COMPOSITIO MATHEMATICA

Continued fractions with low complexity: transcendence measures and quadratic approximation

Yann Bugeaud

Compositio Math. **148** (2012), 718–750.

doi:10.1112/S0010437X11007524

FOUNDATION COMPOSITIO MATHEMATICA

LONDON MATHEMATICAL SOCIETY

Compositio Math. 148 (2012) 718–750
doi:10.1112/S0010437X11007524

Continued fractions with low complexity: transcendence measures and quadratic approximation

Yann Bugeaud

Abstract

We establish measures of non-quadraticity and transcendence measures for real numbers whose sequence of partial quotients has sublinear block complexity. The main new ingredient is an improvement of Liouville's inequality giving a lower bound for the distance between two distinct quadratic real numbers. Furthermore, we discuss the gap between Mahler's exponent w_2 and Koksma's exponent w_2^*.

1. Introduction

A well-known open question in Diophantine approximation asks whether the continued fraction expansion of an irrational algebraic number ξ either is ultimately periodic (this is the case if, and only if, ξ is a quadratic irrational), or contains arbitrarily large partial quotients. As a very small step towards its resolution, we have recently established in [Bug10] two new combinatorial transcendence criteria for continued fraction expansions (we refer the reader to [AB10b] for references to earlier works). One of these criteria implies Theorem Bu below, which states that the sequence of partial quotients $a_1, a_2, \ldots$ of an algebraic number $[0; a_1, a_2, \ldots]$ of degree at least three cannot be too simple, in the following sense.

The complexity function of an infinite sequence $\mathbf{a} = (a_\ell)_{\ell \geqslant 1}$ of positive integers, which we will often view as the infinite word $\mathbf{a} = a_1 a_2 \ldots$, is the function $n \mapsto p(n, \mathbf{a})$ defined by

$$p(n, \mathbf{a}) = \mathrm{Card}\{(a_\ell, a_{\ell+1}, \ldots, a_{\ell+n-1}) : \ell \geqslant 1\} \quad \text{for } n \geqslant 1,$$

which counts the number of distinct blocks of length n in the word $\mathbf{a}$. Observe that the sequence $(p(n, \mathbf{a}))_{n \geqslant 1}$ is non-decreasing and that $p(n, \mathbf{a})$ is infinite for every $n \geqslant 1$ if the sequence $(a_\ell)_{\ell \geqslant 1}$ is unbounded. Furthermore, Morse and Hedlund [MH38, MH40] proved that $(p(n, \mathbf{a}))_{n \geqslant 1}$ is bounded if $\mathbf{a}$ is ultimately periodic and that, otherwise, it satisfies $p(n, \mathbf{a}) \geqslant n + 1$ for $n \geqslant 1$. The latter inequality is sharp since there exist uncountably many words $\mathbf{a}$ for which $p(n, \mathbf{a}) = n + 1$ for $n \geqslant 1$.

Theorem Bu. *Let $\mathbf{a} = (a_\ell)_{\ell \geqslant 1}$ be a sequence of positive integers which is not ultimately periodic and satisfies*

$$\liminf_{n \to +\infty} \frac{p(n, \mathbf{a})}{n} < +\infty. \tag{1.1}$$

Received 25 March 2011, accepted in final form 10 October 2011, published online 19 March 2012.
2010 Mathematics Subject Classification 11J70, 11J82, 11J87 (primary).
Keywords: continued fraction, transcendence, Schmidt's subspace theorem.

Then the real number

$$[0; a_1, a_2, \ldots, a_\ell, \ldots]$$

is transcendental.

One of the purposes of the present work is to study the accuracy with which real numbers whose sequence of partial quotients satisfies a slightly stronger assumption than (1.1) are approximated by algebraic numbers of bounded degree. The quality of approximation is measured by means of the functions w_d introduced in 1932 by Mahler [Mah32]. For every integer $d \geqslant 1$ and every real number ξ, we denote by $w_d(\xi)$ the supremum of the real numbers w for which

$$0 < |P(\xi)| < H(P)^{-w}$$

has infinitely many solutions in integer polynomials $P(X)$ of degree at most d. Here, $H(P)$ stands for the naïve height of the polynomial $P(X)$, that is, the maximum of the absolute values of its coefficients. Theorem 3.2 below gives a necessary and sufficient condition on the infinite word $\mathbf{a} = (a_\ell)_{\ell \geqslant 1}$ with

$$\limsup_{n \to +\infty} \frac{p(n, \mathbf{a})}{n} < +\infty, \tag{1.2}$$

which ensures that the real number $\xi := [0; a_1, a_2, \ldots]$ satisfies $w_d(\xi) < +\infty$ for every $d \geqslant 1$. Its proof splits into two parts. To bound $w_d(\xi)$ for $d \geqslant 3$, we use a general method described in [AB10a], based on a quantitative version of the Schmidt subspace theorem, and which has been already applied successfully in [AB10b] to a certain class of continued fractions. The main novelty in the present paper is the method developed to control $w_2(\xi)$, based on a refinement of Liouville's inequality giving a lower bound for the distance between two distinct quadratic real numbers. Theorem 2.2 below shows that, if (1.2) holds, then $w_2(\xi)$ is finite if, and only if, the Diophantine exponent of $\mathbf{a}$ (a purely combinatorial quantity associated with $\mathbf{a}$; see § 2) is finite. Theorems 3.2 and 3.3 give transcendence measures for a class of transcendental numbers defined by their continued fraction expansion. The first results of this type were proved by Baker [Bak64] in 1964.

Since (1.2) is satisfied when $\mathbf{a}$ is an automatic sequence (see §§ 5 and 8), we get straightforwardly new results on algebraic approximation to real numbers whose sequence of partial quotients can be generated by a finite automaton. We show that these numbers are either quadratic, or S- or T-numbers in Mahler's classification, which is recalled in § 3.

Shortly after Mahler, Koksma [Kok39] introduced in 1939 the exponents of approximation w_d^*. For every integer $d \geqslant 1$ and every real number ξ, we denote by $w_d^*(\xi)$ the supremum of the real numbers w^* for which

$$0 < |\xi - \alpha| < H(\alpha)^{-w^* - 1}$$

has infinitely many solutions in algebraic numbers α of degree at most d. Here, $H(\alpha)$ stands for the naïve height of the minimal defining polynomial of α over $\mathbf{Z}$. The exponents w_1 and w_1^* coincide and, for every real number ξ, we have

$$w_d^*(\xi) \leqslant w_d(\xi) \leqslant w_d^*(\xi) + d - 1, \quad \text{for } d \geqslant 2. \tag{1.3}$$

For more results on w_d and w_d^*, the reader is directed to [Bug04a, ch. 3].

R. C. Baker [Bak76] was the first to establish the existence of real numbers ξ for which $w_d(\xi)$ differs from $w_d^*(\xi)$ for every integer $d \geqslant 2$. To this end, he used a variant of Schmidt's construction of T-numbers [Sch71]. By means of a slight modification of Baker's proof, Bugeaud [Bug04] established that $w_2 - w_2^*$ takes any value in $[0, 1)$. In view of (1.3), this is nearly best possible.

Another purpose of the present work is to provide a new, fairly simple and constructive proof of the latter result. For any δ in $(0, 1]$, we give explicit examples of real numbers ξ defined by their continued fraction expansion satisfying $w_2(\xi) = w_2^*(\xi) + \delta$.

The present paper is organized as follows. Our new results are stated in §§ 2–5. Measures of non-quadraticity and transcendence measures for continued fractions with low complexity are given in §§ 2 and 3, respectively. Section 4 is devoted to the study of the gap between the functions w_2 and w_2^*. The results of §§ 2 and 3 are applied in § 5 to automatic continued fractions and to a class of morphic continued fractions, these two notions being defined in § 8. Various results on continued fractions are gathered in § 6. Section 7 is devoted to our improvement on Liouville's inequality and to two of its applications to bound, under various assumptions, the values of the functions w_2 and w_2^*. A combinatorial auxiliary lemma is the object of § 9. Our main results are proved in §§ 10 and 11. Section 12 is devoted to an extension of Theorems 2.2 and 3.2 to a family of continued fractions with unbounded partial quotients.

Unless otherwise specified, we use the notation $A \ll B$ (respectively $A \ll_a B$) when A is less than some absolute constant (respectively some constant depending at most on a) times B. We write $A \asymp B$ when $A \ll B$ and $B \ll A$ hold simultaneously.

2. Quadratic approximation to continued fractions with low complexity

The following notation will be used throughout this paper. Let $\mathcal{A}$ be a finite or infinite set. The length of a word W on the alphabet $\mathcal{A}$, that is, the number of letters composing W, is denoted by $|W|$. For any positive integer k, we write W^k for the word $W \dots W$ (k times repeated concatenation of the word W). More generally, for any positive real number x, we denote by W^x the word $W^{\lfloor x \rfloor} W'$, where W' is the prefix of W of length $\lceil (x - \lfloor x \rfloor)|W| \rceil$. Here, and throughout, $\lfloor y \rfloor$ and $\lceil y \rceil$ denote, respectively, the integer part and the upper integer part of the real number y.

Let $\mathbf{a} = (a_\ell)_{\ell \geqslant 1}$ be a sequence of elements from $\mathcal{A}$ that we identify with the infinite word $a_1 a_2 \dots a_\ell \dots$ Let $\rho \geqslant 1$ be a real number. We say that $\mathbf{a}$ satisfies condition $(*)_\rho$ if there exist two sequences of finite words $(U_n)_{n \geqslant 1}$, $(V_n)_{n \geqslant 1}$, and a sequence $(w_n)_{n \geqslant 1}$ of real numbers such that:

(i) for $n \geqslant 1$, the word $U_n V_n^{w_n}$ is a prefix of $\mathbf{a}$;

(ii) for $n \geqslant 1$, we have $|U_n V_n^{w_n}| / |U_n V_n| \geqslant \rho$;

(iii) the sequence $(|V_n^{w_n}|)_{n \geqslant 1}$ is increasing.

The *Diophantine exponent* of $\mathbf{a}$, introduced in [AB07a] and denoted by $\mathrm{Dio}(\mathbf{a})$, is the supremum of the real numbers ρ for which $\mathbf{a}$ satisfies condition $(*)_\rho$. It is clear from the definition that

$$1 \leqslant \mathrm{Dio}(\mathbf{a}) \leqslant +\infty$$

and that the Diophantine exponent of an ultimately periodic sequence is infinite. The converse is not true: it is easy to construct sequences whose Diophantine exponent is infinite but which are not ultimately periodic. The Diophantine exponent of $\mathbf{a}$ can be viewed as a measure of the periodicity of $\mathbf{a}$. We stress that it is independent of the alphabet on which $\mathbf{a}$ is written.

We define the *Diophantine exponent* of an irrational real number to be the Diophantine exponent of its sequence of partial quotients.

https://doi.org/10.1112/S0010437X11007524 Published online by Cambridge University Press

DEFINITION 2.1. Let $\xi := [0; a_1, a_2, \ldots, a_\ell, \ldots]$ be an irrational real number. The Diophantine exponent of ξ, denoted by $\mathrm{Dio}(\xi)$, is the Diophantine exponent of the infinite word $a_1 a_2 \ldots$.

By truncating the continued fraction expansion of an irrational real number ξ and then completing by periodicity, one can construct good quadratic approximations to ξ which allow us to bound $w_2^*(\xi)$ from below. An easy calculation (see § 11) shows that

$$w_2^*(\xi) \geqslant \mathrm{Dio}(\xi) - 1, \tag{2.1}$$

if $(q_\ell^{1/\ell})_{\ell \geqslant 1}$ converges, where $(p_\ell/q_\ell)_{\ell \geqslant 1}$ denotes the sequence of convergents to ξ. This simple argument does not yield any upper bound for $w_2^*(\xi)$. However, Theorem 2.2 asserts that, when the continued fraction expansion of ξ has sublinear complexity, it is possible to bound $\mathrm{Dio}(\xi)$ from below in terms of $w_2^*(\xi)$.

THEOREM 2.2. *Let $\kappa \geqslant 2$ and $A \geqslant 3$ be integers. Let $\mathbf{a} = (a_\ell)_{\ell \geqslant 1}$ be a sequence of positive integers bounded by A for which there exists an integer n_0 such that*

$$p(n, \mathbf{a}) \leqslant \kappa n, \quad \text{for } n \geqslant n_0. \tag{2.2}$$

If the Diophantine exponent of $\mathbf{a}$ is finite, then the real number

$$\xi := [0; a_1, a_2, \ldots, a_\ell, \ldots]$$

satisfies

$$\max\{2, \mathrm{Dio}(\xi) - 1\} \leqslant w_2^*(\xi) \leqslant w_2(\xi) \leqslant 118\,000\, \kappa^3 \ \mathrm{Dio}(\xi)(\log(A+1))^4. \tag{2.3}$$

Let $(p_\ell/q_\ell)_{\ell \geqslant 1}$ denote the sequence of convergents to ξ. If $(q_\ell^{1/\ell})_{\ell \geqslant 1}$ converges, then we have

$$w_2(\xi) \leqslant 118\,000\, \kappa^3 \ \mathrm{Dio}(\xi). \tag{2.4}$$

Let $b \geqslant 2$ be an integer and $\mathbf{a} = (a_\ell)_{\ell \geqslant 1}$ an infinite word on $\{0, 1, \ldots, b-1\}$ satisfying (1.2) and which is not ultimately periodic. Théorème 2.1 in [AB11] asserts that the real number

$$\zeta := \sum_{\ell \geqslant 1} \frac{a_\ell}{b^\ell}$$

is a Liouville number (that is, it satisfies $w_1(\zeta) = +\infty$) if, and only if, the Diophantine exponent of $\mathbf{a}$ is infinite. Theorem 2.2 above provides the analogue of this result for continued fraction expansions.

The fact that $w_2^*(\xi)$ always exceeds 2 when ξ is irrational and not quadratic was proved by Davenport and Schmidt [DS67].

It is explained at the end of the proofs of Lemma 9.1 and Theorem 2.2 that one can replace $\log(A+1)$ in (2.3) by the quantity

$$\left(\limsup_{\ell \to +\infty} \frac{\log q_\ell}{\ell} \right) \Big/ \left(\liminf_{\ell \to +\infty} \frac{\log q_\ell}{\ell} \right).$$

Consequently, one gets the upper bound (2.4) when the sequence $(q_\ell^{1/\ell})_{\ell \geqslant 1}$ converges. We stress that this bound does not depend on the alphabet on which $\mathbf{a}$ is written, a fact not previously pointed out.

At the end of [AC06], the authors noted that the classical argument based on triangle inequalities and Liouville's inequality

$$|\alpha - \beta| \geqslant 0.03 \cdot H(\alpha)^{-2} \cdot H(\beta)^{-2}, \tag{2.5}$$

https://doi.org/10.1112/S0010437X11007524 Published online by Cambridge University Press

valid for distinct quadratic numbers α and β, is not powerful enough to yield Theorem 2.2. Fortunately, in the present situation, we are able to considerably improve (2.5), since one of the quadratic numbers involved is very close to its Galois conjugate; see Lemma 7.1.

The assumption (2.2) can be slightly relaxed and Theorem 2.2 can be extended to a class of continued fractions with unbounded partial quotients and having repetitive patterns, provided that, however, the sequence $(q_\ell^{1/\ell})_{\ell\geqslant 1}$ remains bounded; see § 12.

By combining (1.3) and (2.1), every irrational number ξ satisfies $w_2(\xi) \geqslant \mathrm{Dio}(\xi) - 1$. We conclude this section with a sharpening of this inequality.

Theorem 2.3. *Let ξ be an irrational number with bounded partial quotients. Let $(p_\ell/q_\ell)_{\ell\geqslant 1}$ denote the sequence of convergents to ξ. Set*

$$m = \liminf_{\ell\to+\infty} q_\ell^{1/\ell} \quad \text{and} \quad M = \limsup_{\ell\to+\infty} q_\ell^{1/\ell}.$$

Then

$$w_2(\xi) \geqslant \frac{\log m}{\log M}(\mathrm{Dio}(\xi) + 1) - 1.$$

In particular, when the sequence $(q_\ell^{1/\ell})_{\ell\geqslant 1}$ converges,

$$w_2(\xi) \geqslant \mathrm{Dio}(\xi). \tag{2.6}$$

The proof of Theorem 2.3 depends on Lemma 6.1. We show in § 4 that inequalities (2.1) and (2.6) are sharp.

