to be the required space. The next remark illustrates an application of Theorem 2.1.

Remark. X is almost realcompact.

Indeed, X is expressed as the union of a Lindelöf space $\tilde{W} \times N$ and a realcompact dense subspace $D \times \omega N$. Hence, by 2.1, X is almost realcompact.

4. Examples with better topological properties. In view of 1.7(1) and the fact that the Dieudonné plank in Example A is not 1-st countable, we now present 1-st countable counterexamples for (Q.1) and (Q.2) in § 0. By those examples we will establish the following theorems.

THEOREM 4.1. *There exist* 1-*st countable spaces* X *and* Y *and a finite-to-one closed map* $f: X \to Y$ *such that:*

(1) X *is* N-*compact hence realcompact, but* Y *is not realcompact.*

(2) *The set* $M(f)$ *of multiple points of* f *is a countable discrete closed subset of* Y.

Recall that a space is called *submetrizable* if it admits a continuous bijection onto some metric space.

THEOREM 4.2. *There exists a* 1-*st countable, non-realcompact space* $Y_\Delta = G_\Delta \cup N$ *such that* G_Δ *is a separable submetrizable (hence realcompact) dense subspace and* N *is a countable discrete closed subset of* Y_Δ.

THEOREM 4.3. *There exists a* 1-*st countable, non-realcompact space* $Y_\Box = G_\Box \cup N$ *such that* $G_\Box$ *is a metrizable dense subspace* ($G_\Box$ *is of cardinal* $\mathfrak{c}$, *hence realcompact*) *and* N *is a countable discrete closed subset of* $Y_\Box$.

The next lemma 4.4, proved in [**13**], plays an essential role in our construction of examples. A map $f: (X, A) \to (Y, B)$ is called a *relative homeomorphism* if $f \mid (X \setminus A)$ maps $X \setminus A$ homeomorphically onto $Y \setminus B$.

LEMMA 4.4. [**13**, Theorem 7.1] *Let* $\varphi: (X, A) \to (Y, B)$ *be a relative homeomorphism. If* φ *is a quotient map, then its Stone extension*

$$\Phi: (\beta X, \mathrm{cl}_{\beta X} A) \to (\beta Y, \mathrm{cl}_{\beta Y} B)$$

is a relative homeomorphism.

LEMMA 4.5. (1) *Suppose a* 0-*dimensional space* X *admits a (continuous) map* $f: X \to M$ *onto a hereditarily* N-*compact space* M *such that each fiber* $f^{-1}(y)$, $y \in M$, *is compact. Then* X *is* N-*compact.*

(2) *Let* M *be a metric space of non-measurable cardinal. If the covering dimension* $\dim M = 0$, *then* M *is hereditarily* N-*compact.*

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Proof. (1) This is an "N-compact" version of [**10**, 8.17]. See Remark 0 in the introduction, and [**19**, p. 180].

(2) dim $M = 0$ implies dim $\beta M = 0$. The hypothesis implies that M is realcompact. Hence, by [**19**, 3.1], M is N-compact. Since each point of M is G_δ, it follows that M is hereditarily N-compact.

Now we present Example C the idea of which is fundamental throughout the subsequent examples.

Example C. A realcompact space X_Δ, a non-realcompact space Y_Δ as in 4.2, *and a perfect map $\varphi_\Delta: X_\Delta \to Y_\Delta$.*

Let $\omega N = N \cup \{\omega\}$ be the one-point compactification of N. Let $cN = N \cup C$ be a metrizable compactification of N with the remainder C, where C denotes the Cantor set. (For a concrete construction of cN, see the later Remark 4.6. See also [**26**, 6.15].) Put

$$X = N \times cN, \ Y = N \times \omega N \text{ and } K = \omega N \times cN.$$

Note that K is a compactification of X, and let $\pi: \beta X \to K$ be the Stone extension of the identity map $1_X: X \to X$. Let $g: cN \to \omega N$ be the map that keeps the points of N fixed and maps C to $\{\omega\}$. Put $\varphi = 1_N \times g: X \to Y$ and let $\Phi: \beta X \to \beta Y$ be its Stone extension. Notice that g and φ are perfect. Since X is normal, every closed subset A of X is C^*-embedded; so we can identify $\mathrm{cl}_{\beta X} A$ with βA, consequently, $\mathrm{cl}_{\beta X} A \backslash A$ with $A^* = \beta A \backslash A$. (These sorts of identifications will be done in the sequel whenever they are possible and clear.) Let us put $U = \beta X \backslash \beta(N \times C)$ and $U^+ = U \cap X^*$. Define the family of clopen subsets of X by

$$\mathscr{A} = \{A \mid A \text{ is an infinite subset of } N \times N \text{ such that } A \cap (\{n\} \times N) \text{ is finite for each } n \in N\}$$

and put $\mathscr{A}^* = \{A^* \mid A \in \mathscr{A}\}$. To proceed with our construction, we need lemmas.

LEMMA C.1. (1) *Every member of $\mathscr{A}^*$ is a clopen* (*in X^**) *subset contained in U^+.*

(2) *Each $Z \in Z(\beta X)$ with $\emptyset \neq Z \subset X^*$ contains some member of $\mathscr{A}^*$.*

Proof. (1) Let $A \in \mathscr{A}$. Then A and $N \times C$ are completely separated in X, i.e., $\mathrm{cl}_{\beta X} A \cap \mathrm{cl}_{\beta X}(N \times C) = \emptyset$; hence $A^* \subset U^+$. Since A is clopen in X, $\mathrm{cl}_{\beta X} A$ is also clopen in βX. Hence A^* is clopen in X^*.

