ON THE NON-EXISTENCE OF A TYPE OF REGULAR GRAPHS OF GIRTH 5

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1. $f(k, 5)$ is defined to be the smallest integer n for which there exists a regular graph of valency k and girth 5, having n vertices. In **(3)** it was shown that

$$k^2 + 1 \leqslant f(k, 5) \leqslant 4(k-1)(k^2 - k + 1). \tag{1.1}$$

Hoffman and Singleton proved in **(4)** that equality holds in the lower bound of (1.1) only for $k = 2, 3, 7$, and possibly 57. Robertson showed in **(6)** that $f(4, 5) = 19$ and constructed the unique minimal graph.

In this paper we shall prove that

$$f(k, 5) \neq k^2 + 2 \qquad \text{for all } k, \tag{1.2}$$

and provide an improved upper bound. Familiarity with **(2)** is required only in §4, where we discuss the upper bound.

Our methods in §§1–3 will be those of **(4)**: we investigate the eigenvalues of the adjacency matrix of graphs of the prescribed types. The author is indebted to Professors B. Moyls and W. McWorter for several enlightening conversations during the course of this research.

2. An equation satisfied by the adjacency matrix. A graph G having n vertices will be said to be *labelled* if its vertices have been assigned labels $v_1, \ldots, v_n$. The *adjacency matrix* of a labelled graph G is a symmetric matrix $A = (a_{ij})$ defined by

$$a_{ij} = \begin{cases} 1 & \text{if } v_i \text{ and } v_j \text{ are connected by an edge in } G, \\ 0 & \text{otherwise.} \end{cases}$$

It is well known that the entry in the (i, j)th position in A^k is the number of distinct directed paths of length k from v_i to v_j in G; cf. (**1**, p. 127).

Suppose G is regular of valency k. Consider all directed paths of length 2 from some vertex v: their end points, exclusive of v, are $k(k-1)$ in number; the end points of directed paths of length 1 from v are k in number. No two of these k^2 vertices can coincide if G is to have girth 5. We assume henceforth that G has $n = k^2 + 2$ vertices; thus there exists in G exactly one vertex which cannot be reached from v along a path of length less than 3; this vertex we shall

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denote by v^* for any given v. Clearly $(v^*)^* = v$. We thus obtain the matrix equation

$$A^2 + A - (k-1)I = J - B \tag{2.1}$$

where J is the $n \times n$ matrix all of whose entries are 1, and B is a symmetric permutation matrix with zeros on the main diagonal. By a suitable relabelling of G we can arrange that B be a direct sum of matrices $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. It follows incidentally that n is even and so

$$k \equiv 0 \pmod 2. \tag{2.2}$$

3. The solutions of (2.1). The eigenvalues of $(J - B)^2 = (n-2)J + I$ are $(n-1)^2$ and 1, having multiplicities 1 and $n - 1$ respectively (**5**, p. 36). As in (**4**) we observe that

$$u =_{\text{def}} \begin{bmatrix} 1 \\ 1 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix},$$

an nth-order column vector of 1's, is an eigenvector of A and $J - B$ with eigenvalues k and $n - 1$ respectively. Thus, in order that $J - B$ have trace n, its remaining eigenvalues must be 1 and -1, of multiplicities $n/2$ and $n/2 - 1$ respectively. Any eigenvector v of A having eigenvalue r must also be an eigenvector of $J - B$ with eigenvalue $r^2 + r - (k-1)$. As A is real and symmetric, it must have (**5**, p. 302) $n/2$ eigenvalues r satisfying

$$r^2 + r - (k-1) = 1, \qquad \text{i.e. } r = (-1 \pm s)/2 \quad \text{where } s = \sqrt{(4k+1)}, \tag{3.1}$$

and $n/2 - 1$ eigenvalues r satisfying

$$r^2 + r - (k-1) = -1, \qquad \text{i.e. } r = (-1 \pm t)/2 \quad \text{where } t = \sqrt{(4k-7)}. \tag{3.2}$$

We shall impose on k the condition that the trace of A be zero; four cases must be considered.

Case 1: *s and t both rational,* hence both integral. The only two odd positive integers whose squares differ by 8 are 1 and 9, so $k = 2$. But G would then be a hexagon, whose girth is 6, not 5.

Case 2: *s and t both irrational.* First suppose s and t are linearly dependent over the rationals. Then s^2 and t^2 must have the same square-free part α, which must divide their difference, 8. By hypothesis, $\alpha > 1$; but α cannot be even, since s^2 and t^2 are odd. Thus s and t must be linearly independent. But this implies that the eigenvalues $(-1 \pm s)/2$ occur in pairs; so also do the eigenvalues $(-1 \pm t)/2$. This is impossible, since one of $n/2$, $n/2 - 1$ is odd.

Case 3: *s irrational, t rational.* Here t is an odd integer, so $-1 \pm t$ is even. Thus the sum of the eigenvalues $(-1 \pm s)/2$ is an integer. These eigenvalues occur in pairs, and so their sum is $-n/4$. But $4|n$ implies that $k^2 \equiv 2 \pmod 4$, which is impossible.