3. Transcendence measures for continued fractions with low complexity

Mahler's classification of real numbers is based on the functions w_d defined in the Introduction. For a real number ξ, we set $w(\xi) = \limsup_{d\to\infty}(w_d(\xi)/d)$ and, according to Mahler [Mah32], we say that ξ is:

- an A-number, if $w(\xi) = 0$;
- an S-number, if $0 < w(\xi) < \infty$;
- a T-number, if $w(\xi) = \infty$ and $w_d(\xi) < \infty$ for any integer $d \geqslant 1$;
- a U-number, if $w(\xi) = \infty$ and $w_d(\xi) = \infty$ for some integer $d \geqslant 1$.

Two transcendental real numbers belonging to different classes are algebraically independent. The A-numbers are precisely the algebraic numbers and, in the sense of the Lebesgue measure, almost all numbers are S-numbers. The existence of T-numbers remained an open problem for nearly 40 years, until it was confirmed by Schmidt; see [Bug04a, ch. 3] for references and further results. The set of U-numbers can be further divided into countably many subclasses according to the value of the smallest integer d for which $w_d(\xi)$ is infinite.

Definition 3.1. Let $m \geqslant 1$ be an integer. A real number ξ is a U_m-number if, and only if, $w_m(\xi)$ is infinite and $w_d(\xi)$ is finite for $d = 1, \ldots, m-1$.

We establish the following result, which can be viewed as the analogue for continued fraction expansions of [AB11, Théorème 1.1].

https://doi.org/10.1112/S0010437X11007524 Published online by Cambridge University Press

THEOREM 3.2. *Let* $\mathbf{a} = (a_\ell)_{\ell \geqslant 1}$ *be a sequence of positive integers such that*

$$\limsup_{n \to +\infty} \frac{p(n, \mathbf{a})}{n} < +\infty, \tag{3.1}$$

and set

$$\xi := [0; a_1, \ldots, a_\ell, \ldots].$$

If $\mathrm{Dio}(\xi)$ *is finite, then* ξ *is either an* S*-number or a* T*-number; otherwise,* ξ *is either quadratic or a* U_2*-number. Moreover, if* $\mathrm{Dio}(\xi)$ *is finite, then there exists a constant* c*, depending only on* ξ*, such that*

$$w_d(\xi) \leqslant \exp(c(\log 3d)^5 (\log\log 3d)^4) \quad \textit{for } d \geqslant 1.$$

The proof of Theorem 3.2 splits into two parts. Since the approximation by quadratic numbers has been dealt with in Theorem 2.2, it only remains for us to control the quality of approximation by algebraic numbers of fixed degree at least equal to three. To do this, we use the quantitative subspace theorem, following the general method introduced in [AB10a] and already applied to a restricted classes of continued fractions in [AB10b]. There are, however, some additional technical difficulties. Section 10 is devoted to this part of the proof of Theorem 3.2.

The dependence on ξ of the constant c can be made more precise and expressed only in terms of $\mathrm{Dio}(\xi)$ and $\limsup_{n \to +\infty} p(n, \mathbf{a})/n$, provided that the sequence $(q_\ell^{1/\ell})_{\ell \geqslant 1}$ converges, where q_ℓ is the denominator of the ℓth convergent to ξ.

A. Baker [Bak64][1] was the first to establish transcendence measures for a class of continued fractions. He proved the following result.

THEOREM BA. *Consider the continued fraction*

$$\xi = [0; a_1, \ldots, a_{n_0 - 1}, \underbrace{a_{n_0}, \ldots, a_{n_0 + r_0 - 1}}_{\lambda_0 \text{ times}}, \underbrace{a_{n_1}, \ldots, a_{n_1 + r_1 - 1}}_{\lambda_1 \text{ times}}, \ldots],$$

where the notation implies that $n_k = n_{k-1} + \lambda_{k-1} r_{k-1}$ *and the* λ_k*s indicate the number of times a block of partial quotients is repeated (it is understood that two blocks which correspond to consecutive* k *are not identical). Suppose that the sequences* $(a_n)_{n \geqslant 1}$ *and* $(r_n)_{n \geqslant 0}$ *are respectively bounded by* A *and* K*. Set*

$$L = \limsup_{k \to +\infty} \lambda_k / \lambda_{k-1}, \quad \ell = \liminf_{k \to +\infty} \lambda_k / \lambda_{k-1}.$$

If L *is infinite and* $\ell > 1$*, then* ξ *is a* U_2*-number. Furthermore, if* L *is finite and* $\ell > \exp(4A^K)$*, then* ξ *is either an* S*-number or a* T*-number.*

Theorem Ba was extended in [AB10b], where it is proved that the assumption $\ell > \exp(4A^K)$ can be replaced by the weaker condition $\ell > 1$. The techniques of the present work allow us to make a further generalization, that is, to remove the assumption that the sequence $(r_n)_{n \geqslant 0}$ has to be bounded.

THEOREM 3.3. *Consider the continued fraction*

$$\xi = [0; a_1, \ldots, a_{n_0 - 1}, \underbrace{a_{n_0}, \ldots, a_{n_0 + r_0 - 1}}_{\lambda_0 \text{ times}}, \underbrace{a_{n_1}, \ldots, a_{n_1 + r_1 - 1}}_{\lambda_1 \text{ times}}, \ldots],$$

[1] Baker's paper quoted in [AB10b, p. 884] is not the paper containing Theorem Ba.

https://doi.org/10.1112/S0010437X11007524 Published online by Cambridge University Press

where the notation implies that $n_k = n_{k-1} + \lambda_{k-1} r_{k-1}$ *and the* λ_k*s indicate the number of times a block of partial quotients is repeated (it is understood that two blocks which correspond to consecutive* k *are not identical). Suppose that the sequence* $(a_n)_{n \geqslant 1}$ *is bounded and that* λ_k *tends to infinity with* k*. Set*

$$L' = \limsup_{k \to +\infty} \frac{\lambda_k r_k}{\lambda_1 r_1 + \cdots + \lambda_{k-1} r_{k-1}}, \quad \ell' = \liminf_{k \to +\infty} \frac{\lambda_k r_k}{\lambda_1 r_1 + \cdots + \lambda_{k-1} r_{k-1}}.$$

If L' *is infinite, then* ξ *is a* U_2*-number. If* L' *is finite and* ℓ' *is positive, then* ξ *is either an* S*-number or a* T*-number.*

Theorem 3.3 extends [AB10b, Theorem 3.2]. It is not a particular case of Theorem 3.2 above since the assumption (3.1) may not be satisfied. The important point, however, is that Lemma 9.1 can be applied, since the following condition is satisfied: *There is a positive integer* κ *such that, for every sufficiently large integer* n*, there is a word of length* n *having two occurrences in the prefix of length* $(\kappa + 1)n$ *of the infinite word composed of the partial quotients of* ξ*.*

Assumption (3.1) can be slightly relaxed and Theorem 3.2 can be extended to a class of continued fractions with unbounded partial quotients and having repetitive patterns, provided that the sequence $(q_\ell^{1/\ell})_{\ell \geqslant 1}$ remains bounded; see § 12.

4. On the gap between the exponents w_2 and w_2^*

The gap between the exponents w_2 and w_2^* defined in the Introduction was investigated in [Bak76, Bug03, Bug04]. It follows from (1.3) that the set of values taken by the function $w_2 - w_2^*$ is contained in the closed interval $[0, 1]$, and we proved in [Bug04] that this set includes the half open interval $[0, 1)$. The constructions in [Bak76, Bug03, Bug04] are variants of Schmidt's complicated construction of T-numbers [Sch71] and do not give explicit real numbers ξ for which $w_2(\xi)$ differs from $w_2^*(\xi)$.

Lemma 6.1 below, on the continued fraction expansions of conjugate quadratic numbers, enables us to construct quite easily explicit examples of real numbers ξ with prescribed values for $w_2(\xi)$, $w_2^*(\xi)$ and their difference $w_2(\xi) - w_2^*(\xi)$.

THEOREM 4.1. *Let* $w \geqslant 3$ *be a real number. Let* b, c *be distinct positive integers. Define the sequence* $(a_{n,w})_{n \geqslant 1}$ *by setting* $a_{n,w} = c$ *if there exists an integer* j *such that* $n = \lfloor w^j \rfloor$ *and* $a_{n,w} = b$ *otherwise. Set*

$$\xi_w := [0; a_{1,w}, a_{2,w}, a_{3,w}, \ldots].$$

Then $\mathrm{Dio}(\xi_w) = w$ *and* ξ_w *is either an* S*- or a* T*-number. Furthermore, if* $w \geqslant (5 + \sqrt{17})/2$*, then*

$$w_2^*(\xi_w) = w - 1 \quad \textit{and} \quad w_2(\xi_w) = w. \tag{4.1}$$

It is very likely that (4.1) remains true for $3 \leqslant w < (5 + \sqrt{17})/2$, but this seems to be difficult to prove.

Theorem 4.1 shows that inequalities (2.1) and (2.6) are both sharp.

We can modify our construction to give, for every δ in $(0, 1)$, explicit real numbers ξ satisfying $w_2(\xi) - w_2^*(\xi) = \delta$.

THEOREM 4.2. *Let* $w \geqslant 3$ *be a real number. Let* b, c, d *be distinct positive integers. Let* η *be a positive real number with* $\eta < \sqrt{w}/4$*. For* $j \geqslant 1$*, set*

$$m_j = \lfloor (\lfloor w^{j+1} \rfloor - \lfloor w^j - 1 \rfloor) / \lfloor \eta w^j \rfloor \rfloor.$$

Define the sequence $(a_{n,w,\eta})_{n\geqslant 1}$ *by setting* $a_{n,w,\eta}=c$ *if there exists an integer* j *such that* $n=\lfloor w^j\rfloor$, *by setting* $a_{n,w,\eta}=d$ *if there exist positive integers* j *and* $m=1,\ldots,m_j$ *such that* $n=\lfloor w^j\rfloor+m\lfloor\eta w^j\rfloor$, *and by setting* $a_{n,w,\eta}=b$ *otherwise. Set*

$$\xi_{w,\eta}:=[0;a_{1,w,\eta},a_{2,w,\eta},a_{3,w,\eta},\ldots].$$

Then $\mathrm{Dio}(\xi_{w,\eta})=w/(1+\eta)$ *and* $\xi_{w,\eta}$ *is either an* S*- or a* T*-number. Furthermore, if* $w\geqslant 16$, *then*

$$w_2^*(\xi_{w,\eta})=\frac{2w-2-\eta}{2+\eta}\quad\text{and}\quad w_2(\xi_{w,\eta})=\frac{2w-\eta}{2+\eta},$$

hence

$$w_2(\xi_{w,\eta})-w_2^*(\xi_{w,\eta})=\frac{2}{2+\eta}.$$

We state an immediate consequence of the combination of (1.3) with Theorems 4.1 and 4.2 (or with [Bug04]).

THEOREM 4.3. *The set of values taken by the function* $w_2-w_2^*$ *is equal to the closed interval* $[0,1]$.

The real numbers defined in Theorems 4.1 and 4.2 are the first explicit examples of real numbers ξ for which $w_2^*(\xi)$ and $w_2(\xi)$ differ. They are also the first examples of badly approximable numbers with this badly.

5. Algebraic approximation to automatic continued fractions

An infinite sequence $\mathbf{a}=(a_\ell)_{\ell\geqslant 1}$ is an automatic sequence if it can be generated by a finite automaton, that is, if there exists an integer $k\geqslant 2$ such that a_ℓ is a finite-state function of the representation of ℓ in base k, for every $\ell\geqslant 1$. Let $b\geqslant 2$ be an integer. In 1968, Cobham [Cob68] asked whether a real number whose b-ary expansion can be generated by a finite automaton (in the following, such a real number is called a *b-ary automatic number*) is always either rational or transcendental. A positive answer to Cobham's question was recently given in [AB07b], by means of a combinatorial transcendence criterion established in [ABL04]. We addressed in [AB05] the analogous question for continued fraction expansions and gave a positive answer to it in [Bug10]. Namely, we proved that a real number whose continued fraction expansion can be generated by a finite automaton (in the following, such a real number is called an *automatic number*) is always either quadratic or transcendental.

More Diophantine properties of b-ary irrational automatic real numbers are known. Adamczewski and Cassaigne [AC06] established that the Diophantine exponent of any non-ultimately periodic automatic sequence is finite (Lemma 8.3 below) and deduced that no b-ary automatic number is a Liouville number. By Theorem 2.2, we obtain likewise a measure of non-quadraticity for automatic real numbers which are not quadratic.

Precise definitions of automatic and morphic sequences and of various quantities and notions associated with them are postponed to § 8.

THEOREM 5.1. *Let* $k\geqslant 2$ *be an integer and* $\mathbf{a}=(a_\ell)_{\ell\geqslant 1}$ *an infinite sequence of positive integers generated by a* k*-automaton. Let* $A\geqslant 3$ *be an upper bound for the sequence* $\mathbf{a}=(a_\ell)_{\ell\geqslant 1}$. *Let* m *be the cardinality of the* k*-kernel of the sequence* $\mathbf{a}$ *and let* I *be the internal alphabet associated with* $\mathbf{a}$. *Then, the real number*

$$\xi:=[0;a_1,a_2,\ldots,a_\ell,\ldots]$$

satisfies

$$w_2(\xi) \leqslant 31\,300\,(\text{Card}\, I)^6\, k^{m+3}(\log(A+1))^4. \tag{5.1}$$

Theorem 5.1 extends [AC06, Theorem 7.3], valid only for a restricted class of automatic numbers.

Theorem 5.1 is an immediate consequence of Theorem 2.2, inequality (8.1) and Lemma 8.3. We can slightly improve Theorem 5.1 by arguing directly (as was done in [AC06] to show that b-ary automatic numbers are not Liouville numbers) rather than applying the general Lemma 9.1.

The bound (5.1) depends on three parameters, Card I, k and m, which appear naturally in the study of automatic sequences. It also depends, and this is rather annoying, on the alphabet on which the sequence is written. However, one can remove the latter dependence when the automatic sequence is generated by a primitive morphism.

Theorem 5.2. *Let $k \geqslant 2$ be an integer and let $\mathbf{a} = (a_\ell)_{\ell \geqslant 1}$ be a non-ultimately periodic infinite sequence of positive integers generated by a primitive morphism σ on an alphabet of cardinality $b \geqslant 2$. Let v denote the width of σ. Then the real number*

$$\xi := [0; a_1, a_2, \ldots, a_\ell, \ldots]$$

satisfies

$$w_2(\xi) \leqslant 3 \cdot 10^5\, v^{12b-6}\, b^{12}\ \text{Dio}(\xi).$$

We stress that the upper bound in Theorem 5.2 does not depend on the alphabet on which the morphic sequence is written. Theorem 5.2 is an immediate consequence of Theorem 2.2 combined with Lemmas 8.4–8.6.

The Thue–Morse sequence

$$abbabaabbaababbabaab\ldots$$

defined on the alphabet $\{a, b\}$ is a classical example of an automatic sequence satisfying the assumption of Theorem 5.2. Further examples are given in [AS03].

Adamczewski and Bugeaud [AB11] showed that irrational b-ary automatic numbers are either S- or T-numbers in Mahler's classification of numbers. Since the complexity function of every automatic sequence $\mathbf{a}$ grows at most linearly with n (see inequality (8.1) below), Theorem 3.2 implies the analogous result for automatic numbers which are not quadratic.

Theorem 5.3. *Let $\mathbf{a} = (a_\ell)_{\ell \geqslant 1}$ be an automatic sequence (or a sequence generated by a primitive morphism) of positive integers which is not ultimately periodic and set*

$$\xi := [0; a_1, a_2, \ldots, a_\ell, \ldots].$$

Then ξ is a transcendental number and there exists a constant c, depending only on ξ, such that

$$w_d(\xi) \leqslant \exp(c(\log 3d)^5(\log\log 3d)^4), \quad \text{for } d \geqslant 1.$$

In particular, ξ is either an S-number or a T-number.

The part of Theorem 5.3 dealing with sequences generated by a primitive morphism follows from Theorem 3.2 combined with Lemmas 8.4–8.6.