(2) Let Z be a zero-set in βX such that $\emptyset \neq Z \subset X^*$. Write $Z = Z(h)$ for some continuous function $h: \beta X \to I$. Put $H_n = h^{-1}[0, 1/n)$ for each $n \in N$. Then, since $N \times N$ is dense in βX, we can choose a point $x_n \in H_n \cap (N \times N)$. Put $A = \{x_n\}_{n \in N}$. It is easy to see that $A^* \subset Z$ and $A \in \mathscr{A}$.

The next lemma follows from 4.4.

LEMMA C.2. $\Phi: (\beta X, \beta(N \times C)) \to (\beta Y, \beta(N \times \{\omega\}))$ *is a relative homeomorphism, i.e., Φ maps U homeomorphically onto $\Phi(U)$.*

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Now, let us choose an arbitrary point p from the subset $\beta(N \times \{\omega\})\backslash(N \times \{\omega\})$ of βY, and fix it in the sequel. Consider the family $\mathscr{Z}_p$ of all zero-sets in βY containing p. Then, since βY is separable, we have

$$|\mathscr{Z}_p| \leqq |Z(\beta Y)| \leqq |C^*(\beta Y)| = \mathfrak{c}.$$

So, we can index $\mathscr{Z}_p = \{Z_t | t \in C\}$. On the other hand, put $Z_t' = \pi^{-1}(\{(\omega, t)\})$ for each $t \in C$, where $(\omega, t) \in \{\omega\} \times C \subset K$. Since K is a metric space, each point of K is a zero-set; hence Z_t' is a zero-set in βX. Here notice that $\{Z_t'\}_{t \in C}$ is disjoint in βX, while $\{Z_t\}_{t \in C}$ is not disjoint in βY. Put $\tilde{Z}_t = Z_t' \cap \Phi^{-1}(Z_t)$ for each $t \in C$. Then Z_t is nonempty: Indeed, since

$$\Phi(\mathrm{cl}_{\beta X}(N \times \{t\})) = \mathrm{cl}_{\beta Y}(N \times \{\omega\}) \ni p,$$

for each $t \in C$ there exists $p_t \in \mathrm{cl}_{\beta X}(N \times \{t\})\backslash(N \times \{t\})$ such that $\Phi(p_t) = p$. Clearly, $\pi(p_t) = (\omega, t)$. Hence $p_t \in \tilde{Z}_t$, and consequently $\tilde{Z}_t \neq \emptyset$. Thus, $\{\tilde{Z}_t\}_{t \in C}$ is a family of disjoint zero-sets in βX such that $\emptyset \neq \tilde{Z}_t \subset X^*$ and $\Phi(\tilde{Z}_t) \subset Z_t$. Now, by Lemma C.1 we can choose $A_t^* \in \mathscr{A}^*$ such that $A_t^* \subset \tilde{Z}_t$ for each $t \in C$. Define subspaces G and X_0 of βX by

$$G = (N \times N) \cup (\oplus_{t \in C} A_t^*) \quad \text{and} \quad X_0 = G \cup (N \times C).$$

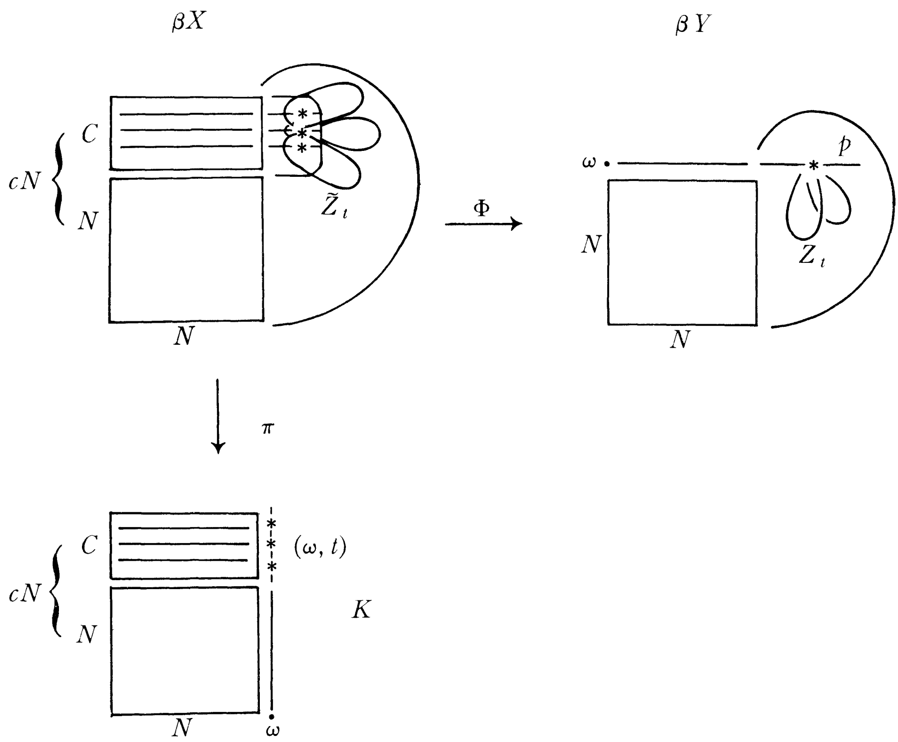

FIGURE 1.