Case 4: *s rational, t irrational.* The eigenvalues $(-1 \pm t)/2$ must occur in pairs, so their sum is $(-1/2)(n/2 - 1)$. Suppose the multiplicity of the eigenvalue $(-1 + s)/2$ is m. Then the trace of A is

$$0 = k + m(-1 + s)/2 + (n/2 - m)(-1 - s)/2 + (-1/2)(n/2 - 1).$$

Because $n = k^2 + 2$ and $k = (s^2 - 1)/4$, this yields a quintic equation for s:

$$s^5 + 2s^4 - 2s^3 - 20s^2 + (33 - 64m)s + 50 = 0. \tag{3.3}$$

Any positive rational solutions s must be among the integral divisors of 50, viz. 1, 2, 5, 10, 25, 50. Of these, only three yield solutions, namely

$s = 1$	$s = 5$	$s = 25$
$m = 1$	$m = 12$	$m = 6565$
$k = 0$	$k = 6$	$k = 156$

The case $s = 1$ is obviously of no interest: it yields a graph of infinite girth.

We eliminate the cases $s = 5$, $s = 25$ by the following argument. As G has girth 5 it cannot contain any triangles. Hence A^3 has only zeros on its main diagonal. The eigenvalues of A^3 are the cubes of the eigenvalues of A. Hence we can impose on the eigenvalues of A the additional condition that the sum of their cubes be zero. In neither of the remaining cases is this condition satisfied.

4. An upper bound for $f(k, 5)$**.** We shall here provide an upper bound for $f(k, 5)$ which is an improvement over that given in (1.1) but is probably not best possible, even in an asymptotic sense. In **(2)**, we constructed for each prime power q and each integer r such that $0 \leqslant r \leqslant q - 3$ a graph which we denoted by ${}_rL(\mathrm{PG}(2, q))$ of girth 6, regular of valency $q - r$. Assume $q \geqslant 5$. In each of $[P_i]$, $[l_i]$ $(i = r + 1, \ldots, q)$ let a q-gon be described, arbitrarily. The resulting graph evidently has girth 5 and valency $q - r + 2$. Thus

$$f(k, 5) \leqslant 2q(k - 2) \tag{4.1}$$

where $k > 5$ and q is a prime power not less than 5 or $k - 2$. As in **(2, §7)** we can conclude that

$$f(k, 5) \leqslant 2(2k - 1)(k - 2) \qquad \text{for all } k > 5. \tag{4.2}$$

And, moreover, to each $\epsilon > 0$ there exists an integer K_ϵ such that

$$f(k, 5) < 2(1 + \epsilon)k(k - 2) \qquad \text{for all } k > K. \tag{4.3}$$

Thus $f(k, 5) = O(k^2)$, which is certainly an improvement over (1.1). In **(2)** we proved that

$$\lim_{k\to\infty} f(k, 6)/k^2 = 2;$$

we have not been able to prove even the existence of

$$\lim_{k\to\infty} f(k, 5)/k^2.$$

5. Regular graphs of valency k, girth 5, having $n = k^2 + 3$ vertices. Graphs of this type certainly exist, Robertson's graph **(6)** being an example for $k = 4$.

The theory of the preceding sections can be generalized here. We again obtain an equation of form (2.1), where now B is a symmetric zero–one matrix in which the main diagonal consists entirely of zeros, and the row and column sums are 2. By a suitable relabelling, B becomes a direct sum of symmetric ath-order circulants of the form

$$D_a = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \qquad (3 \leqslant a).$$

If the orders of these circulants are $a_1, a_2, \ldots, a_b$ (so that $\sum_{i=1}^{b} a_i = n$), then it can be shown that the eigenvalues of $J - B$ are $n - 2$, -2, and

$$-2\cos\left(\frac{2\pi c_i}{a_i}\right), \qquad c_i = 1, \ldots, a_i - 1; i = 1, \ldots, b,$$

of multiplicities 1, $b - 1$, and 1. Also the eigenvalues of A are k, $b - 1$ roots of $r^2 + r - (k - 1) = -2$, and one root of each of

$$r^2 + r - (k-1) = -2\cos\left(\frac{2\pi c_i}{a_i}\right), \qquad c_i = 1, \ldots, a_i - 1; i = 1, \ldots, b.$$

The general solution over all partitions, subject to the condition that the sum of the eigenvalues be zero, appears very difficult. Indeed, solutions do exist for certain $k > 4$: we have found some empirically for $k = 5$ and $k = 8$. It thus becomes necessary to consider the sum of the cubes of the possible eigenvalues in an attempt to exclude irrelevant cases. In all solutions we have found for $k = 5$ and $k = 8$, the sum of the cubes has not been zero.

In conclusion, we remark without proof that the relevant partition for Robertson's graph is $19 = 4 + 12 + 3$, and the eigenvalues are

$$\frac{-1 \pm \sqrt{21}}{2}, \quad \frac{-1 \pm \sqrt{5}}{2}, \quad \frac{-1 \pm \sqrt{17}}{2}, \quad \frac{-1 \pm \sqrt{17}}{2}, \quad \frac{-1 \pm \sqrt{13}}{2},$$
$$\frac{-1 \pm \sqrt{13}}{2}, \quad \pm\sqrt{3}, \quad \pm\sqrt{3}, \quad 4, \quad 1, \quad 1.$$

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