There have recently been several new results on the rational approximation to real numbers whose expansion in some integer base is an automatic sequence [AR09, Bug08, BKS11]. Motivated by these works, we address and briefly discuss the following problem.

Problem 5.4. Determine the set of values taken by the exponents w_2 and w_2^* at automatic continued fractions.

With a suitable modification of the construction given in [Bug08], one can construct explicitly, for every sufficiently large rational number p/q, an automatic continued fraction ξ satisfying $w_2^*(\xi) = p/q$. It seems to be difficult and challenging to show that the sets of values taken by the functions w_2^* and w_2 at automatic continued fractions include every rational number greater than or equal to 2.

Bugeaud and Laurent [BL05] computed the values of $w_2(\xi)$ and $w_2^*(\xi)$ for certain morphic continued fractions ξ and found that these values are quadratic numbers.

It was proved in [Bug] that the Thue–Morse–Mahler number $\xi_{\mathbf{t}}$, whose sequence of binary digits is the Thue–Morse sequence on $\{0, 1\}$ starting with 0, satisfies $w_1(\xi_{\mathbf{t}}) = 1$. We address the analogous question for continued fraction expansions.

Problem 5.5. Let a and b be distinct positive integers. Let $\xi_{\mathbf{t},a,b}$ be the real number whose sequence of partial quotients is the Thue–Morse sequence on $\{a, b\}$ starting with a. Compute $w_2^*(\xi_{\mathbf{t},a,b})$ and $w_2(\xi_{\mathbf{t},a,b})$.

It is known [Que00] that $w_2^*(\xi_{\mathbf{t},a,b}) \geqslant 7/3$. In view of Schmidt's theorem [Sch67], this is sufficient to conclude that $\xi_{\mathbf{t},a,b}$ is transcendental, a result first proved by Queffélec [Que98] (see also [ADQZ01]). Since this lower bound is small, one cannot prove that this is the exact value by arguing as in Lemma 7.3. We suspect, however, that 7/3 is not the correct value and plan to return to this question in a future work.

6. Continued fractions

We gather in this section various results on continued fraction expansions, but we assume that the reader is already quite familiar with the subject (otherwise, he is directed, for example, to [Per29] or to [Bug04a, ch. 1]).

In this and the next sections, we use the notation

$$[0; a_1, \ldots, a_r, \overline{a_{r+1}, \ldots, a_{r+s}}] := [0; U, \overline{V}],$$

where $U = a_1 \ldots a_r$ and $V = a_{r+1} \ldots a_{r+s}$, to indicate that the block of partial quotients $a_{r+1}, \ldots, a_{r+s}$ is repeated infinitely many times. We also denote by ζ' the Galois conjugate of a quadratic real number ζ.

The key observation for our main results is given in the following lemma.

Lemma 6.1. *Let ξ be a quadratic real number with ultimately periodic continued fraction expansion*

$$\xi = [0; a_1, \ldots, a_r, \overline{a_{r+1}, \ldots, a_{r+s}}],$$

with $r \geqslant 3$ and $s \geqslant 1$, and denote by ξ' its Galois conjugate. Let $(p_\ell/q_\ell)_{\ell \geqslant 1}$ denote the sequence of convergents to ξ. If $a_r \neq a_{r+s}$, then

$$|\xi - \xi'| \ll a_r^2 \max\{a_{r-2}, a_{r-1}\} q_r^{-2}.$$

Proof. By a theorem of Galois (see [Per29, p. 83]), the Galois conjugate of

$$\tau = [a_{r+1}; \overline{a_{r+2}, \ldots, a_{r+s}, a_{r+1}}]$$

is the quadratic number

$$\tau' = -[0; \overline{a_{r+s}, \dots, a_{r+2}, a_{r+1}}].$$

Since

$$\xi = \frac{p_r \tau + p_{r-1}}{q_r \tau + q_{r-1}} \quad \text{and} \quad \xi' = \frac{p_r \tau' + p_{r-1}}{q_r \tau' + q_{r-1}},$$

we get

$$|\xi - \xi'| = \frac{\tau - \tau'}{(q_r \tau + q_{r-1}) \cdot |q_r \tau' + q_{r-1}|} \ll \frac{1}{q_r \cdot |q_r \tau' + q_{r-1}|}. \tag{6.1}$$

Assume that $a_r \neq a_{r+s}$. Using the mirror formula

$$q_{r-1}/q_r = [0; a_r, a_{r-1}, \dots, a_1],$$

we see that

$$|q_r \tau' + q_{r-1}| = |[0; a_r, a_{r-1}, \dots, a_1] - [0; \overline{a_{r+s}, \dots, a_{r+2}, a_{r+1}}]| \cdot q_r.$$

If $\max\{a_r, a_{r+s}\} \geqslant 2 \min\{a_r, a_{r+s}\}$, then

$$|q_r \tau' + q_{r-1}| \gg q_r / \min\{a_r, a_{r+s}\}.$$

Otherwise, if $a_r < a_{r+s}$, then an easy computation shows that

$$\begin{aligned} |q_r \tau' + q_{r-1}| &\geqslant \left| \frac{1}{a_r + 1/(1 + 1/(a_{r-2} + 1))} - \frac{1}{a_{r+s} + 1/(a_{r+s-1} + 1)} \right| \cdot q_r \\ &\gg \frac{q_r}{a_r^2} \max\{a_{r-2}^{-1}, a_{r+s-1}^{-1}\} \\ &\gg \frac{q_r}{a_r^2} \times \frac{1}{a_{r-2}}, \end{aligned}$$

while, if $a_r > a_{r+s}$, then the similar estimate

$$|q_r \tau' + q_{r-1}| \gg \frac{q_r}{a_r^2} \max\{a_{r-1}^{-1}, a_{r+s-2}^{-1}\} \gg \frac{q_r}{a_r^2} \times \frac{1}{a_{r-1}}$$

holds. By (6.1), this completes the proof of the lemma. □

We give an elementary result on ultimately periodic continued fraction expansions, whose proof can be found in [Per29].

Lemma 6.2. *Let ξ be a quadratic real number with ultimately periodic continued fraction expansion*

$$\xi = [0; a_1, \dots, a_r, \overline{a_{r+1}, \dots, a_{r+s}}], \tag{6.2}$$

and denote by $(p_\ell / q_\ell)_{\ell \geqslant 1}$ the sequence of its convergents. Then, ξ is a root of the polynomial

$$\begin{aligned} &(q_{r-1} q_{r+s} - q_r q_{r+s-1}) X^2 - (q_{r-1} p_{r+s} - q_r p_{r+s-1} + p_{r-1} q_{r+s} - p_r q_{r+s-1}) X \\ &\quad + (p_{r-1} p_{r+s} - p_r p_{r+s-1}). \end{aligned} \tag{6.3}$$

In particular, if the continued fraction expansion of ξ is purely periodic, that is, if

$$\xi = [0; \overline{a_1, \dots, a_s}],$$

then ξ is a root of the polynomial

$$q_{s-1} X^2 - (p_{s-1} - q_s) X - p_s. \tag{6.4}$$

https://doi.org/10.1112/S0010437X11007524 Published online by Cambridge University Press

The polynomials (6.3) and (6.4) may not be primitive, as is the case in particular when $a_r = a_{r+s}$. They provide us only with an upper bound for the height of the real number ξ defined by (6.2).

For the proof of Theorems 4.1 and 4.2 we need a precise estimate of the height of quadratic numbers of a special form.

Lemma 6.3. *Let b, c and d be distinct positive integers. Let $n \geqslant 3$ be an integer and $a_1, \dots, a_{n-2}$ be positive integers. Set*

$$\xi := [0; a_1, \dots, a_{n-2}, b, c, \overline{b}].$$

Then, the height of ξ satisfies

$$q_n^2 \ll_{b,c} H(\xi) \ll_{b,c} q_n^2,$$

where q_n is the denominator of the rational number $p_n/q_n := [0; a_1, \dots, a_{n-2}, b, c]$. Let $m \geqslant 3$ be an integer and set

$$\zeta := [0; a_1, \dots, a_{n-2}, b, c, \overline{b, b, \dots, b, d}],$$

where the periodic part $b, b, \dots, b, d$ has length m. Then the height of ζ satisfies

$$q_n q_{n+m} \ll_{b,c,d} H(\zeta) \ll_{b,c,d} q_n q_{n+m}.$$

where q_{n+m} is the denominator of the rational number $[0; a_1, \dots, a_{n-2}, b, c, b, b, \dots, b, d]$.

Proof. We deduce from Lemma 6.2 that

$$H(\xi) \ll_{b,c} q_n^2 \quad \text{and} \quad H(\zeta) \ll_{b,c,d} q_n q_{n+m}.$$

Let

$$P_\xi(X) = AX^2 + BX + C = A(X - \xi)(X - \xi')$$

denote the minimal defining polynomial of ξ over $\mathbf{Z}$. Since $|P_\xi(p_n/q_n)|$ is a non-zero rational number of denominator dividing q_n^2, we have $|P_\xi(p_n/q_n)| \geqslant q_n^{-2}$. It follows from Lemma 6.1 and the definition of p_n/q_n that

$$|\xi - p_n/q_n| < q_n^{-2}, \quad |\xi' - p_n/q_n| \ll_{b,c} q_n^{-2}.$$

Consequently, we get $A \gg_{b,c} q_n^2$, thus $H(\xi) \gg_{b,c} q_n^2$. This completes the proof of the first assertion of the lemma.

Since the resultant of the minimal defining polynomials of ξ and ζ is a non-zero integer, we get

$$\begin{aligned} |\zeta - \xi| &\geqslant H(\xi)^{-2} \cdot H(\zeta)^{-2} \cdot |\zeta - \xi'|^{-1} \cdot |\zeta' - \xi|^{-1} \cdot |\zeta' - \xi'|^{-1} \\ &\gg_{b,c,d} H(\xi)^{-2} \cdot H(\zeta)^{-2} \cdot q_n^6, \end{aligned}$$

by Lemma 6.1. Using $|\zeta - \xi| \ll_{b,c,d} q_{n+m}^{-2}$ and $H(\xi) \ll_{b,c} q_n^2$, this gives

$$H(\zeta)^2 \gg_{b,c,d} q_n^2 \cdot q_{n+m}^2.$$

This finishes the proof of the lemma. □

For the proof of Theorem 9.1 we need classical results on continuants which we recall below (see [Per29] for a proof). If $a_1, \dots, a_m$ are positive integers, then the continuant $K_m(a_1, \dots, a_m)$ is the denominator of the rational number $[0; a_1, \dots, a_m]$.

https://doi.org/10.1112/S0010437X11007524 Published online by Cambridge University Press

Lemma 6.4. *For any positive integers* $a_1, \ldots, a_m$ *and any integer* k *with* $1 \leqslant k \leqslant m-1$,

$$K_m(a_1, \ldots, a_m) = K_m(a_m, \ldots, a_1),$$
$$K_m(a_1, \ldots, a_m) \leqslant (1 + \max\{a_1, \ldots, a_m\})^m,$$
$$K_m(a_1, \ldots, a_m) \geqslant \max\{(\min\{a_1, \ldots, a_m\})^m, 2^{(m-1)/2}\},$$

and

$$K_k(a_1, \ldots, a_k) \cdot K_{m-k}(a_{k+1}, \ldots, a_m) \leqslant K_m(a_1, \ldots, a_m) \leqslant 2\, K_k(a_1, \ldots, a_k) \cdot K_{m-k}(a_{k+1}, \ldots, a_m).$$

7. Liouville's inequality and applications

Observe that if α is a real quadratic number and α' denotes its Galois conjugate, then

$$H(\alpha)^{-1} \leqslant |\alpha - \alpha'| \leqslant 2H(\alpha). \tag{7.1}$$

To see this, it is sufficient to note that, if the minimal defining polynomial of α over $\mathbf{Z}$ is $aX^2 + bX + c$, then

$$|\alpha - \alpha'| = \frac{\sqrt{b^2 - 4ac}}{a}.$$

Note that $|\alpha - \alpha'|$ can be as small as $\sqrt{5}H(\alpha)^{-1}$. Indeed, for any integer $m \geqslant 2$, the discriminant of the polynomial

$$(m^2 + m - 1)X^2 - (2m + 1)X + 1$$

is equal to 5, thus, the distance between its two roots is equal to $\sqrt{5}$ divided by its height.

Lemma 7.1. *Let* α *and* β *be real quadratic numbers. Denote by* $P_\alpha(X) := a(X - \alpha)(X - \alpha')$ *and* $P_\beta(X) := b(X - \beta)(X - \beta')$ *their minimal defining polynomials over* $\mathbf{Z}$. *Assume that* $\alpha, \alpha', \beta, \beta'$ *are distinct. Then*

$$|\alpha - \beta| \geqslant 0.03 \cdot \max\{|\alpha - \alpha'|^{-1}, 1\} \cdot H(\alpha)^{-2} \cdot H(\beta)^{-2}. \tag{7.2}$$

Under the assumption of Lemma 7.1, the usual form of Liouville's inequality (see, for example, [Bug04a, Theorem A.1]) implies that

$$|\alpha - \beta| \geqslant 0.03 \cdot H(\alpha)^{-2} \cdot H(\beta)^{-2}. \tag{7.3}$$

Lemma 7.1 shows that this estimate can be improved if α and its Galois conjugate are close to each other. Roughly speaking, in the most favourable cases, the exponent -2 of $H(\alpha)$ in (7.3) can be replaced by -1.

Proof. In view of (7.3), we assume that $|\alpha - \alpha'| < 1$. Since the resultant of $P_\alpha(X)$ and $P_\beta(X)$ is a non-zero integer, we get

$$ab^2|\alpha - \beta| \cdot |\alpha' - \beta| \cdot |P_\alpha(\beta')| \geqslant 1. \tag{7.4}$$

If $|\beta'| \geqslant 2$, then

$$|b\beta'| \leqslant |b\beta\beta'| \leqslant H(\beta) \quad \text{if } |\beta| \geqslant 1,$$

while

$$|b\beta'| \leqslant 2|b(\beta + \beta')| \leqslant 2H(\beta) \quad \text{if } |\beta| \leqslant 1.$$

Consequently, regardless the value of $|\beta'|$, this gives

$$b^2\, |P_\alpha(\beta')| \leqslant 3H(\alpha)\, b^2 \max\{1, |\beta'|\}^2 \leqslant 12 \cdot H(\alpha) \cdot H(\beta)^2,$$

https://doi.org/10.1112/S0010437X11007524 Published online by Cambridge University Press

thus, by (7.4),

$$|\alpha - \beta| \geqslant |\alpha' - \beta|^{-1} \cdot H(\alpha)^{-2} \cdot H(\beta)^{-2}/12. \tag{7.5}$$

With no loss of generality, we may assume that $|\alpha - \beta| \leqslant |\alpha' - \beta|$.

If $|\alpha' - \beta| \leqslant 2|\alpha - \beta|$, then (7.5) implies that

$$|\alpha - \beta|^2 \geqslant H(\alpha)^{-2} \cdot H(\beta)^{-2}/24,$$

which, by (7.1), yields a much better lower bound than (7.2).

If $|\alpha' - \beta| \geqslant 2|\alpha - \beta|$, then, using the triangle inequality

$$|\alpha - \alpha'| \geqslant |\alpha' - \beta| - |\alpha - \beta| \geqslant |\alpha' - \beta|/2,$$

it follows from (7.5) that

$$|\alpha - \beta| \geqslant |\alpha - \alpha'|^{-1} \cdot H(\alpha)^{-2} \cdot H(\beta)^{-2}/24.$$

Combined with (7.3), this gives the lemma. □

We point out two important consequences of Lemma 7.1. The first states that if a real number ξ is quite close to a dense (in a suitable sense) sequence of quadratic numbers having a close conjugate, then ξ cannot be too well approximated by quadratic numbers.