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(Notice that $\{A_t^*\}$ is disjoint and each A_t^* is clopen in X^* so that the union $\cup A_t^*$ is just the topological sum of A_t^*'s.) Put $Y_0 = \Phi(X_0)$. By Lemma C.2 we henceforth identify G with $\Phi(G)$, and we make a convention that A_t^* denotes $\mathrm{cl}_{\beta X}A_t \backslash A_t$ and $\mathrm{cl}_{\beta Y}A_t \backslash A_t$ interchangeably. So we can write $Y_0 = G \cup (N \times \{\omega\}) \subset \beta Y$. Let $\varphi_0 : X_0 \to Y_0$ be the restriction of the map Φ. φ_0 is perfect because $X_0 = \Phi^{-1}(Y_0)$. Let G_Δ, X_Δ and Y_Δ be the quotient spaces obtained from G, X_0 and Y_0 respectively by shrinking each A_t^*, $t \in C$, to one point p_t. (Of course these quotient spaces are completely regular Hausdorff.)

$$X_0 = G \cup (N \times C), \quad Y_0 = G \cup (N \times \{\omega\}),$$
$$X_\Delta = G_\Delta \cup (N \times C), \quad Y_\Delta = G_\Delta \cup (N \times \{\omega\}).$$

Let $\rho_1 : X_0 \to X_\Delta$ and $\rho_2 : Y_0 \to Y_\Delta$ be the quotient maps, and let $\varphi_\Delta : X_\Delta \to Y_\Delta$ be the map induced from $\varphi_0 : X_0 \to Y_0$ such that $\varphi_\Delta \circ \rho_1 = \rho_2 \circ \varphi_0$. Observe that ρ_1, ρ_2 and φ_Δ are perfect. The required objects in our construction are these spaces X_Δ and Y_Δ, and the map φ_Δ.

ASSERTION C.3. *X_0, X_Δ, G and G_Δ are N-compact hence realcompact. In particular, X_Δ and G_Δ are submetrizable.*

Proof. We prove the assertion for X_0 and X_Δ; the proof for G and G_Δ is similar. Note that any 0-dimensional, perfect pre-image of an N-compact space is N-compact [**11**]. So, it suffices to show that X_Δ is N-compact and submetrizable, because ρ_1 is perfect. Put $M = \pi(X_0) \subset K$ and $\pi_0 = \pi | X_0 : X_0 \to M$. Since $\pi(A_t^*) \subset \pi(\tilde{Z}_t) \subset \pi(Z_t') = (\omega, t)$, i.e., $\pi(A_t^*)$ is singleton for each $t \in C$, π_0 induces a mapping $\pi_\Delta : X_\Delta \to M$ such that $\pi_\Delta \circ \rho_1 = \pi_0$. Observe that π_Δ is a bijection onto the metric space M with $\dim M = 0$. Thence it follows from 4.5 that X_Δ is N-compact, and of course, submetrizable. It might be interesting to note that N-compactness of X_0 follows directly from 4.5(1), because each fiber of π_0 is compact.

$$\begin{tikzcd}
& X_\Delta \arrow[r, "\varphi_\Delta"] \arrow[dl, dashed, "\pi_\Delta"'] & Y_\Delta \\
K \supset M & X_0 \arrow[l, "\pi_0"] \arrow[u, "\rho_1"'] \arrow[r, "\varphi_0"] & Y_0 \arrow[u, "\rho_2"']
\end{tikzcd}$$

ASSERTION C.4. *Y_0 and Y_Δ are not realcompact.*

Proof. As Y_0 is a perfect pre-image of Y_Δ, it suffices to show that Y_0 is not realcompact. Note that $\beta Y_0 = \beta Y$ since $Y \subset Y_0 \subset \beta Y$. Every zero-set in βY containing p coincides with some Z_t, $t \in C$, and meets Y_0 because

$$\emptyset \neq A_t^* = \Phi(A_t^*) \subset \Phi(\tilde{Z}_t) \subset Z_t.$$

Hence $p \in \upsilon Y_0 \backslash Y_0$ and consequently Y_0 is not realcompact.

Now we can summarize the properties of X_Δ, Y_Δ and φ_Δ.

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(1) *X_Δ and Y_Δ are separable,* 1-*st countable and locally compact. X_Δ is N-compact and submetrizable.*

(2) *The* 1-*st countable, almost realcompact space Y_Δ is the union of a realcompact dense subspace G_Δ and a countable discrete closed subset $N \times \{\omega\}$, and yet it is not realcompact.*

(3) *φ_Δ is a perfect irreducible map with the countable, discrete closed set $N \times \{\omega\}$ of multiple points.*

Clearly (2) establishes Theorem 4.2, and the map φ_Δ satisfies all conditions in Theorem 4.1 except that it is not finite-to-one (this defect will be improved in Example C^*).

Remark C.5. Some properties of the space Y_Δ depend on the choice of $\{A_\iota^*\}$. If we assume the Continuum Hypothesis and carefully select $\{A_\iota^*\}$, the following properties can be added to Y_Δ:

(i) *The closed subset $N \times \{\omega\}$ is C^*-embedded in Y_Δ.*
(ii) *$vY_\Delta \setminus Y_\Delta$ consists of one point.*

Proof. In the above construction of Y_Δ, we considered the family $\mathscr{Z}_p = \{Z_\iota\}_{\iota \in C}$ and chose $\{A_\iota^*\}_{\iota \in C}$ such that $A_\iota^* \subset Z_\iota$. Well-order $\mathscr{Z}_p = \{Z_\alpha\}_{\alpha < \omega_1}$ and define $\mathscr{Z} = \{\bar{Z}_\alpha\}_{\alpha < \omega_1}$ by $\bar{Z}_\alpha = \bigcap_{\beta \leqq \alpha} Z_\beta$ for each $\alpha < \omega_1$. In the above construction replace $\mathscr{Z}_p$ by this $\mathscr{Z}$: if one indexes $\mathscr{Z}$ by C, the construction proceeds the same as in the case of $\mathscr{Z}_p$. Then we show the resultant space Y_Δ satisfies the conditions (i) and (ii).