Lemma 7.2. *Let ξ be a real number. Let $C > 1$ be a real number and $(Q_j)_{j\geqslant 1}$ an increasing sequence of integers such that $Q_{j+1} \leqslant Q_j^C$ for $j \geqslant 1$. Let $w > 2$ and $0 < \varepsilon < 1$ be real numbers. Assume that there exists a sequence $(\alpha_j)_{j\geqslant 1}$ of quadratic numbers such that, denoting by α'_j the Galois conjugate of α_j,*

$$|\xi - \alpha_j| < Q_j^{-2-\varepsilon} \cdot \max\{|\alpha_j - \alpha'_j|^{-1}, 1\},$$
$$H(\alpha_j) \leqslant Q_j,$$

and

$$|\xi - \alpha_j| \geqslant Q_j^{-w},$$

for $j \geqslant 1$. Then $w_2^*(\xi)$ is finite and

$$w_2^*(\xi) \leqslant w - 1 + 2wC\varepsilon^{-1}.$$

Proof. Set $A = 1 + 2C\varepsilon^{-1}$. Let α be a quadratic real number and let j be the integer defined by the inequalities

$$Q_j \leqslant H(\alpha)^A < Q_{j+1}.$$

By Lemma 7.1, if $\alpha \neq \alpha_j$, then

$$|\alpha - \alpha_j| \geqslant 0.03 \cdot \max\{|\alpha_j - \alpha'_j|^{-1}, 1\} \cdot Q_j^{-2} H(\alpha)^{-2}. \tag{7.6}$$

For j sufficiently large, the choice of A implies that

$$H(\alpha) \leqslant Q_{j+1}^{1/A} \leqslant Q_j^{C/A} \leqslant 0.1 \cdot Q_j^{\varepsilon/2},$$

hence,

$$\begin{aligned} 0.01 \cdot \max\{|\alpha_j - \alpha'_j|^{-1}, 1\} \cdot Q_j^{-2} H(\alpha)^{-2} &\geqslant \max\{|\alpha_j - \alpha'_j|^{-1}, 1\} \cdot Q_j^{-2-\varepsilon} \\ &> |\xi - \alpha_j|. \end{aligned} \tag{7.7}$$

https://doi.org/10.1112/S0010437X11007524 Published online by Cambridge University Press

Using (7.6) and (7.7), the triangle inequality then gives

$$\begin{aligned}|\xi - \alpha| \geqslant |\alpha - \alpha_j| - |\xi - \alpha_j| &\geqslant 0.02 \cdot \max\{|\alpha_j - \alpha_j'|^{-1}, 1\} \cdot Q_j^{-2} H(\alpha)^{-2} \\ &\geqslant 0.02 \cdot Q_j^{-2} \cdot H(\alpha)^{-2} \\ &\geqslant 0.02 \cdot H(\alpha)^{-4-4C/\varepsilon},\end{aligned}$$

if j is sufficiently large. It remains for us to consider the case where $\alpha = \alpha_j$. Then

$$|\xi - \alpha| \geqslant Q_j^{-w} \geqslant H(\alpha)^{-wA} \geqslant H(\alpha)^{-w-2wC/\varepsilon}.$$

Since $w \geqslant 2$, this proves the lemma. □

Let us briefly explain the novelty in Lemma 7.2. If nothing is known on the Galois conjugates of the quadratic approximants α_j, then, in order to get an upper bound for $w_2^*(\xi)$, we have to assume that α_j is very close to ξ, namely, roughly speaking, that $|\xi - \alpha_j| < H(\alpha_j)^{-2-\varepsilon}$ for some positive real number ε. This is the strong assumption made in [AC06, p. 1370] and in [AB10b, p. 896]. Fortunately, we can considerably weaken it and replace it even by $|\xi - \alpha_j| < H(\alpha_j)^{-1-\varepsilon}$ (in the most favourable cases), provided that α_j is very close to its Galois conjugate α_j'. This is crucial for the applications we have in mind, especially for the proof of Theorem 2.2.

Lemma 7.3. *Let ξ be a real number. Assume that there exist positive real numbers $c_1, c_2, c_3, \delta, \rho, \theta$ and a sequence $(\alpha_j)_{j\geqslant 1}$ of quadratic numbers such that*

$$c_1 H(\alpha_j)^{-\rho-1} \leqslant |\xi - \alpha_j| \leqslant c_2 H(\alpha_j)^{-\delta-1} \tag{7.8}$$

and

$$H(\alpha_j) \leqslant H(\alpha_{j+1}) \leqslant c_3 H(\alpha_j)^{\theta},$$

for $j \geqslant 1$. Set $\varepsilon = 0$ or assume that there exist $c_4 \geqslant 1$ and $0 < \varepsilon \leqslant 1$ such that

$$|\alpha_j - \alpha_j'| \leqslant c_4 H(\alpha_j)^{-\varepsilon}, \tag{7.9}$$

for $j \geqslant 1$, where α_j' denotes the Galois conjugate of α_j. Then

$$\delta \leqslant w_2^*(\xi) \leqslant \rho$$

if

$$(\rho - 1)(\delta - 1 + \varepsilon) \geqslant 2\theta(2 - \varepsilon). \tag{7.10}$$

Furthermore, if $\varepsilon > 0$, then

$$\delta \leqslant w_2^*(\xi) \leqslant \rho \quad \text{and} \quad w_2(\xi) = w_2^*(\xi) + \varepsilon,$$

if

$$(\delta - 2 + \varepsilon)(\delta - 1 + \varepsilon) \geqslant 2\theta(2 - \varepsilon)$$

and

$$\lim_{j\to+\infty} \frac{\log |\alpha_j - \alpha_j'|}{\log H(\alpha_j)} = -\varepsilon. \tag{7.11}$$

Proof. Set $Q_j = H(\alpha_j)$ for $j \geqslant 1$. Let α be either a rational number, or a quadratic real number not belonging to the sequence $(\alpha_j)_{j\geqslant 1}$. Let j be the integer defined by the inequalities

$$Q_{j-1} < (c_5 H(\alpha))^{2/(\varepsilon+\delta-1)} \leqslant Q_j,$$

where

$$c_5 = (70 c_2 c_4)^{1/2}. \tag{7.12}$$

https://doi.org/10.1112/S0010437X11007524 Published online by Cambridge University Press

We assume that $H(\alpha)$, hence j, is sufficiently large. By Lemma 7.1 and (7.12),

$$\begin{aligned}|\alpha - \alpha_j| &\geqslant 0.03 \cdot |\alpha_j - \alpha_j'|^{-1} Q_j^{-2} \cdot H(\alpha)^{-2}\\ &\geqslant 0.03 \cdot c_4^{-1} c_5^2\, Q_j^{-1-\delta}\\ &\geqslant 2c_2 Q_j^{-1-\delta}.\end{aligned}$$

Consequently,

$$|\xi - \alpha| \geqslant |\alpha - \alpha_j| - |\xi - \alpha_j| \geqslant |\alpha - \alpha_j|/2.$$

Since

$$Q_j \leqslant c_3 Q_{j-1}^{\theta} \leqslant c_3 (c_5 H(\alpha))^{2\theta/(\varepsilon+\delta-1)},$$

we conclude that

$$\begin{aligned}|\xi - \alpha| &\geqslant 0.01 \cdot c_4^{-1} Q_j^{-2+\varepsilon} \cdot H(\alpha)^{-2}\\ &\geqslant 0.01 \cdot c_4^{-1} c_3^{-2+\varepsilon} c_5^{-2\theta(2-\varepsilon)/(\varepsilon+\delta-1)} H(\alpha)^{-2-2\theta(2-\varepsilon)/(\varepsilon+\delta-1)},\end{aligned} \tag{7.13}$$

thus

$$w_2^*(\xi) \leqslant \max\{\rho, 1 + 2\theta(2-\varepsilon)/(\varepsilon+\delta-1)\}.$$

Combined with (7.8), this implies that $w_2^*(\xi) \leqslant \rho$ when (7.10) is satisfied. Furthermore, assumption (7.8) ensures that $w_2^*(\xi) \geqslant \delta$ and $w_2(\xi) \geqslant w_2^*(\xi) + \varepsilon$, by (7.9). This proves the first assertion of the lemma.

If, moreover, $(\delta - 2 + \varepsilon)(\delta - 1 + \varepsilon) \geqslant 2\theta(2-\varepsilon)$, then it follows from (7.13) that

$$|\xi - \alpha| \geqslant 0.01 \cdot c_4^{-1} c_3^{-2+\varepsilon} c_5^{-2\theta(2-\varepsilon)/(\varepsilon+\delta-1)} H(\alpha)^{-\delta-\varepsilon}. \tag{7.14}$$

This means that the best algebraic approximants to ξ of degree at most two belong to the sequence $(\alpha_j)_{j\geqslant 1}$. Assume that α is irrational and denote by $P_\alpha(X)$ its minimal defining polynomial, by α' its Galois conjugate, and by $a_\alpha \geqslant 1$ the leading coefficient of $P_\alpha(X)$. Note that $a_\alpha \cdot |\alpha - \alpha'| \geqslant 1$. Using (7.14), the fact that $\delta \geqslant 2$ and the triangle inequality, we then get

$$\begin{aligned}|P_\alpha(\xi)| &\geqslant a_\alpha \cdot |\xi - \alpha| \cdot |\xi - \alpha'|\\ &\geqslant a_\alpha \cdot |\xi - \alpha| \cdot |\alpha - \alpha'|/2\\ &\geqslant 0.005 \cdot c_4^{-1} c_3^{-2+\varepsilon} c_5^{-2\theta(2-\varepsilon)/(\varepsilon+\delta-1)} H(\alpha)^{-\delta-\varepsilon}.\end{aligned} \tag{7.15}$$

For $j \geqslant 1$, denoting by $P_j(X)$ the minimal defining polynomial of α_j, we deduce, again from the triangle inequality, that

$$|P_j(\xi)| \leqslant 2\, H(\alpha_j) \cdot |\xi - \alpha_j| \cdot |\alpha_j - \alpha_j'|. \tag{7.16}$$

It thus follows from (7.8), (7.11) and (7.16) that

$$w_2(\xi) \geqslant w_2^*(\xi) + \varepsilon \geqslant \delta + \varepsilon. \tag{7.17}$$

We conclude from (7.15) that the first inequality in (7.17) is an equality. This finishes the proof of the lemma. □

8. Automatic and morphic sequences

We recall in this section basic definitions and several results on automatic and morphic sequences. For more information, the reader is advised to consult the monograph [AS03]. As in [AS03], but unlike in the rest of the present paper, we index the sequences from $\ell = 0$.

https://doi.org/10.1112/S0010437X11007524 Published online by Cambridge University Press

Let $k \geqslant 2$ be an integer and denote by Σ_k the set $\{0, 1, \ldots, k-1\}$. A k-automaton is a 6-tuple

$$A = (Q, \Sigma_k, \delta, q_0, \Delta, \tau),$$

where Q is a finite set of states, Σ_k is the input alphabet, $\delta : Q \times \Sigma_k \to Q$ is the transition function, q_0 is the initial state, Δ is the output alphabet and $\tau : Q \to \Delta$ is the output function.

For a state q in Q and for a finite word $W = w_1 w_2 \ldots w_n$ on the alphabet Σ_k, we recursively define $\delta(q, W)$ by $\delta(q, W) = \delta(\delta(q, w_1 w_2 \ldots w_{n-1}), w_n)$. Let $n \geqslant 0$ be an integer and let $w_r w_{r-1} \ldots w_1 w_0$ in $(\Sigma_k)^{r+1}$ be the representation of n in base k, meaning that $n = w_r k^r + \cdots + w_0$. We denote by W_n the word $w_0 w_1 \ldots w_r$. Then a sequence $\mathbf{a} = (a_\ell)_{\ell \geqslant 0}$ is said to be k-automatic if there exists a k-automaton A such that $a_\ell = \tau(\delta(q_0, W_\ell))$ for all $\ell \geqslant 0$.

For a finite set $\mathcal{A}$, we denote by $\mathcal{A}^*$ the free monoid generated by $\mathcal{A}$. The empty word ε is the neutral element of $\mathcal{A}^*$. Let $\mathcal{A}$ and $\mathcal{B}$ be two finite sets. An application from $\mathcal{A}$ to $\mathcal{B}^*$ can be uniquely extended to a homomorphism between the free monoids $\mathcal{A}^*$ and $\mathcal{B}^*$. We call such a homomorphism a morphism from $\mathcal{A}$ to $\mathcal{B}$. If there is a positive integer k such that each element of $\mathcal{A}$ is mapped to a word of length k, then the morphism is called k-uniform (or, simply, uniform). Similarly, an application from $\mathcal{A}$ to $\mathcal{B}$ can be uniquely extended to a homomorphism between the free monoids $\mathcal{A}^*$ and $\mathcal{B}^*$. Such an application is called a coding (or a 'letter-to-letter' morphism).

A morphism σ from $\mathcal{A}$ into itself is said to be prolongable if there exists a letter a such that $\sigma(a) = aW$, where the word W is such that $\sigma^n(W)$ is not the empty word for every $n \geqslant 0$. In that case, the sequence of finite words $(\sigma^n(a))_{n \geqslant 1}$ converges in $\mathcal{A}^{\mathbf{Z}_{\geqslant 0}}$ (endowed with the product topology of the discrete topology on each copy of $\mathcal{A}$) to an infinite word $\mathbf{a} := \sigma^\infty(a)$. This infinite word is clearly a fixed point for σ, and we say that $\mathbf{a}$ is generated by the morphism σ. The *width* of the morphism σ is the maximum of the lengths of the words $\sigma(a)$ for a in $\mathcal{A}$. A morphism σ is *primitive* if there exists a positive integer n such that a occurs in $\sigma^n(a')$ for all a, a' in $\mathcal{A}$.

Cobham [Cob72] established that uniform morphisms and automatic sequences are strongly connected.

THEOREM 8.1. *Let $k \geqslant 2$ be an integer. A sequence is k-automatic if, and only if, it is the image under a coding of a fixed point of a k-uniform morphism.*

Theorem 8.1 means that one can always associate with a k-automatic sequence $\mathbf{a}$ a 5-tuple (ϕ, σ, i, A, I), where σ is a k-uniform morphism defined over a finite alphabet I, i is a letter of I, ϕ is a coding from I into A, and such that $\mathbf{a} = \phi(\mathbf{i})$, with $\mathbf{i} = \sigma^\infty(i)$. The set I and the sequence $\mathbf{i}$ are respectively called the internal alphabet and the internal sequence associated with the 5-tuple (ϕ, σ, i, A, I). With a slight abuse of language, we say that I (respectively, $\mathbf{i}$) is the internal alphabet (respectively, internal sequence) associated with $\mathbf{a}$. Indeed, Cobham gives in fact a canonical way to associate with $\mathbf{a}$ a 5-tuple (ϕ, σ, i, A, I). He also proved [Cob72, Theorem 2] that

$$p(n, \mathbf{a}) \leqslant k(\text{Card } I)^2 n \quad \text{for } n \geqslant 1. \tag{8.1}$$

The k-kernel of a sequence $\mathbf{a} = (a_\ell)_{\ell \geqslant 0}$ is the set of all sequences $(a_{k^i \cdot \ell + j})_{\ell \geqslant 0}$, where $i \geqslant 0$ and $0 \leqslant j < k^i$. This notion gives rise to another useful characterization of k-automatic sequences which was first proved by Eilenberg [Eil74].

THEOREM 8.2. *Let $k \geqslant 2$ be an integer. A sequence is k-automatic if, and only if, its k-kernel is finite.*

We reproduce [AC06, Lemma 5.1], which gives an upper bound for the Diophantine exponent of a non-ultimately periodic automatic sequence.

https://doi.org/10.1112/S0010437X11007524 Published online by Cambridge University Press

Lemma 8.3. *Let $k \geqslant 2$ be an integer. Let* $\mathbf{a}$ *be a k-automatic sequence which is not ultimately periodic. Let m be the cardinality of the k-kernel of* $\mathbf{a}$. *Then the Diophantine exponent of* $\mathbf{a}$ *is less than k^m.*

We conclude this section with three results on fixed points of primitive morphisms.

Mossé [Mos92] established that a fixed point of a primitive morphism either is ultimately periodic, or contains no occurrence of words of the form W^x, with W finite and non-empty and x sufficiently large (independently of the length of W). Her result immediately implies the next lemma.

Lemma 8.4. *Let* $\mathbf{a}$ *be a fixed point of a primitive morphism and assume that* $\mathbf{a}$ *is not ultimately periodic. Then the Diophantine exponent of* $\mathbf{a}$ *is finite.*

A second result was proved by Queffélec [Que00].