(i) Put $\bar{Y}_0 = G \cup \beta(N \times \{\omega\}) \subset \beta Y$ where $G = (N \times N) \cup (\oplus_{\alpha < \omega_1} A_\alpha^*)$, and let $\bar{Y}_\Delta$ be the quotient space obtained from $\bar{Y}_0$ by shrinking each A_α^* to one point. Then, since

$$\mathrm{cl}_{\beta Y}(G \cap Y^*) \cap \mathrm{cl}_{\beta Y}(N \times \{\omega\}) = \{p\},$$

$\bar{Y}_\Delta$ becomes a space in our sense, i.e., a completely regular Hausdorff space. Since $N \times \{\omega\} \subset \beta(N \times \{\omega\}) \subset \bar{Y}_\Delta$, $N \times \{\omega\}$ is C^*-embedded in $\bar{Y}_\Delta$. Clearly Y_Δ is a subspace of $\bar{Y}_\Delta$; and hence $N \times \{\omega\}$ is C^*-embedded in Y_Δ.

(ii) Observe that the subspace $(\oplus_{\alpha < \omega_1} A_\alpha^*) \cup \{p\}$ of βY is Lindelöf. Hence $Y_0 \cup \{p\}$ and its continuous image $Y_\Delta \cup \{p\}$ are also Lindelöf. Consequently we have $vY_\Delta \subset Y_\Delta \cup \{p\}$, and hence, by Assertion C.4, $vY_\Delta = Y_\Delta \cup \{p\}$.

Remark C.6. In general, consider spaces of the form $N \cup \mathscr{R}$ [**20**, § 3]. Here $\mathscr{R}$ is a collection of almost disjoint infinite subsets of N, and the space $N \cup \mathscr{R}$ has the following topology: each point of N is isolated and each point $A \in \mathscr{R}$ has a neighborhood basis $\{\{A\} \cup (A \setminus F) | F$ is a finite subset of $N\}$. Then the spaces G_Δ and Y_Δ constructed above are of this form. In fact, if we put

$$\mathscr{R}_1 = \{A_\iota\}_{\iota \in C} \text{ and } \mathscr{R}_2 = \mathscr{R}_1 \cup \{\{n\} \times N | \, n \in N\},$$

then $G_\Delta \cong (N \times N) \cup \mathscr{R}_1$ and $Y_\Delta \cong (N \times N) \cup \mathscr{R}_2$. Hence the following assertion is proved:

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There exist collections $\mathscr{R}_1$ and $\mathscr{R}_2$ of almost disjoint infinite subsets of N such that $N \cup \mathscr{R}_1$ is N-compact, $\mathscr{R}_1 \subset \mathscr{R}_2$ and $|\mathscr{R}_2 \backslash \mathscr{R}_1| = \aleph_0$, but $N \cup \mathscr{R}_2$ is not realcompact.

This answers Mrówska's question [**19**, p. 181] in the negative.

The next example continues the construction of Example C, so we keep the notation the same as in Example C. Its purpose is to deform the map φ_Δ into a finite-to-one map φ_*.

Example C. The finite-to-one closed map $\varphi_*: X_* \to Y_\Delta$ where X_* is N-compact and Y_Δ is the non-realcompact space in Example C.*

For each $n \in N$ let $\pi_n: C = D^\omega \to D^n$, $D = \{0, 1\}$ be the projection, and for each $(i_1, \ldots i_n) \in D^n$ put

$$C(i_1 \ldots i_n) = \{n\} \times \pi_n^{-1}(i_1 \ldots i_n) \subset N \times C.$$

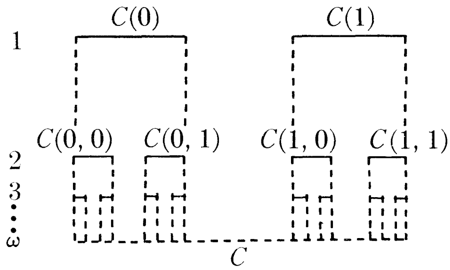

FIGURE 2.

Let L and cL be the quotient spaces obtained from $N \times C$ and $\omega N \times C$, respectively, by shrinking each $C(i_1 \ldots i_n)$, for $(i_1 \ldots i_n) \in D^n$ and $n \in N$, to one point. Then L is discrete and cL is compact and metrizable. Let $f_1: N \times C \to L$ and $f_2: \omega N \times C \to cL$ be the quotient maps. Clearly, they are both perfect and f_2 maps $\{\omega\} \times C$ homeomorphically onto $cL \backslash L$. Let $X_* = G_\Delta \cup L$ be the adjunction space determined by $X_\Delta = G_\Delta \cup (N \times C)$ and f_1, i.e., X_* is the quotient space with the identification map $q_1: X_\Delta \to X_*$ such that $q_1(x) = x$ if $x \in G_\Delta$ and $q_1(x) = f_1(x)$ if $x \in N \times C$. Similarly let $M_* = (N \times N) \cup cL$ be the adjunction space determined by $M = (N \times N) \cup (\omega N \times C) \subset K$ (this M is the same as the $M = \pi(X_0)$ in the proof of Assertion C.3) and f_2, and let $q_2: M \to M_*$ be the quotient map such that $q_2(x) = x$ if $x \in N \times N$ and $q_2(x) = f_2(x)$ if $x \in \omega N \times C$. Observe that q_1 and q_2 are perfect and that M_* is metrizable. Let $\varphi_*: X_* \to Y_\Delta$ and $\pi_*: X_* \to M_*$ be the maps induced from φ_Δ and π_Δ respectively such that the following diagram commutes.

$$\begin{tikzcd} & M_* & X_* \arrow[l, "\pi_*"'] \arrow[dr, "\varphi_*"] & \\ K \supset & M \arrow[u, "q_2"] & X_\Delta \arrow[l, "\pi_\Delta"] \arrow[u, "q_1"'] \arrow[r, "\varphi_\Delta"'] & Y_\Delta \end{tikzcd}$$

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Then π_* is a bijection and hence X_* is N-compact (by 4.5) and submetrizable. φ_* is clearly a finite-to-one map, and is perfect because $\varphi_\Delta = \varphi_* \circ q_1$ is perfect. Thus, we have constructed the required map $\varphi_*: X_* \to Y_\Delta$ the properties of which are summarized as follows:

(1) *X_* is separable, 1-st countable and locally compact. Furthermore, it is N-compact and submetrizable.*

(2) *Y_Δ is the non-realcompact space in Example C.*

(3) *φ_* is a finite-to-one, perfect irreducible map such that the set $M(\varphi_*)$ of multiple points is a countable discrete closed subset $N \times \{\omega\}$ of Y_Δ.*

This example proves Theorem 4.1.

Remark 4.6. The compact space cL used in Example C^* is a concrete representation of the compactification cN, because L is a countable discrete space and the remainder $cL \backslash L$ is homeomorphic with C.

Before presenting the next Example D, we need some preliminary results. A subfamily $\mathscr{Z}$ of $Z(X)$ is called a *zero-set base at* $x \in X$ if each member of $\mathscr{Z}$ contains x and for each $Z \in Z(X)$ containing x there exists some $Z' \in \mathscr{Z}$ with $x \in Z' \subset Z$. The *zero-set character* at the point $x \in X$, denoted by $\chi_Z(x, X)$, is defined as the least cardinal of zero-set bases at this point. Let us denote by $\chi(A, X)$, as usual, the *character* of A in X. The following lemma is proved easily.

LEMMA 4.7. *For any space X and any point $x \in X$,*

$$\chi_Z(x, X) = \chi(x, pX) \leqq \chi(x, X)^{\aleph_0}$$

where pX is the P-space defined before 2.6.

LEMMA 4.8. *Let F be a closed subset of a normal space X. Then we have $\chi(\beta F, \beta X) = \chi(F, X)$ where $\beta F = \mathrm{cl}_{\beta X} F$.*

Proof. Let $\mathfrak{B}_1$ and $\mathfrak{B}_2$ be the minimal bases for $\beta F \subset \beta X$ and $F \subset X$ respectively. For an open subset U of X, set $\mathrm{Ex}\, U = \beta X \backslash \mathrm{cl}_{\beta X}(X \backslash U)$ (cf. [**7**, p. 269]). Put $\mathfrak{B}_1' = \{V \cap X | V \in \mathfrak{B}_1\}$ and $\mathfrak{B}_2' = \{\mathrm{Ex}\, U | U \in \mathfrak{B}_2\}$. Then it follows from the normality of X that $\mathfrak{B}_1'$ and $\mathfrak{B}_2'$ are bases for $F \subset X$ and $\beta F \subset \beta X$ respectively. Therefore, $|\mathfrak{B}_2| \leqq |\mathfrak{B}_1'| = |\mathfrak{B}_1|$ and $|\mathfrak{B}_1| \leqq |\mathfrak{B}_2'| = |\mathfrak{B}_2|$. Consequently, $|\mathfrak{B}_1| = |\mathfrak{B}_2|$ which proves the Lemma.

PROPOSITION 4.9. *Let F be a closed subset of a normal space X. Then, for every $p \in \beta F = \mathrm{cl}_{\beta X} F$,*

$$\chi_Z(p, \beta X) \leqq [\chi(p, \beta F) \cdot \chi(F, X)]^{\aleph_0};$$

especially, if F is separable and $\chi(F, X) \leqq \mathfrak{c}$, then

$$\chi_Z(p, \beta X) \leqq \mathfrak{c}.$$

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Proof. Note the well-known inequality

$$\chi(p, \beta X) \leqq \chi(p, \beta F) \cdot \chi(\beta F, \beta X).$$

It follows from 4.7 and 4.8, respectively, that

$$\chi_Z(p, \beta X) \leqq [\chi(p, \beta X)]^{\aleph_0} \text{ and } \chi(\beta F, \beta X) = \chi(F, X).$$

Hence an easy calculation leads to the first inequality in our proposition. To prove the second half of the proposition, let F be separable and $\chi(F, X) \leqq \mathfrak{c}$. Then

$$\chi(p, \beta F) \leqq w(\beta F) \leqq \mathfrak{c}$$

where $w(\beta F)$ denotes the weight of βF. Hence $\chi_Z(p, \beta X) \leqq [\mathfrak{c} \cdot \mathfrak{c}]^{\aleph_0} = \mathfrak{c}$.