Lemma 8.5. *Let $\mathbf{a} = (a_\ell)_{\ell \geqslant 1}$ be a fixed point of a primitive morphism on the alphabet $\{1, 2, \ldots\}$. For $\ell \geqslant 1$, let q_ℓ denote the denominator of the ℓth convergent to $[0; a_1, a_2, \ldots]$. Then the sequence $(q_\ell^{1/\ell})_{\ell \geqslant 1}$ converges.*

A third result, deduced from [AS03, proof of Theorem 10.4.12], shows that the complexity of a sequence generated by a primitive morphism is sublinear.

Lemma 8.6. *Let $\mathbf{a} = (a_\ell)_{\ell \geqslant 1}$ be a fixed point of a primitive morphism σ on an alphabet of cardinality $b \geqslant 2$. Let v denote the width of σ. Then*

$$p(n, \mathbf{a}) \leqslant 2v^{4b-2} b^3 n \quad \text{for } n \geqslant 1.$$

The reader is referred to [AS03] for many examples of sequences generated by an automaton or by a primitive morphism.

9. A combinatorial lemma

Throughout this section and the next, we use the following notation. If $\xi = [0; a_1, a_2, \ldots]$ is an irrational real number whose sequence of convergents is $(p_\ell/q_\ell)_{\ell \geqslant -1}$, then, for integers $r \geqslant 0$ and $s \geqslant 1$, we define the integer polynomial $P_{\xi,r,s}(X)$ by

$$\begin{aligned} P_{\xi,r,s}(X) := {} & (q_{r-1} q_{r+s} - q_r q_{r+s-1}) X^2 \\ & - (q_{r-1} p_{r+s} - q_r p_{r+s-1} + p_{r-1} q_{r+s} - p_r q_{r+s-1}) X \\ & + (p_{r-1} p_{r+s} - p_r p_{r+s-1}). \end{aligned} \tag{9.1}$$

By Lemma 6.2, the quadratic number

$$[0; a_1, \ldots, a_r, \overline{a_{r+1}, \ldots, a_{r+s}}]$$

is a root of the polynomial $P_{\xi,r,s}(X)$. For the proofs of Theorems 2.2 and 3.2, we need the following auxiliary result.

Lemma 9.1. *Let $A \geqslant 3$ be an integer. Let $\mathbf{a} = (a_\ell)_{\ell \geqslant 1}$ be an infinite sequence of positive integers at most equal to A and set*

$$\xi := [0; a_1, a_2, \ldots, a_\ell, \ldots].$$

Assume that the Diophantine exponent of ξ is finite. Let $(p_\ell/q_\ell)_{\ell \geqslant 1}$ be the sequence of convergents to ξ. Let M be an upper bound for the sequence $(q_\ell^{1/\ell})_{\ell \geqslant 1}$. Assume that there are integers $n_0 \geqslant 4$

https://doi.org/10.1112/S0010437X11007524 Published online by Cambridge University Press

and $\kappa \geqslant 3$ such that, for every integer $n \geqslant n_0$, there is a word of length n having two occurrences in the prefix of length $(\kappa+1)n$ of $\mathbf{a}$*. Then there exist non-negative integers $r_1, r_2, \ldots$ and positive integers $s_1, s_2, \ldots$ such that, for $j \geqslant 1$, the word* $\mathbf{a}$ *begins with a word of length r_j followed by a word of length s_j to the power $1+1/\kappa$ and:*

(i) $r_j \leqslant (2\kappa+1)s_j$; $s_{j+1} \geqslant 2s_j$;

(ii) $a_{r_j} \neq a_{r_j+s_j}$;

(iii) $(2q_{r_j}q_{r_j+s_j})^2 \leqslant 2q_{r_{j+1}}q_{r_{j+1}+s_{j+1}} \leqslant (2q_{r_j}q_{r_j+s_j})^{650\kappa^2(\log M)^2}$;

(iv) $|P_{\xi,r_j,s_j}(\xi)| \leqslant (2q_{r_j}q_{r_j+s_j})^{-1-1/(15\kappa\log M)}$.

Furthermore, if

$$\alpha_j = [0; a_1, \ldots, a_{r_j}, \overline{a_{r_j+1}, \ldots, a_{r_j+s_j}}],$$

then α_j is the root of $P_{\xi,r_j,s_j}(X)$ which is the closest to ξ and, denoting by α_j' its Galois conjugate,

$$H(\alpha_j) \leqslant 2q_{r_j}q_{r_j+s_j}, \tag{9.2}$$

$$|\xi - \alpha_j| < (2q_{r_j}q_{r_j+s_j})^{-2-1/(15\kappa\log M)} \max\{|\alpha_j - \alpha_j'|^{-1}, 1\}, \tag{9.3}$$

and

$$|\xi - \alpha_j| > (2q_{r_j}q_{r_j+s_j})^{-6(\log M)\,\mathrm{Dio}(\xi)}. \tag{9.4}$$

Estimate (9.2) holds since the quantity $2q_rq_{r+s}$ associated with the polynomial $P_{\xi,r,s}(X)$ defined in (9.1) is an obvious upper bound for its height. Note, however, that the height of $P_{\xi,r,s}(X)$ can be much smaller than $2q_rq_{r+s}$ when q_r/q_{r-1} is close to q_{r+s}/q_{r+s-1}; see § 12.

By Lemma 6.4, we can choose $M = A+1$ in Lemma 9.1. We have decided to introduce the quantity M since the numerical values occurring in Lemma 9.1 heavily depend on the behaviour of the sequence $(q_\ell^{1/\ell})_{\ell\geqslant 1}$, as is explained after its proof. Moreover, it allows us to adapt Lemma 9.1 more easily to the case when the sequence $(a_\ell)_{\ell\geqslant 1}$ is unbounded; see § 12. However, in all the applications presented in §§ 2–5, the sequence $(a_\ell)_{\ell\geqslant 1}$ is bounded.

Proof. The first part is purely combinatorial and we argue as in [AB11, proof of Lemma 9.1]. Let $\ell \geqslant 2$ be an integer, and denote by $A(\ell)$ the prefix of $\mathbf{a}$ of length ℓ. By assumption, for $\ell \geqslant n_0$, there exists a word W_ℓ of length ℓ having at least two occurrences in $A((\kappa+1)\ell)$. In other words, there exist (possibly empty) words B_ℓ, D_ℓ, E_ℓ and a non-empty word C_ℓ such that

$$A((\kappa+1)\ell) = B_\ell W_\ell D_\ell E_\ell = B_\ell C_\ell W_\ell E_\ell.$$

We choose these words in such a way that, if B_ℓ is non-empty, then the last letter of B_ℓ differs from the last letter of C_ℓ.

Assume first that $|C_\ell| \geqslant |W_\ell|$. Then, there exists a word F_ℓ such that

$$A((\kappa+1)\ell) = B_\ell W_\ell F_\ell W_\ell E_\ell.$$

Setting $U_\ell = B_\ell$, $V_\ell = W_\ell F_\ell$, and $w_\ell = |W_\ell F_\ell W_\ell|/|W_\ell F_\ell|$, we observe that $U_\ell V_\ell^{w_\ell}$ is a prefix of $\mathbf{a}$. Furthermore, we check that $w_\ell \geqslant w$ and

$$\frac{|U_\ell V_\ell^{w_\ell}|}{|U_\ell V_\ell|} = 1 + \frac{|W_\ell|}{|U_\ell V_\ell|} \geqslant 1 + \frac{1}{\kappa}.$$

Assume now that $|C_\ell| < |W_\ell|$. This means that the two occurrences of W_ℓ do overlap, hence there exists a rational number $d_\ell > 1$ such that

$$W_\ell = C_\ell^{d_\ell}.$$

https://doi.org/10.1112/S0010437X11007524 Published online by Cambridge University Press

Setting $U_\ell = B_\ell$ and $V_\ell = C_\ell^{\lceil d_\ell/2\rceil}$, and noticing that $3\lceil d_\ell/2\rceil/2 \leqslant d_\ell + 1$, we observe that $U_\ell V_\ell^{3/2}$ is a prefix of $\mathbf{a}$. Furthermore, we check that $|V_\ell| \leqslant (\ell+2)/2$ and

$$\frac{|V_\ell^{1/2}|}{|U_\ell V_\ell|} \geqslant \frac{|V_\ell|}{2(|V_\ell| + \kappa\ell)} \geqslant \frac{\ell+2}{2\ell+4+4\kappa\ell} \geqslant \frac{1}{4\kappa+3}.$$

To summarize, setting $w = 1 + 1/\kappa$, we have shown that, for $\ell \geqslant n_0$, there exist two finite words U_ℓ, V_ℓ and a rational number w_ℓ such that $w \leqslant w_\ell \leqslant 3/2$ and:

(v) $U_\ell V_\ell^{w_\ell}$ is a prefix of $\mathbf{a}$;

(vi) $|U_\ell| \leqslant (2\kappa+1)|V_\ell|$;

(vii) $\ell/2 \leqslant |V_\ell| \leqslant \kappa\ell$;

(viii) if U_ℓ is not the empty word, then the last letter of U_ℓ differs from the last letter of V_ℓ;

(ix) $|U_\ell V_\ell^{w_\ell}|/|U_\ell V_\ell| \geqslant 1 + 1/(4\kappa+3)$.

The words U_ℓ, $\ell \geqslant n_0$, constructed above may not be all distinct. For $\ell \geqslant n_0$, let ρ_ℓ and σ_ℓ denote the lengths of U_ℓ and V_ℓ, respectively. The definition of M and Lemma 6.4 imply that

$$2^{(j-1)/2} \leqslant q_j \leqslant M^j \quad \text{for } j \geqslant 3. \tag{9.5}$$

Set $f = \lceil 4(\log M)/(\log 2)\rceil + 1$. Since

$$\frac{\ell}{2} \leqslant 2\rho_\ell + \sigma_\ell \leqslant 2(\kappa+1)\ell,$$

we have, for every $\ell \geqslant n_0$,

$$\begin{aligned} 2f(2\rho_\ell + \sigma_\ell) \leqslant 4f(\kappa+1)\ell &\leqslant 4\rho_{4f(\kappa+1)\ell} + 2\sigma_{4f(\kappa+1)\ell} \\ &\leqslant 16f(\kappa+1)^2\ell \leqslant 32f(\kappa+1)^2(2\rho_\ell+\sigma_\ell), \end{aligned}$$

by using (vi) and (vii). Setting $c = 32f(\kappa+1)^2$, $r_j = \rho_{(4f)^j(\kappa+1)^j}$, and $s_j = \sigma_{(4f)^j(\kappa+1)^j}$, we thus get

$$2f(2r_j + s_j) \leqslant 2(2r_{j+1} + s_{j+1}) \leqslant c(2r_j + s_j)$$

and

$$s_j \leqslant \kappa(4f)^j(\kappa+1)^j < \frac{(4f)^{j+1}(\kappa+1)^{j+1}}{2} \leqslant \frac{s_{j+1}}{2}.$$

It follows from (9.5) that

$$(2q_{r_j}q_{r_j+s_j})^2 \leqslant 2q_{r_{j+1}}q_{r_{j+1}+s_{j+1}} \leqslant (2q_{r_j}q_{r_j+s_j})^{c(\log M)/(\log 2)}.$$

Since $r_j \leqslant (2\kappa+1)s_j$, conditions (i) and (iii) of the lemma are satisfied. Furthermore, condition (ii) follows from (viii).

Estimate (9.2) directly follows from Lemma 6.1. Set $w'_j = w_{(4f)^j(\kappa+1)^j}$. Since, by condition (v), the real numbers ξ and

$$\alpha_j = [0; U_{(4f)^j(\kappa+1)^j}, \overline{V_{(4f)^j(\kappa+1)^j}}]$$

have the same first $r_j + \lfloor w'_j s_j\rfloor$ partial quotients, we get

$$|\xi - \alpha_j| \leqslant q_{r_j+\lfloor w'_j s_j\rfloor}^{-2} \ll q_{r_j+s_j}^{-2} 2^{-s_j(w'_j-1)}, \tag{9.6}$$

by Lemma 6.4. The inequality

$$|\alpha_j - \alpha'_j| \geqslant (q_{r_j}q_{r_j+s_j})^{-1},$$

https://doi.org/10.1112/S0010437X11007524 Published online by Cambridge University Press

combined with (9.6), shows that α_j is the root of $P_{\xi,r_j,s_j}(X)$ which is the closest to ξ, provided that j is large enough.

It follows from (ix) that

$$\lceil s_j(w'_j - 1)\rceil \geqslant (r_j + s_j)/(4\kappa + 3), \tag{9.7}$$

and, by (9.5) and Lemma 6.4, we get

$$\begin{aligned} 2^{r_j+s_j} &\gg (q_{r_j} q_{r_j+s_j})^{(r_j+s_j)(\log 2)/((2r_j+s_j)\log M)} \\ &\gg (q_{r_j} q_{r_j+s_j})^{(\log 2)/(2\log M)}. \end{aligned} \tag{9.8}$$

Putting together (9.6), (9.7) and (9.8), we deduce that

$$|\xi - \alpha_j| \ll q_{r_j}^2 (2q_{r_j} q_{r_j+s_j})^{-2-(\log 2)/(2(4\kappa+3)\log M)}. \tag{9.9}$$

Furthermore, it follows from Lemma 6.1 that

$$|\alpha_j - \alpha'_j| \ll A^3 q_{r_j}^{-2}, \tag{9.10}$$

thus

$$q_{r_j}^2 \ll A^3 \max\{|\alpha_j - \alpha'_j|^{-1}, 1\}. \tag{9.11}$$

Combining (9.9) and (9.11) with $\kappa \geqslant 3$, we get (9.3) for j large enough. We also deduce from (9.9), (9.10) and the triangle inequality that

$$\begin{aligned} |P_{\xi,r_j,s_j}(\xi)| &\leqslant 2\cdot(2q_{r_j} q_{r_j+s_j})\cdot|\xi - \alpha_j|\cdot|\alpha_j - \alpha'_j| \\ &\leqslant 2A^3 (2q_{r_j} q_{r_j+s_j})^{-1-(\log 2)/(2(4\kappa+3)\log M)}. \end{aligned}$$

This shows that (iv) holds for j large enough.

If $\mathrm{Dio}(\xi)$ is bounded from above by δ, then the continued fraction expansions of α_j and ξ agree at most until the $\lceil\delta(r_j + s_j)\rceil$ partial quotient. Consequently, using [AB06, Lemma 5], we get

$$\begin{aligned} |\xi - \alpha_j| &\geqslant (A+2)^{-3} q_{\lceil\delta(r_j+s_j)\rceil}^{-2} \\ &\geqslant (A+2)^{-3} M^{-2} M^{-2\delta(r_j+s_j)} \\ &\geqslant (A+2)^{-3} M^{-2} (2q_{r_j} q_{r_j+s_j})^{-4\delta(r_j+s_j)(\log M)/((2r_j+s_j)(\log 2))} \\ &\geqslant (A+2)^{-3} M^{-2} (2q_{r_j} q_{r_j+s_j})^{-4\delta(\log M)/(\log 2)}. \end{aligned} \tag{9.12}$$

This shows that (9.4) holds for j large enough. This completes the proof of the lemma. □

It is apparent in the proof of Lemma 9.1 that the estimates can be improved if, instead of (9.5), we use the fact that, for some real numbers m, M with $M > m \geqslant \sqrt{2}$,

$$m^j \leqslant q_j \leqslant M^j,$$

for every sufficiently large j. The quantity $(\log M)/(\log \sqrt{2})$ coming from (9.5) and occurring in the proof of Lemma 9.1 can then be replaced everywhere by $(\log M)/(\log m)$. In particular, when the sequence $(q_\ell^{1/\ell})_{\ell\geqslant 1}$ converges, then m and M can be taken arbitrarily close, thus $(\log M)/(\log m)$ can be taken arbitrarily close to 1. This shows that, under this assumption, Lemma 9.1 holds with $\log M$ replaced by 1 (and even by $1/(\log \sqrt{2})$) in (iii), (iv), (9.3), and (9.4).

https://doi.org/10.1112/S0010437X11007524 Published online by Cambridge University Press

10. Second part of the proof of Theorem 3.2

The proof of Theorem 3.2 partly relies on the quantitative version of the Schmidt subspace theorem. Theorem E below statement was proved by Evertse [Eve96]. We refer to his paper for the definition of the height $H(L)$ of the linear form

$$L(\mathbf{x}) = \alpha_1 x_1 + \cdots + \alpha_m x_m,$$

where $\alpha_1, \ldots, \alpha_m$ are real algebraic numbers all belonging to the same number field of degree d. For our purpose, it is sufficient to stress that $H(L)$ can be bounded from above in terms of the heights of the coefficients of L. More precisely, inequality (6.6) from [AB10a] asserts that

$$H(L) \leqslant m^{d/2}(d+1)^{m/2} \prod_{i=1}^{m} H(\alpha_i). \tag{10.1}$$

Theorem E. *Let $m \geqslant 2$, H and d be positive integers. Let $L_1, \ldots, L_m$ be linearly independent (over $\overline{\mathbf{Q}}$) linear forms in m variables with real algebraic coefficients. Assume that $H(L_i) \leqslant H$ for $i = 1, \ldots, m$ and that the number field generated by all the coefficients of these linear forms has degree at most d. Let ε be a real number with $0 < \varepsilon < 1$. Then the primitive integer vectors $\mathbf{x} = (x_1, \ldots, x_m)$ in $\mathbf{Z}^m$ with $H(\mathbf{x}) \geqslant H$ and such that*

$$\prod_{i=1}^{m} |L_i(\mathbf{x})| < |\det(L_1, L_2, \ldots, L_m)| \cdot (\max\{|x_1|, \ldots, |x_m|\})^{-\varepsilon}$$

lie in at most

$$c_{m,\varepsilon}(\log 3d)(\log\log 3d)$$

proper subspaces of $\mathbf{Q}^m$, where $c_{m,\varepsilon}$ is a constant which only depends on m and ε.

We retain the notation from § 9. First, we check that the entries of the quadruple

$$(q_{r-1}q_{r+s} - q_r q_{r+s-1}, q_{r-1}p_{r+s} - q_r p_{r+s-1}, p_{r-1}q_{r+s} - p_r q_{r+s-1}, p_{r-1}p_{r+s} - p_r p_{r+s-1}) \tag{10.2}$$

are relatively prime. To see this, assume that a positive integer m divides $p_{r-1}q_{r+s} - p_r q_{r+s-1}$ and $p_{r-1}p_{r+s} - p_r p_{r+s-1}$. Then it also divides

$$p_{r+s-1}(p_{r-1}q_{r+s} - p_r q_{r+s-1}) - q_{r+s-1}(p_{r-1}p_{r+s} - p_r p_{r+s-1}) = \pm p_{r-1}$$

and

$$p_{r+s-1}(q_{r-1}q_{r+s} - q_r q_{r+s-1}) - q_{r+s-1}(q_{r-1}p_{r+s} - q_r p_{r+s-1}) = \pm q_{r-1}.$$

Since p_{r-1} and q_{r-1} are relatively prime, the quadruple (10.2) is primitive.

Incidentally, this shows that when $q_{r-1}p_{r+s} - q_r p_{r+s-1}$ is equal to $p_{r-1}q_{r+s} - p_r q_{r+s-1}$, the triple

$$(q_{r-1}q_{r+s} - q_r q_{r+s-1}, q_{r-1}p_{r+s} - q_r p_{r+s-1}, p_{r-1}p_{r+s} - p_r p_{r+s-1})$$

is primitive. These results will be used in the following.

Second part of the proof of Theorem 3.2. Let ξ be as in the statement of Theorem 3.2. Assume that the Diophantine exponent of ξ is finite. By Theorem 2.2, this implies that $w_2(\xi)$ is also finite. To prove that ξ is either an S- or a T-number, it remains for us to control the accuracy of the approximation to ξ by algebraic numbers of exact degree d, for every integer $d \geqslant 3$. We follow [AB10b, proof of Theorem 2.1]. Since there are a few technical difficulties, we give some details.

https://doi.org/10.1112/S0010437X11007524 Published online by Cambridge University Press

Let $d \geqslant 3$ be an integer. Let α be an algebraic number of degree d. At several places in the proof below, it is convenient to assume that the height of α is sufficiently large. Let χ be a positive real number such that

$$|\xi - \alpha| < H(\alpha)^{-\chi}.$$

Our aim is to prove that

$$\chi < \exp(c(\log 3d)^5 \, (\log\log 3d)^4) \tag{10.3}$$

for some constant c which does not depend on d.

Let $\kappa \geqslant 3$ and n_0 be integers such that $p(n, \mathbf{a}) \leqslant \kappa n$ for $n \geqslant n_0$. The *Schubfachprinzip* implies that the assumption of Lemma 9.1 is satisfied. We retain the notation of that lemma and let $(r_j)_{j\geqslant 1}$ and $(s_j)_{j\geqslant 1}$ be the sequences obtained by applying it. Let k be the unique positive integer such that

$$2q_{r_k} q_{r_k+s_k} \leqslant H(\alpha) < 2q_{r_{k+1}} q_{r_{k+1}+s_{k+1}}. \tag{10.4}$$

Denote by M_1 the largest integer such that

$$(2q_{r_k} q_{r_k+s_k})^{\chi} > (2q_{r_{k+M_1}} q_{r_{k+M_1}+s_{k+M_1}})^3$$

and observe that

$$|\xi - \alpha| < (2q_{r_{k+h}} q_{r_{k+h}+s_{k+h}})^{-3} \tag{10.5}$$

for every $h = 1, \ldots, M_1$. From the definition of M_1 and using condition (iii) of Lemma 9.1,

$$\begin{aligned}(2q_{r_k} q_{r_k+s_k})^{\chi} &\leqslant (2q_{r_{k+M_1+1}} q_{r_{k+M_1+1}+s_{k+M_1+1}})^3 \\ &\leqslant (2q_{r_k} q_{r_k+s_k})^{3(650\kappa^2(\log M)^2)^{M_1+1}}.\end{aligned}$$

Consequently, inequality (10.3) holds if we have

$$M_1 < c_0(\log 3d)^5(\log\log 3d)^4$$

for some constant c_0 which does not depend on d.

We will argue by contradiction. From now on, we assume that

$$M_1 > c_1(\log 3d)^5(\log\log 3d)^4, \tag{10.6}$$

for some constant c_1, and we will derive a contradiction if c_1 is sufficiently large.

For simplicity, for every integer $j \geqslant 1$, we write $P_j(X)$ instead of $P_{\xi,r_j,s_j}(X)$, defined by (9.1). Let $j \geqslant 1$ be an integer. Observe that, by condition (iv),

$$|P_j(\xi)| \ll (2q_{r_j} q_{r_j+s_j})^{-1-1/(15\kappa \log M)}, \quad j \geqslant 1. \tag{10.7}$$

Furthermore, for $j \geqslant 1$, we infer from the theory of continued fractions that

$$\begin{aligned}&|(q_{r_j-1} q_{r_j+s_j} - q_{r_j} q_{r_j+s_j-1})\xi - (q_{r_j-1} p_{r_j+s_j} - q_{r_j} p_{r_j+s_j-1})| \\ &\quad \leqslant q_{r_j-1} |q_{r_j+s_j}\xi - p_{r_j+s_j}| + q_{r_j} |q_{r_j+s_j-1}\xi - p_{r_j+s_j-1}| \\ &\quad \ll q_{r_j} q_{r_j+s_j}^{-1}\end{aligned} \tag{10.8}$$

and, likewise,

$$|(q_{r_j-1} q_{r_j+s_j} - q_{r_j} q_{r_j+s_j-1})\xi - (p_{r_j-1} q_{r_j+s_j} - p_{r_j}\, q_{r_j+s_j-1})| \ll q_{r_j}^{-1} q_{r_j+s_j}. \tag{10.9}$$

Using Rolle's theorem, inequalities (10.5) and (10.7) imply that

$$\begin{aligned}|P_{k+h}(\alpha)| &< |P_{k+h}(\xi)| + 3q_{r_{k+h}} q_{r_{k+h}+s_{k+h}} \, |\xi - \alpha| \\ &\ll (2q_{r_{k+h}} q_{r_{k+h}+s_{k+h}})^{-1-1/(15\kappa \log M)}, \quad 1 \leqslant h \leqslant M_1.\end{aligned} \tag{10.10}$$

We apply Theorem E to the system of linear forms

$$\begin{aligned} L_1(X_1, X_2, X_3, X_4) &= \alpha^2 X_1 - \alpha(X_2 + X_3) + X_4,\\ L_2(X_1, X_2, X_3, X_4) &= \alpha X_1 - X_2,\\ L_3(X_1, X_2, X_3, X_4) &= \alpha X_1 - X_3,\\ L_4(X_1, X_2, X_3, X_4) &= X_1. \end{aligned}$$

These linear forms are linearly independent and with algebraic coefficients. For $j \geqslant 1$, define the quadruple

$$\begin{aligned} \underline{x}_j := (&q_{r_j-1}q_{r_j+s_j} - q_{r_j}q_{r_j+s_j-1}, q_{r_j-1}p_{r_j+s_j} - q_{r_j}p_{r_j+s_j-1},\\ &p_{r_j-1}q_{r_j+s_j} - p_{r_j}q_{r_j+s_j-1}, p_{r_j-1}p_{r_j+s_j} - p_{r_j}p_{r_j+s_j-1}), \end{aligned}$$

Set $\mathcal{N}_1 = \{k+h, \lfloor M_1/2 \rfloor \leqslant h \leqslant M_1\}$ and $\mathcal{P}_1 = \{\underline{x}_j, j \in \mathcal{N}_1\}$. Let j be in $\mathcal{N}_1$. Evaluating the linear forms above at the integer point $\underline{x}_j$, we infer from inequalities (10.8), (10.9) and (10.10) that

$$\begin{aligned} \prod_{1\leqslant h\leqslant 4} |L_h(\underline{z}_j)| &\ll |\det(L_1, L_2, L_3, L_4)|\, (q_{r_j}q_{r_j+s_j})^{-1/(15\kappa \log M)}\\ &< |\det(L_1, L_2, L_3, L_4)|\, (q_{r_j}q_{r_j+s_j})^{-1/(16\kappa \log M)}. \end{aligned}$$

Furthermore, as noted after the statement of Theorem E, all the elements of the set $\mathcal{P}_1$ are primitive and, by (10.1), the maximal absolute value of the entries of $\underline{x}_j$ exceeds $H(L_i)$ for $i = 1, \ldots, 4$ and $j \in \mathcal{N}_1$. Moreover, the coefficients of the linear forms $L_1, \ldots, L_4$ generate a number field of degree d. Let T_1 be the upper bound for the number of exceptional subspaces given by Theorem E applied with $m = 4$ and $\varepsilon = 1/(16\kappa \log M)$. Set

$$M_2 := \lfloor M_1/T_1 \rfloor.$$

Inequality (10.6) ensures the existence of a constant c_2 such that

$$M_2 > c_2(\log 3d)^4(\log\log 3d)^3. \tag{10.11}$$

By the *Schubfachprinzip*, there exists a proper subspace of $\mathbf{Q}^4$ containing at least M_2 points of $\mathcal{P}_1$. Thus, there exist a non-zero integer vector (z_1, z_2, z_3, z_4) and a set of integers $\mathcal{N}_2 \subset \mathcal{N}_1$ of cardinality M_2 such that

$$\begin{aligned} &z_1(q_{r_j-1}q_{r_j+s_j} - q_{r_j}q_{r_j+s_j-1}) + z_2(q_{r_j-1}p_{r_j+s_j} - q_{r_j}p_{r_j+s_j-1})\\ &\quad + z_3(p_{r_j-1}q_{r_j+s_j} - p_{r_j}q_{r_j+s_j-1}) + z_4(p_{r_j-1}p_{r_j+s_j} - p_{r_j}p_{r_j+s_j-1}) = 0, \end{aligned} \tag{10.12}$$

for every $j \in \mathcal{N}_2$. Let $l_1 < l_2 < \cdots < l_{M_2}$ denote the elements of $\mathcal{N}_2$ once ordered.

Let j be in $\mathcal{N}_2$ such that $r_j \geqslant 1$. Dividing (10.12) by $q_{r_j}q_{r_j+s_j-1}$ and setting

$$Q_j := \frac{q_{r_j-1}q_{r_j+s_j}}{q_{r_j}q_{r_j+s_j-1}},$$

we get

$$\begin{aligned} &z_1(Q_j - 1) + z_2\left(Q_j \frac{p_{r_j+s_j}}{q_{r_j+s_j}} - \frac{p_{r_j+s_j-1}}{q_{r_j+s_j-1}}\right)\\ &\quad + z_3\left(Q_j \frac{p_{r_j-1}}{q_{r_j-1}} - \frac{p_{r_j}}{q_{r_j}}\right) + z_4\left(Q_j \frac{p_{r_j-1}}{q_{r_j-1}}\frac{p_{r_j+s_j}}{q_{r_j+s_j}} - \frac{p_{r_j}}{q_{r_j}}\frac{p_{r_j+s_j-1}}{q_{r_j+s_j-1}}\right) = 0. \end{aligned}$$

Setting

$$Z = \max\{|z_1|, |z_2|, |z_3|, |z_4|\}$$

https://doi.org/10.1112/S0010437X11007524 Published online by Cambridge University Press

and using

$$\left|\alpha - \frac{p_h}{q_h}\right| \leqslant \frac{1}{q_h q_{h+1}},$$

for

$$h \in \{r_j - 1, r_j, r_j - 1 + s_j - 1, r_j + s_j\},$$

it then follows that

$$|(Q_j - 1)(z_1 + (z_2 + z_3)\alpha + z_4\alpha^2)| \ll Q_j Z q_{r_j}^{-1} q_{r_j-1}^{-1}.$$

Let A be an upper bound for the sequence $(a_\ell)_{\ell \geqslant 1}$. We deduce from the fact that a_{r_j} differs from $a_{r_j+s_j}$ that $Q_j/|Q_j - 1| \ll A^2$, thus

$$|z_1 + (z_2 + z_3)\alpha + z_4\alpha^2| \ll A^2 Z q_{r_j}^{-1} q_{r_j-1}^{-1}. \tag{10.13}$$

At this point, there is a difficulty to overcome which did not occur in [AB10b]. Indeed, r_j and $r_j - 1$ can both be small and (10.13) may not imply that $z_1 + (z_2 + z_3)\alpha + z_4\alpha^2$ is equal to 0.

As in [AB10b], we first have to distinguish two cases.

Assumption $\mathcal{A}$. There exist three integers $1 \leqslant a < b < c \leqslant \lfloor M_2/4 \rfloor$ such that the vectors $\underline{x}_{l_a}$, $\underline{x}_{l_b}$ and $\underline{x}_{l_c}$ are linearly independent.

As explained in [AB10b], if Assumption $\mathcal{A}$ is satisfied, then there exist $z_1, \ldots, z_4$ as above with

$$Z \leqslant 12(2q_{r_{l_c}} q_{r_{l_c}+s_{l_c}})^3. \tag{10.14}$$

Furthermore, Liouville's inequality asserts that

$$|z_1 + (z_2 + z_3)\alpha + z_4\alpha^2| \gg H(\alpha)^{-2} \cdot Z^{-d+1}, \tag{10.15}$$

if $z_1 + (z_2 + z_3)\alpha + z_4\alpha^2$ is non-zero. The combination of (10.13), (10.14) and (10.15) gives

$$q_{r_j} q_{r_j-1} \ll A^2 H(\alpha)^2 (q_{r_{l_c}} q_{r_{l_c}+s_{l_c}})^{3d}, \tag{10.16}$$

for j in $\mathcal{N}_2$, if $z_1 + (z_2 + z_3)\alpha + z_4\alpha^2$ is non-zero. Note that (10.16) trivially holds if $r_j = 0$, since $q_{-1} = 0$. There are two subcases to distinguish. For simplicity, put $\lambda = l_{\lceil M_2/2 \rceil}$.