Example D. *A realcompact space $X_\square$, a non-realcompact space $Y_\square$ as in* 4.3 *and a perfect map $\varphi_\square : X_\square \to Y_\square$.*

Let C be the Cantor set and put $N_\square = \oplus_{t \in C} N_t$ where each N_t is a copy of N. Let $cN_\square = N_\square \cup C$ be a metrizable space such that $N_\square$ is discrete, open and dense, and for each $t \in C$ the subspace $N_t \cup \{t\}$ is the one-point compactification of N_t. The existence of such $cN_\square$ is assured if one considers the subspace $C \times (\{0\} \cup \{1/n | n \in N\})$ of the retopologized plane $\mathbf{R}^2$ with the metric "river" (cf. [**7**, Example 4.1.4]). Let $\omega N_\square = N_\square \cup \{\omega\}$ be the quotient space obtained from $cN_\square$ by shrinking the compact set C to one point ω. Let $g : cN_\square \to \omega N_\square$ be the quotient map. Then g is perfect and hence $\omega N_\square$ is metrizable (in fact, $\omega N_\square$ is a subspace of the hedgehog with $\mathfrak{c}$ prickles [**7**, Example 4.1.3]). It should be noted here that neither $cN_\square$ nor $\omega N_\square$ is compact; however, the following construction is almost parallel to Example C if the N of cN in Example C is replaced by $N_\square$. Put $X = N \times cN_\square$, $Y = N \times \omega N_\square$, $K = \omega N \times cN_\square$ and $\varphi = 1_N \times g : X \to Y$. Let $\Phi : \beta X \to \beta Y$ and $\pi : \beta X \to \beta K$ be the Stone extension of the perfect map φ and the inclusion map $X \to X \subset K$, respectively. For each $t \in C$ define

$$\mathscr{A}_t = \{A | A \text{ is an infinite subset of } N \times N_t \text{ such that } A \cap (\{n\} \times N_t) \text{ is finite for each } n \in N\},$$

and $\mathscr{A}_t^* = \{A^* | A \in \mathscr{A}_t\}$ where $A^* = \mathrm{cl}_{\beta X} A \backslash A$. Put $U = \beta X \backslash \beta(N \times C)$ and $U^+ = U \cap X^*$. Then the same statements as in Lemma C.1(1) (just replace $\mathscr{A}^*$ by $\mathscr{A}_t^*$) and Lemma C.2 are valid. The statement corresponding to Lemma C.1(2) reads as follows.

LEMMA D. *Let $t \in C$. Then each $Z \in Z(\beta X)$ with*

$$Z \cap (\mathrm{cl}_{\beta X}(N \times \{t\}) \backslash (N \times \{t\})) \neq \emptyset$$

contains some member of $\mathscr{A}_t^*$.

Proof. Suppose $p_t \in Z \cap (\mathrm{cl}_{\beta X}(N \times \{t\}) \backslash (N \times \{t\}))$. Since $p_t \notin \mathrm{cl}_{\beta X}(\{n\} \times N_\square)$ for each $n \in N$, we can assume that Z misses $\cup_{n \in N} \mathrm{cl}_{\beta X}(\{n\} \times N_\square)$.

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Write $Z = Z(h)$ for some continuous function $h:\beta X \to I$, and put $H_n = h^{-1}[0, 1/n)$ for each $n \in N$. Note that $N \times \{t\} \subset \mathrm{cl}_X(N \times N_t)$; hence

$$p_t \in \mathrm{cl}_{\beta X}(N \times \{t\}) \subset \mathrm{cl}_{\beta X}(N \times N_t).$$

Therefore, the open set H_n meets $N \times N_t$. Choose a point x_n from $H_n \cap (N \times N_t)$. Put $A = \{x_n\}_{n \in N}$. Then it is easy to see that $A^* \subset Z$ and $A \subset \mathscr{A}_t$. This completes the proof.

Now, let us choose an arbitrary point p from the subset $\beta(N \times \{\omega\}) \setminus (N \times \{\omega\})$ of βY, and fix it in the sequel. Observe that Y is normal (metrizable) and $\chi(N \times \{\omega\}, Y) \leqq \mathfrak{c}$. Hence it follows from 4.9 that the zero-set character at $p \in \beta Y$ does not exceed $\mathfrak{c}$. So, we can select a zero-set base at $p \in \beta Y$ so that it can be indexed as $\{Z_t | t \in C\}$. On the other hand, put $Z_t' = \pi^{-1}(\{(\omega, t)\})$ for each $t \in C$, where $(\omega, t) \in \{\omega\} \times C \subset K$. Since K is 1-st countable (metrizable), each point of K is a zero-set in βK (cf. [**10**, 9.7]); hence Z_t' is a zero-set in βX. Put $\tilde{Z}_t = Z_t' \cap \Phi^{-1}(Z_t)$ for each $t \in C$. Then, by the same reason as in Example C, we can claim that $\tilde{Z}_t$ meets $\mathrm{cl}_{\beta X}(N \times \{t\}) \setminus (N \times \{t\})$. Hence, by Lemma D we can choose $A_t \in \mathscr{A}_t$ such that $A_t^* \subset \tilde{Z}_t$ for each $t \in C$. Here notice that A_t, $t \in C$, are mutually disjoint (in Example C this is not the case), which follows from the definition of $N_\square$ and the families $\mathscr{A}_t$, $t \in C$. Define subspaces G and X_0 of βX by

$$G = (N \times N_\square) \cup (\oplus_{t \in C} A_t^*) \text{ and } X_0 = G \cup (N \times C).$$

Put $Y_0 = \Phi^{-1}(X_0)$. Let $\varphi_0: X_0 \to Y_0$ be the restriction of Φ. Let $G_\square$, $X_\square$ and $Y_\square$ be the quotient spaces obtained from G, X_0 and Y_0 respectively by shrinking each A_t^*, $t \in C$, to one point ω_t. Let $\varphi_\square: X_\square \to Y_\square$ be the map induced from φ_0. Then it is easy to see that the same statements as Assertions C.3 and C.4 hold if the X_Δ and G_Δ are replaced by $X_\square$ and $G_\square$. (Note $\dim K = 0$, so that Lemma 4.5 is applicable as in Assertion C.3.) Furthermore

ASSERTION D. *$G_\square$ is homeomorphic with the topological sum of $\mathfrak{c}$ copies of the one-point compactification ωN of N. Hence $G_\square$ is metrizable.*

Proof. First note that the selected subsets A_t, $t \in C$, are mutually disjoint, as was noticed before. Put $\omega A_t = A_t \cup \{\omega_t\} \subset G_\square$ for each $t \in C$. Then ωA_t is a one-point compactification of A_t, and $G_\square$ can be expressed as

$$G_\square = (N \times N_\square \setminus \oplus_{t \in C} A_t) \oplus (\oplus_{t \in C} \omega A_t).$$

Write $N \times N_\square \setminus \oplus_{t \in C} A_t = \{x_t | t \in C'\}$ for some subset $C' \subset C$, where the x_t's are distinct points. Put

$$\omega A_t' = \begin{cases} \omega A_t \oplus \{x_t\} & \text{if } t \in C' \\ \omega A_t & \text{if } t \in C \setminus C'. \end{cases}$$

Then $G_\square = \oplus_{t \in C} \omega A_t'$. Hence $G_\square$ is homeomorphic with the topological sum of $\mathfrak{c}$ copies of the one-point compactification ωN of N.

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We conclude this example by summarizing the properties of $X_{\square}$, $Y_{\square}$ and $\varphi_{\square}$.

(1) *$X_{\square}$ and $Y_{\square}$ are both* 1-*st countable. $X_{\square}$ is N-compact and submetrizable.*

(2) *The* 1-*st countable, almost realcompact space $Y_{\square}$ is the union of the metrizable dense subspace $G_{\square}$ of cardinal $\mathfrak{c}$ and the countable discrete closed subset $N \times \{\omega\}$, and yet it is not realcompact.*

(3) *$\varphi_{\square}$ is a perfect irreducible map with the countable, discrete closed subset $N \times \{\omega\}$ of multiple points.*

Certainly, (2) proves Theorem 4.3.

Note that both the space $Y_{\square}$ and the Dieudonné plank T_D (in Example A) have the form of the union of a countable discrete closed space and a metric space of weight $\leqq \mathfrak{c}$.

Remark 4.10. The map $\varphi_{\square}: X_{\square} \rightarrow Y_{\square}$ in Example D can be deformed into a finite-to-one map using the same technique as the construction of Example C^* from Example C. Then the deformed map will satisfy Theorem 4.1.

5. Concluding Remarks. Bearing in mind the results in §§ 3 and 4, we study the class $\mathfrak{R}$ of realcompact spaces X such that the union of X with a Lindelöf space is always realcompact if the Lindelöf space is closed in the union. Let $\mathfrak{R}_0$ be the class of spaces X of the form $X = L \cup P$ where L is Lindelöf closed and P is a P-space such that every clopen subset missing L is realcompact.

(1) Theorem 1.7(2) implies that $\mathfrak{R}_0$ is a subclass of $\mathfrak{R}$.

(2) Example C shows that there exists a separable submetrizable space that does not belong to $\mathfrak{R}$.

(3) Example D or Example A shows that there exists a metric space of weight $\leqq \mathfrak{c}$ that does not belong to $\mathfrak{R}$.

In view of the facts (1), (2) and (3) it is plausible to conjecture that $\mathfrak{R}$ coincides with $\mathfrak{R}_0$. Though we don't know yet the general answer, under some restriction of spaces the answer is affirmative as the following theorem shows.

THEOREM 5.1. *Let X be a paracompact* 1-*st countable space of nonmeasurable cardinal. Then X belongs to $\mathfrak{R}$ if and only if it belongs to $\mathfrak{R}_0$, i.e., the derived set X^d is Lindelöf.*

Proof. We need to prove the "only if" part. Suppose X^d is not Lindelöf. Then, since X is paracompact, there exists an uncountable discrete family $\{U_\alpha | \alpha < \omega_1\}$ of open sets such that each U_α meets X^d. Since X is 1-st countable, each U_α contains a copy of ωN. Consequently X contains a closed subspace that is homeomorphic with $D \times \omega N$ where D is a discrete space of cardinal $\aleph_1$. Example A shows that $D \times \omega N$ does not belong to $\mathfrak{R}$. Hence 5.1 follows from the next lemma.