If (10.16) is not satisfied for some j in $\mathcal{N}_2$ with $j \geqslant \lambda$, we get immediately that

$$z_1 + (z_2 + z_3)\alpha + z_4\alpha^2 = 0, \tag{10.17}$$

a case to which we will return later. Otherwise, we deduce from (10.4), (10.16) and (iii) of Lemma 9.1 that, for every j in $\mathcal{N}_2$ with $j \geqslant l_{\lfloor 3M_2/4 \rfloor}$,

$$\begin{aligned} q_{r_j} q_{r_j-1} &\ll A^2 H(\alpha)^2 (q_{r_\lambda} q_{r_\lambda+s_\lambda})^{(3d)\cdot 2^{-M_2/4}} \\ &\ll A^2 q_{r_{k+1}+s_{k+1}}^2 (q_{r_\lambda} q_{r_\lambda+s_\lambda})^{1/M_2}, \end{aligned}$$

thus $r_j \leqslant s_\lambda/(2\kappa)$, for every j in $\mathcal{N}_2$ with $j \geqslant l_{\lfloor 3M_2/4 \rfloor}$.

In this case, we replace ξ by a suitable real number equivalent to it in order to reach a situation where the results of [AB10b] are applicable. Set $u = \lceil s_\lambda/(2\kappa) \rceil$,

$$\tilde{\xi} := [0; a_{u+1}, a_{u+2}, \ldots] = \frac{p_u - q_u\xi}{q_{u-1}\xi - p_{u-1}},$$

and

$$\tilde{\alpha} := [0; a_{u+1}, a_{u+2}, \ldots] = \frac{p_u - q_u\alpha}{q_{u-1}\alpha - p_{u-1}}.$$

https://doi.org/10.1112/S0010437X11007524 Published online by Cambridge University Press

Let $j = \lfloor 3M_2/4 \rfloor, \ldots, M_2$. Let W_j be the prefix of the word $\mathbf{a}_u = a_{u+1}a_{u+2}\ldots$ of length s_{l_j}. Since $\mathbf{a}$ begins with a word of length r_{l_j} followed by a word of length s_{l_j} to the power $1+1/\kappa$, the word $W_j^{1+1/(2\kappa)}$ is also a prefix of $\mathbf{a}_u$. Denoting by $(\tilde{p}_\ell/\tilde{q}_\ell)_{\ell \geqslant 1}$ the sequence of convergents to $\tilde{\xi}$, we get from Lemma 6.2 that

$$|\tilde{q}_{s_{l_j}-1}\tilde{\xi}^2 - (p_{s_{l_j}-1} - q_{s_{l_j}})\tilde{\xi} - p_{s_{l_j}}| \ll \tilde{q}_{s_{l_j}}\ \tilde{q}^{-2}_{s_{l_j}+\lfloor s_{l_j}/(2\kappa)\rfloor}.$$

Since

$$|\tilde{\xi} - \tilde{\alpha}| \ll q_u^2|\xi - \alpha|,$$

we also get

$$|\tilde{q}_{s_{l_j}-1}\tilde{\alpha}^2 - (p_{s_{l_j}-1} - q_{s_{l_j}})\tilde{\alpha} - p_{s_{l_j}}| \ll \tilde{q}_{s_{l_j}}\ \tilde{q}^{-2}_{s_{l_j}+\lfloor s_{l_j}/(2\kappa)\rfloor}.$$

Since

$$M_2 > c_2(\log 3d)^3(\log\log 3d)^2,$$

we can now follow [AB10b, Proof on pp. 898–903] to reach a contradiction.

We return to the case where (10.17) holds. Since α is algebraic of degree at least three, we get that $z_1 = z_4 = 0$ and $z_2 = -z_3$. Then, z_2 is non-zero and, for any j in $\mathcal{N}_2$, the polynomial $P_j(X)$ can simply be expressed as

$$\begin{aligned} P_j(X) := {} & (q_{r_j-1}q_{r_j+s_j} - q_{r_j}q_{r_j+s_j-1})X^2 \\ & - 2(q_{r_j-1}p_{r_j+s_j} - q_{r_j}p_{r_j+s_j-1})X + (p_{r_j-1}p_{r_j+s_j} - p_{r_j}p_{r_j+s_j-1}). \end{aligned}$$

Consider now the three linearly independent linear forms

$$\begin{aligned} L'_1(X_1, X_2, X_3) &= \alpha^2 X_1 - 2\alpha X_2 + X_3, \\ L'_2(X_1, X_2, X_3) &= \alpha X_1 - X_2, \\ L'_3(X_1, X_2, X_3) &= X_1. \end{aligned}$$

Evaluating them on the triple

$$\underline{x}'_j := (q_{r_j-1}q_{r_j+s_j} - q_{r_j}q_{r_j+s_j-1}, q_{r_j-1}p_{r_j+s_j} - q_{r_j}p_{r_j+s_j-1}, p_{r_j-1}p_{r_j+s_j} - p_{r_j}p_{r_j+s_j-1}),$$

it follows from (10.8), (10.9) and (10.10) that

$$\prod_{1\leqslant h\leqslant 3} |L'_h(\underline{x}'_j)| < |\det(L'_1, L'_2, L'_3)|\,(q_{r_j}q_{r_j+s_j})^{-1/(16\kappa\log M)},$$

by arguing as above.

Applying Theorem E, we get that the points $\underline{x}'_j$ lie in a finite number of proper subspaces of $\mathbf{Q}^3$. Thus, by (10.11), there exist a non-zero integer triple (z'_1, z'_2, z'_3) and an infinite set of distinct positive integers $\mathcal{N}_3$ included in $\mathcal{N}_2$, of cardinality

$$M_3 > c_3(\log 3d)^3(\log\log 3d)^2, \tag{10.18}$$

and such that

$$\begin{aligned} & z'_1(q_{r_j-1}q_{r_j+s_j} - q_{r_j}q_{r_j+s_j-1}) + z'_2(q_{r_j-1}p_{r_j+s_j} - q_{r_j}p_{r_j+s_j-1}) \\ & \quad + z'_3(p_{r_j-1}p_{r_j+s_j} - p_{r_j}p_{r_j+s_j-1}) = 0, \end{aligned} \tag{10.19}$$

for any j in $\mathcal{N}_3$.

https://doi.org/10.1112/S0010437X11007524 Published online by Cambridge University Press

Let j be in $\mathcal{N}_3$ with $r_j \geqslant 1$. Divide (10.19) by $q_{r_j} q_{r_j+s_j-1}$. This gives

$$z_1'(Q_j - 1) + z_2'\left(Q_j \frac{p_{r_j+s_j}}{q_{r_j+s_j}} - \frac{p_{r_j+s_j-1}}{q_{r_j+s_j-1}}\right) + z_3'\left(Q_j \frac{p_{r_j-1}}{q_{r_j-1}} \frac{p_{r_j+s_j}}{q_{r_j+s_j}} - \frac{p_{r_j}}{q_{r_j}} \frac{p_{r_j+s_j-1}}{q_{r_j+s_j-1}}\right) = 0.$$

It then follows that

$$|z_1' + z_2'\alpha + z_3'\alpha^2| \ll A^2 Z' Q_j q_{r_j}^{-1} q_{r_j-1}^{-1},$$

where $Z' := \max\{|z_1'|, |z_2'|, |z_3'|\}$. Since every $\underline{x}_j'$ is primitive, the vectors $\underline{x}_j'$ and $\underline{x}_h'$ are not collinear for distinct indices j, h in $\mathcal{N}_3$. This allows us to bound Z'. As above, we get that either

$$z_1' + z_2'\alpha + z_3'\alpha^2 = 0,$$

in which case we get a contradiction since α is algebraic of degree at least three, or r_j is small for every sufficiently large j in $\mathcal{N}_3$. In the latter case, we introduce as above a number $\tilde{\xi}$ equivalent to ξ and, using (10.18), we get a contradiction in a similar way.

To conclude, it remains for us to treat the case where *Assumption* $\mathcal{A}$ is not satisfied. Set $\mathcal{N}_4 := \{\lfloor M_2/8 \rfloor, \ldots, \lfloor M_2/4 \rfloor\}$. Observe that the vectors $\underline{x}_{l_j}$, with j in $\mathcal{N}_4$, belong to the subspace generated by $\underline{x}_{l_1}$ and $\underline{x}_{l_2}$. Arguing as in [AB10b, p. 903], there exists a non-zero integer vector (z_1'', z_2'', z_4'') such that

$$Z'' := \max\{|z_1''|, |z_2''|, |z_4''|\} \leqslant 2 q_{r_{l_2}} q_{r_{l_2}+s_{l_2}} \tag{10.20}$$

and

$$\begin{aligned} &z_1''(q_{r_j-1} q_{r_j+s_j} - q_{r_j} q_{r_j+s_j-1}) + z_2''(q_{r_j-1} p_{r_j+s_j} - q_{r_j} p_{r_j+s_j-1}) \\ &\quad + z_4''(p_{r_j-1} p_{r_j+s_j} - p_{r_j} p_{r_j+s_j-1}) = 0, \end{aligned}$$

for every $j \in \mathcal{N}_4$. Proceeding exactly as in the proof of (10.13), we obtain

$$|z_1'' + z_2''\alpha + z_4''\alpha^2| \ll A^2 Z'' q_{r_j-1}^{-1} q_{r_j}^{-1}, \tag{10.21}$$

for every $j \in \mathcal{N}_4$ with $r_j \geqslant 1$. Since α is of degree at least three, Liouville's inequality gives us that

$$|z_1'' + z_2''\alpha + z_4''\alpha^2| \gg H(\alpha)^{-2} \cdot (Z'')^{-d+1},$$

thus, by (10.20) and (10.21),

$$q_{r_j} \ll A^2 H(\alpha)^2 (q_{r_{l_2}} q_{r_{l_2}+s_{l_2}})^d,$$

for every $j \in \mathcal{N}_4$ (recall that $q_{-1} = 0$). This proves that r_j is small compared to s_{l_2} and we apply the same trick as above, where we have introduced the real numbers $\tilde{\xi}$ and $\tilde{\alpha}$. Since there is no specific difficulty, we omit the details. □

11. Proofs

Proof of (2.1) and Theorem 2.3. Let $\mathbf{a} = (a_\ell)_{\ell \geqslant 1}$ be an infinite sequence of positive integers. Set

$$\xi = [0; a_1, a_2, \ldots, a_\ell, \ldots],$$

and denote by $(p_\ell/q_\ell)_{\ell \geqslant 1}$ the sequence of convergents to ξ. Let m and M be real numbers greater than 1 such that

$$m^\ell < q_\ell < M^\ell,$$

for every integer ℓ greater than some integer ℓ_0. Assume that $\mathrm{Dio}(\xi)$ exceeds 1 and let $\delta > 1$ be a real number less than $\mathrm{Dio}(\xi)$. For every positive integer j, there exist finite words U_j and V_j

https://doi.org/10.1112/S0010437X11007524 Published online by Cambridge University Press

and a real number w_j such that $\mathbf{a}$ begins with the word $U_jV_j^{w_j}$ and $|U_jV_j^{w_j}|/|U_jV_j| \geqslant \delta$. Set $\alpha_j = [0; U_j, \overline{V_j}]$ and denote by r_j and s_j the lengths of the words U_j and V_j, respectively. Assume first that $(r_j)_{j\geqslant 1}$ is strictly increasing and that r_1 exceeds ℓ_0. By Lemma 6.2, α_j is a root of an integer polynomial of height less than $2q_{r_j}q_{r_j+s_j}$, thus

$$H(\alpha_j) \leqslant 2M^{2r_j+s_j}.$$

Since the first $r_j + \lfloor w_j s_j \rfloor$ partial quotients of ξ and α_j are the same, we get

$$\begin{aligned} |\xi - \alpha_j| < q_{r_j+\lfloor w_j s_j \rfloor}^{-2} &\leqslant (H(\alpha_j)/2)^{-2(r_j+w_js_j-1)(\log m)/((2r_j+s_j)(\log M))} \\ &\leqslant (H(\alpha_j)/2)^{-(r_j+w_js_j-1)(\log m)/((r_j+s_j)(\log M))}. \end{aligned} \tag{11.1}$$

We then deduce that

$$w_2^*(\xi) \geqslant \frac{\log m}{\log M}\ \mathrm{Dio}(\xi) - 1.$$

When the sequence $(q_\ell^{1/\ell})_{\ell\geqslant 1}$ converges, m and M can be taken arbitrarily close and we obtain the lower bound (2.1). The case when the sequence $(r_j)_{j\geqslant 1}$ is bounded is easier, and we omit it.

In (11.1), we have used the trivial upper bound $2r_j + s_j \leqslant 2(r_j + s_j)$ which is sharp when r_j is large compare to s_j, but very weak otherwise. Assuming now that the sequence $\mathbf{a}$ is bounded by A, it follows from Lemma 6.1 that

$$\begin{aligned} |\xi - \alpha_j| \cdot |\xi - \alpha_j'| &\ll A^3\, q_{r_j+\lfloor w_j s_j \rfloor}^{-2} \cdot q_{r_j}^{-2} \\ &\ll A^3\, (H(\alpha_j)/2)^{-2(2r_j+w_js_j-1)(\log m)/((2r_j+s_j)(\log M))}. \end{aligned}$$

Since

$$\begin{aligned} \frac{2r_j + w_j s_j}{2r_j + s_j} &\geqslant \frac{r_j + \delta(r_j + s_j)}{2r_j + s_j} \\ &\geqslant \frac{r_j + s_j/2 + \delta(r_j + s_j/2)}{2r_j + s_j} \geqslant \frac{1+\delta}{2}, \end{aligned}$$

we deduce that

$$w_2(\xi) \geqslant \frac{\log m}{\log M}\ (1 + \mathrm{Dio}(\xi)) - 1.$$

In particular, we have shown that $w_2(\xi) \geqslant \mathrm{Dio}(\xi)$ if the sequence $(q_\ell^{1/\ell})_{\ell\geqslant 1}$ converges. □

Proof of Theorem 2.2. Let ξ be as in the statement of the theorem. Since $p(1, \mathbf{a})$ is finite, the sequence $\mathbf{a}$ is bounded, thus ξ is a badly approximable number. In particular, it satisfies $w_1(\xi) = 1$. If $\mathrm{Dio}(\xi)$ is infinite, then $w_2^*(\xi)$ is also infinite, by (2.1). Consequently, we assume that $\mathrm{Dio}(\xi)$ is finite. It follows from the *Schubfachprinzip* that we are in position to apply Lemma 9.1. By Lemma 6.4, the quantity M occurring in its statement can be taken equal to $A+1$. Thus, Lemma 9.1 gives a sequence $(\alpha_j)_{j\geqslant 1}$ of quadratic numbers and a sequence $(Q_j)_{j\geqslant 1}$ of integers such that the assumptions of Lemma 7.2 are satisfied with

$$C = 650\kappa^2(\log(A+1))^2, \quad \varepsilon = 1/(15\kappa \log(A+1)),$$

and

$$w = 6\,\mathrm{Dio}(\xi) \log(A+1).$$

It then follows from (1.3) and Lemma 7.2 that

$$w_2^*(\xi) \leqslant w_2(\xi) \leqslant 6\,\mathrm{Dio}(\xi) \log(A+1)(1 + 19\,500\kappa^3(\log(A+1))^3).$$

https://doi.org/10.1112/S0010437X11007524 Published online by Cambridge University Press

This finishes the proof of (2.3). Furthermore, the discussion following the proof of Lemma 9.1 shows that, if the sequence $(q_\ell^{1/\ell})_{\ell\geqslant 1}$ converges, then one can apply Lemma 7.2 with

$$C = 650\kappa^2, \quad \varepsilon = 1/(15\kappa) \quad \text{and} \quad w = 6\,\mathrm{Dio}(\xi).$$

This proves (2.4). □

Proof of Theorem 3.3. Set $\mathbf{a} = a_1a_2\ldots$ Since ℓ' is positive and $\lambda_k \geqslant 2$ for every sufficiently large integer k, there exists an integer κ such that, for every sufficiently large integer n, there exists a word of length n having two occurrences in the prefix of $\mathbf{a}$ of length $(\kappa+1)n$. This is precisely the assumption needed to apply Lemma 9.1. Furthermore, the Diophantine exponent of ξ is equal to

$$1 + \limsup_{k\to+\infty} \frac{(\lambda_k - 1)r_k}{\lambda_1 r_1 + \cdots + \lambda_{k-1}r_{k-1} + r_k}.$$

Since λ_k tends to infinity with k, the Diophantine exponent of ξ is finite if, and only if, L' is finite. We then apply Lemma 9.1 and follow the proof of Theorem 3.2. □

Proof of Theorem 4.1. Although the numerical constants implied in $\ll$ depend on b and c, we write $\ll$ instead of $\ll_{b,c}$.