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LEMMA 5.2. *If a space X belongs to $\mathfrak{R}$, then every C^*-embedded closed subspace of X belongs to $\mathfrak{R}$.*

Proof. Suppose that X contains a C^*-embedded closed subspace G not belonging to $\mathfrak{R}$. Then there must be a non-realcompact space $G_L = G \cup L$ such that L is Lindelöf and closed in G_L. By 1.10 we can assume that G is dense in G_L. Therefore there exist a space $\tilde{G}_L$ with $G \subset \tilde{G}_L \subset \beta G$ and a perfect map $\varphi: \tilde{G}_L \rightarrow G_L$ that is an extension of the identity map of G (see the proof of Theorem I in § 2). As G is C^*-embedded in X, we identify βG with $\mathrm{cl}_{\beta X} G$. Put $\tilde{X} = X \cup \tilde{G}_L \subset \beta X$, and let $X_L = X \cup G_L$ be the adjunction space determined by $\tilde{X}$ and $\varphi: \tilde{G}_L \rightarrow G_L$. Then $X_L = X \cup L$, and L is closed in X_L. X_L is not realcompact since it contains the non-realcompact closed subset G_L. Hence X does not belong to $\mathfrak{R}$.

It seems an interesting problem to find a space in $\mathfrak{R} \backslash \mathfrak{R}_0$. For example, do the next spaces belong to $\mathfrak{R}$? (1) The square of the Sorgenfrey line. (2) The Niemytzki plane (cf. [**7**]) or its locally compact modification in [**18**]. (3) The Tychonoff product of $\mathfrak{c}$ copies of the real line.

The following problem also remains open in this paper. Let $f: X \rightarrow Y$ be a finite-to-one closed map of a realcompact space X onto a space Y such that (a) the set $M(f)$ of multiple points is a countable closed subset of Y, and (b) $\sup \{|f^{-1}(y)| | y \in Y\}$ is finite. Then, is Y realcompact? Notice that Example C^* does not satisfy the condition (b), while Mrówka's example in [**18**] does not satisfy (a).

REFERENCES

1. R. A. Alo and H. L. Shapiro, *Normal topological spaces*, Cambridge tracts in Mathematics *65* (Cambridge Univ. Press, 1974).

2. R. L. Blair, *On v-embedded sets in topological spaces*, Proc. Second Pittsburgh Symposium on Gen. Top. (Pittsburgh, 1972) Lecture notes in Math. *378* (Berlin-Heidelberg-New York, Springer, 1974), 46–79.

3. ——— *Spaces in which special sets are z-embedded*, Can. J. Math. *28* (1976), 673–690.

4. R. L. Blair and A. W. Hager, *Extensions of zero-sets and of real-valued functions*, Math. Z. *136* (1974), 41–52.

5. N. Dykes, *Mappings and realcompact spaces*, Pacific J. Math. *31* (1969), 347–358.

6. ——— *Generalizations of realcompact spaces*, Pacific J. Math. *33* (1970), 571–581.

7. R. Engelking, *Outline of general topology* (North Holland Publishing Co., Amsterdam-New York, 1968).

8. Z. Frolik, *A generalization of realcompact spaces*, Czechoslovak Math. J. *13* (88) (1963), 127–138.

9. ——— *Realcompactness is a Baire-measurable property*, Bull. Acad. Polon. Sci. *19* (1971), 617–621.

10. L. Gillman and M. Jerison, *Rings of continuous functions* (Van Nostrand, Princeton, 1960).

11. H. Herrlich and J. van der Slot, *Properties which are closely related to compactness*, Indag. Math. *29* (1967), 524–529.

12. M. Henriksen and D. G. Johnson, *On the structure of a class of archimedean lattice-ordered algebras*, Fund. Math. *50* (1961), 73–94.

https://doi.org/10.4153/CJM-1979-104-8 Published online by Cambridge University Press

13. A. Kato, *Various countably-compact-ifications and their applications,* Gen. Top. Appl. *8* (1978), 27–46.

14. J. Mack, M. C. Rayburn and G. Woods, *Lattices of topological extensions,* Trans. Am. Math. Soc. *189* (1974), 163–174.

15. A. K. Misra, *A topological view of P-spaces,* Gen. Top. Appl. *2* (1972), 349–362.

16. W. Moran, *Measures on metacompact spaces,* Proc. London Math. Soc. *20* (1970), 23–40.

17. S. Mrówka, *Further results on E-compact spaces I,* Acta Math. *120* (1968), 161–185.

18. ——— *Some comments on the author's example of a non-R-compact space,* Bull. Acad. Polo. Sci. *18* (1970), 443–448.

19. ——— *Recent results on E-compact spaces and structures of continuous functions,* Proc. Univ. of Oklahoma Topology Conference 1972, 168–221.

20. ——— *Some set-theoretic constructions in topology,* Fund. Math. *94* (1977), 83–92.

21. J. van der Slot, *A survey of realcompactness,* Theory of sets and topology, Berlin (1972), 473–494.

22. L. A. Steen and J. A. Seebach, Jr., *Counterexamples in topology* (Holt, Rinehart and Winston, Inc., New York, 1970).

23. D. P. Strauss, *Extremally disconnected spaces,* Proc. A.M.S. *18* (1967), 305–309.

24. T. Terada, *New topological extension properties,* Proc. A.M.S. *67* (1977), 162–166.

25. R. E. Wheeler, *On separable z-filters,* Gen. Top. Appl. *5* (1975), 333–345.

26. R. C. Walker, *The Stone-Čech compactifications,* Ergebnisse der Mathematik und ihrer Grenzgebiete, Band *83* (Springer-Verlag, Berlin-Heidelberg-New York, 1974).

National Defense Academy,
Yokosuka, Japan

https://doi.org/10.4153/CJM-1979-104-8 Published online by Cambridge University Press