Let $j \geqslant 2$ be an integer. Define the quadratic number

$$\xi_{w,j} := [0; a_{1,w}, \ldots, a_{\lfloor w^j\rfloor,w}, \overline{b}]$$

and denote by $(p_\ell/q_\ell)_{\ell\geqslant 1}$ the sequence of its convergents. Set $\gamma = (b + \sqrt{b^2+4})/2$. There exists an integer M such that

$$\gamma^\ell \ll q_\ell \ll \ell^M \gamma^\ell, \quad \ell \geqslant 1.$$

Clearly, we have

$$|\xi_w - \xi_{w,j}| \asymp q^{-2}_{\lfloor w^{j+1}\rfloor}$$

and, by Lemmas 6.1 and 6.3,

$$|\xi_{w,j} - \xi'_{w,j}|^{-1} \asymp q^2_{\lfloor w^j\rfloor} \asymp H(\xi_{w,j}),$$

where $\xi'_{w,j}$ denotes the Galois conjugate of $\xi_{w,j}$. We conclude by Lemma 7.3 that

$$w_2^*(\xi) + 1 = w, \quad w_2(\xi) = w,$$

provided that $(w-1)(w-2) \geqslant 2w$, that is, provided that $w \geqslant (5+\sqrt{17})/2$. □

Proof of Theorem 4.3. Although the numerical constants implied in $\ll$ depend on b, c and d, we write $\ll$ instead of $\ll_{b,c,d}$.

Let $j \geqslant 2$ be an integer. Define the quadratic number

$$\xi_{w,\eta,j} := [0; a_{1,w,\eta}, \ldots, a_{\lfloor w^j\rfloor,w,\eta}, \overline{b, b, \ldots, b, d}]$$

and denote by $(p_\ell/q_\ell)_{\ell\geqslant 1}$ the sequence of its convergents. Set $\gamma = (b + \sqrt{b^2+4})/2$. There exists an integer M such that

$$\gamma^\ell \ll q_\ell \ll \ell^M \gamma^\ell, \quad \ell \geqslant 1.$$

Since

$$\lfloor w^j\rfloor + (m_j+1)\lfloor \eta w^j\rfloor > \lfloor w^{j+1}\rfloor - 3,$$

we have

$$|\xi_w - \xi_{w,\eta,j}| \asymp q^{-2}_{\lfloor w^{j+1}\rfloor}.$$

https://doi.org/10.1112/S0010437X11007524 Published online by Cambridge University Press

It follows from Lemma 6.3 that

$$H(\xi_{w,\eta,j}) \ll (2w^j)^{2M}\gamma^{w^j(2+\eta)},$$

and from Lemma 6.1 that

$$|\xi_{w,\eta,j} - \xi'_{w,\eta,j}| \asymp q_{\lfloor w^j \rfloor}^{-2},$$

where $\xi'_{w,\eta,j}$ denotes the Galois conjugate of $\xi_{w,\eta,j}$. Thus, assuming that

$$\left(\frac{2w}{2+\eta} - 2 + \frac{2}{2+\eta}\right)\left(\frac{2w}{2+\eta} - 3 + \frac{2}{2+\eta}\right) \geqslant 2w\left(2 - \frac{2}{2+\eta}\right), \tag{11.2}$$

we deduce from Lemma 7.3 that

$$w_2^*(\xi) = -1 + 2w/(2+\eta) = (2w-2-\eta)/(2+\eta)$$

and

$$w_2(\xi) = (2w-\eta)/(2+\eta).$$

In particular,

$$w_2(\xi) - w_2^*(\xi) = 2/(2+\eta).$$

Since $\eta \leqslant \sqrt{w}/4$, inequality (11.2) holds for $w \geqslant 16$, this concludes the proof of the theorem. □

12. An extension of Theorem 3.2

Theorems 2.2 and 3.2 apply to continued fractions with bounded partial quotients. However, their proofs are flexible enough to extend to a wider class of continued fractions with unbounded partial quotients. The methods of the present paper allow us to establish the following more general result.

Theorem 12.1. *Let $\mathbf{a} = (a_\ell)_{\ell \geqslant 1}$ be an infinite sequence of positive integers and set*

$$\xi := [0; a_1, a_2, \ldots, a_\ell, \ldots].$$

Let $(p_\ell/q_\ell)_{\ell \geqslant 1}$ be the sequence of convergents to ξ and assume that $(q_\ell^{1/\ell})_{\ell \geqslant 1}$ is bounded. Assume, furthermore, that there is a positive integer κ such that, for every sufficiently large integer n, there is a word of length n having two occurrences in the prefix of length $(\kappa+1)n$ of $\mathbf{a}$. If $\mathrm{Dio}(\xi)$ is finite, then ξ is either an S-number or a T-number; otherwise, ξ is either quadratic or a U_2-number. Moreover, if $\mathrm{Dio}(\xi)$ is finite, then there exists a constant c, depending only on ξ, such that

$$w_d(\xi) \leqslant \exp(c(\log 3d)^5(\log\log 3d)^4) \quad \textit{for } d \geqslant 1.$$

By Theorem 12.1, the conclusion of Theorem 3.3 still holds if the sequence $(a_n)_{n \geqslant 1}$ is unbounded, provided that $(q_\ell^{1/\ell})_{\ell \geqslant 1}$ is bounded, where q_ℓ denotes the denominator of the ℓth convergent to ξ.

Retaining the notation and the assumption of Theorem 12.1, we first show that ξ is not a Liouville number. Indeed, since $(q_\ell^{1/\ell})_{\ell \geqslant 1}$ is bounded, there exists M such that $q_\ell \leqslant M^\ell$ for $\ell \geqslant 1$. Since, by Lemma 6.4, $q_\ell \geqslant 2^{(\ell-1)/2}$ for $\ell \geqslant 1$, we deduce that

$$q_\ell \leqslant M^2 \cdot (M^2)^{(\ell-2)/2} \leqslant M^2 \cdot q_{\ell-1}^{(\log M^2)/(\log 2)},$$

for $\ell \geqslant 2$. Putting $v = 2(\log M)/(\log 2)$, this can be rewritten as

$$q_\ell \ll q_{\ell-1}^v \quad \text{for } \ell \geqslant 2, \tag{12.1}$$

https://doi.org/10.1112/S0010437X11007524 Published online by Cambridge University Press

where $\ll$ means, until the end of this section, that the implied constant depends on M. The upper bound

$$w_1(\xi) \leqslant (\log M^2)/(\log 2)$$

is an immediate consequence of (12.1).

The strategy to prove Theorem 12.1 is to follow the proofs of Lemma 9.1, Theorems 2.2 and 3.2 and to modify accordingly the few steps where we have used the fact that the sequence $(a_\ell)_{\ell \geqslant 1}$ is bounded. This involves no great difficulty, but some technical complications.

First, we note that the first part of Lemma 9.1 can be proved with the upper bound for q_j given by (9.5), even if the sequence $\mathbf{a}$ is unbounded. The boundedness of $\mathbf{a}$ is used only in (9.10) and (9.12). By condition (ii) of the lemma, α_j and its Galois conjugate may not be very close to each other when $a_{r_j} = a_{r_j+s_j} \pm 1$ and a_{r_j-1} and $a_{r_j+s_j-1}$ are both large. But, in that case, the height of the polynomial $P_{\xi,r_j,s_j}(X)$ is indeed much smaller than $2q_{r_j}q_{r_j+s_j}$. Thus, instead of emphasizing the quantity $2q_rq_{r+s}$ which bounds the height of $P_{\xi,r,s}(X)$, it is better to work with the refined upper bound

$$H_{r,s} := 3q_rq_{r+s} \cdot |q_{r-1}/q_r - q_{r+s-1}/q_{r+s}|,$$

having noticed that the four numbers

$$\begin{aligned} &|q_{r-1}/q_r - q_{r+s-1}/q_{r+s}|, \quad |p_{r-1}/p_r - q_{r+s-1}/q_{r+s}|, \\ &|q_{r-1}/q_r - p_{r+s-1}/p_{r+s}|, \quad |p_{r-1}/p_r - p_{r+s-1}/p_{r+s}| \end{aligned}$$

are very close to each other.

To evaluate $H_{r,s}$, we argue as in the proof of Lemma 6.1 to get

$$\begin{aligned} H_{r,s} &\gg q_rq_{r+s}|q_{r-1}/q_r - q_{r+s-1}/q_{r+s}| \gg \frac{q_rq_{r+s}}{a_{r+s}a_r \max\{a_{r+s-1}, a_{r-1}\}} \\ &\gg (q_rq_{r+s})^{1/v^2}, \end{aligned}$$

with v as in (12.1). This shows that one can get the analogue of (iii). To obtain the analogue of (9.4), one uses Lemma 5.5 of [AB10b].

In the last part of the proof in § 10, we have to bound from above the quantity $Q_j/|Q_j - 1|$. Arguing as in the proof of Lemma 6.1, we show that

$$Q_j/|Q_j - 1| \ll q_{r_j},$$

if $r_j \geqslant 1$. Instead of (10.13), we obtain the inequality

$$|z_1 + (z_2 + z_3)\alpha + z_4\alpha^2| \ll Zq_{r_j-1}^{-1},$$

and (10.16) is replaced by

$$q_{r_j-1} \ll H(\alpha)^2(q_{r_{l_c}}q_{r_{l_c}+s_{l_c}})^{3d},$$

if $r_j \geqslant 0$. We then proceed exactly as in § 10 to reach a contradiction.

Acknowledgement

I am very much indebted to the referee, who pointed out several obscure points in the first version of the present text.

References

AB05 B. Adamczewski and Y. Bugeaud, *On the complexity of algebraic numbers, II. Continued fractions*, Acta Math. **195** (2005), 1–20.

AB06 B. Adamczewski and Y. Bugeaud, *On the Littlewood conjecture in simultaneous Diophantine approximation*, J. Lond. Math. Soc. (2) **73** (2006), 355–366.

AB07a B. Adamczewski and Y. Bugeaud, *Dynamics for β-shifts and Diophantine approximation*, Ergodic Theory Dynam. Systems **27** (2007), 1695–1711.

AB07b B. Adamczewski and Y. Bugeaud, *On the complexity of algebraic numbers I. Expansions in integer bases*, Ann. of Math. (2) **165** (2007), 547–565.

AB10a B. Adamczewski and Y. Bugeaud, *Mesures de transcendance et aspects quantitatifs de la méthode de Thue–Siegel–Roth–Schmidt*, Proc. Lond. Math. Soc. (3) **101** (2010), 1–31.

AB10b B. Adamczewski and Y. Bugeaud, *Transcendence measures for continued fractions involving repetitive or symmetric patterns*, J. Eur. Math. Soc. **12** (2010), 883–914.

AB11 B. Adamczewski and Y. Bugeaud, *Nombres réels de complexité sous-linéaire: mesures d'irrationalité et de transcendance*, J. Reine Angew. Math. **658** (2011), 65–98.

ABL04 B. Adamczewski, Y. Bugeaud and F. Luca, *Sur la complexité des nombres algébriques*, C. R. Acad. Sci. Paris **339** (2004), 11–14.

AC06 B. Adamczewski and J. Cassaigne, *Diophantine properties of real numbers generated by finite automata*, Compositio Math. **142** (2006), 1351–1372.

AR09 B. Adamczewski and T. Rivoal, *Irrationality measures for some automatic real numbers*, Math. Proc. Cambridge Philos. Soc. **147** (2009), 659–678.

ADQZ01 J.-P. Allouche, J. L. Davison, M. Queffélec and L. Q. Zamboni, *Transcendence of Sturmian or morphic continued fractions*, J. Number Theory **91** (2001), 39–66.

AS03 J.-P. Allouche and J. Shallit, *Automatic sequences: theory, applications, generalizations* (Cambridge University Press, Cambridge, 2003).

Bak64 A. Baker, *On Mahler's classification of transcendental numbers*, Acta Math. **111** (1964), 97–120.

Bak76 R. C. Baker, *On approximation with algebraic numbers of bounded degree*, Mathematika **23** (1976), 18–31.

Bug03 Y. Bugeaud, *Mahler's classification of numbers compared with Koksma's*, Acta Arith. **110** (2003), 89–105.

Bug04a Y. Bugeaud, *Approximation by algebraic numbers*, Cambridge Tracts in Mathematics, vol. 160 (Cambridge University Press, Cambridge, 2004).

Bug04 Y. Bugeaud, *Mahler's classification of numbers compared with Koksma's, III*, Publ. Math. Debrecen **65** (2004), 305–316.

Bug08 Y. Bugeaud, *Diophantine approximation and Cantor sets*, Math. Ann. **341** (2008), 677–684.

Bug Y. Bugeaud, *On the rational approximation to the Thue–Morse–Mahler numbers*, Ann. Inst. Fourier., to appear.

Bug10 Y. Bugeaud, *Continued fractions of transcendental numbers*, Preprint (2010), arXiv:1012.1709.

BKS11 Y. Bugeaud, D. Krieger and J. Shallit, *Morphic and automatic words: maximal blocks and diophantine approximation*, Acta Arith. **149** (2011), 181–199.

BL05 Y. Bugeaud and M. Laurent, *Exponents of Diophantine approximation and Sturmian continued fractions*, Ann. Inst. Fourier (Grenoble) **55** (2005), 773–804.

Cob68 A. Cobham, *On the Hartmanis–Stearns problem for a class of tag machines*, in *Conference record of 1968 ninth annual symposium on switching and automata theory*, Schenectady, NY, 1968, 51–60.

Cob72 A. Cobham, *Uniform tag sequences*, Math. Systems Theory **6** (1972), 164–192.

https://doi.org/10.1112/S0010437X11007524 Published online by Cambridge University Press

DS67 H. Davenport and W. M. Schmidt, *Approximation to real numbers by quadratic irrationals*, Acta Arith. **13** (1967), 169–176.

Eil74 S. Eilenberg, *Automata, languages, and machines, Vol. A*, Pure and Applied Mathematics, vol. 58 (Academic Press, New York, 1974).

Eve96 J.-H. Evertse, *An improvement of the quantitative subspace theorem*, Compositio Math. **101** (1996), 225–311.

Kok39 J. F. Koksma, *Über die Mahlersche Klasseneinteilung der transzendenten Zahlen und die Approximation komplexer Zahlen durch algebraische Zahlen*, Monatsh. Math. Phys. **48** (1939), 176–189.

Mah32 K. Mahler, *Zur Approximation der Exponentialfunktionen und des Logarithmus. I, II*, J. Reine Angew. Math. **166** (1932), 118–150.

MH38 M. Morse and G. A. Hedlund, *Symbolic dynamics*, Amer. J. Math. **60** (1938), 815–866.

MH40 M. Morse and G. A. Hedlund, *Symbolic dynamics II*, Amer. J. Math. **62** (1940), 1–42.

Mos92 B. Mossé, *Puissances de mots et reconnaissabilité des points fixes d'une substitution*, Theoret. Comput. Sci. **99** (1992), 327–334.

Per29 O. Perron, *Die Lehre von den Ketterbrüchen* (Teubner, Leipzig, 1929).

Que98 M. Queffélec, *Transcendance des fractions continues de Thue–Morse*, J. Number Theory **73** (1998), 201–211.

Que00 M. Queffélec, *Irrational numbers with automaton-generated continued fraction expansion*, in *Dynamical systems (Luminy-Marseille, 1998)* (World Scientific Publishing, River Edge, NJ, 2000), 190–198.

Sch67 W. M. Schmidt, *On simultaneous approximations of two algebraic numbers by rationals*, Acta Math. **119** (1967), 27–50.

Sch71 W. M. Schmidt, *Mahler's T-numbers*, in *1969 Number Theory Institute – Proceedings of Symposia in Pure Mathematics, Vol. XX, State Univ. New York, Stony Brook, NY, 1969* (American Mathematical Society, Providence, RI, 1971), 275–286.

Yann Bugeaud bugeaud@math.unistra.fr
Mathématiques, Université de Strasbourg, 7, rue René Descartes,
67084 Strasbourg, France