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A QUESTION OF VALDIVIA ON QUASINORMABLE FRÉCHET SPACES

JOSÉ BONET

ABSTRACT. It is proved that a Fréchet space is quasinormable if and only if every null sequence in the strong dual converges equicontinuously to the origin. This answers positively a question raised by Valdivia. As a consequence a positive answer to a problem of Jarchow on Fréchet Schwartz spaces is obtained.

The class of quasinormable Fréchet spaces was studied by Grothendieck in [2] as a class "containing the most usual Fréchet functions spaces" (cf. [2, p. 107]). This class received recently much attention in the context of the structure theory of Fréchet spaces and Köthe echelon spaces (see [1,6,8,9,10]). Valdivia in 1981 [8] asked if every separable Fréchet space such that its strong dual verifies the Mackey convergence condition is quasinormable. This question was also collected in the problem list of [7, problem 13.5.1]. Here we present a positive answer to this problem, even without the assumption of the separability of the Fréchet space.

Let F be a Fréchet space with an increasing fundamental sequence of seminorms $(\|\,.\,\|_n)_{n\in\mathbb{N}}$ such that $U_n := \{x \in F;\ \|x\|_n \leq 1\}$ $(n \in \mathbb{N})$ form a basis of 0-neighbourhoods in F. The system of all closed absolutely convex bounded subsets of F is denoted by $\mathcal{B}(F)$. The dual seminorms are defined by $\|u\|_n^* := \sup\{\,|\langle u,x\rangle\,|;\ x \in U_n\}$, if $u \in F'$. We denote by $F_n' := \{u \in F';\ \|u\|_n^* < \infty\}$ the linear span of U_n° endowed with the normed topology defined by $\|\,.\,\|_n^*$. The symbols F_b' and F_i' stand for the strong and the inductive dual of F respectively, i.e., $F_i' := \operatorname{ind} F_n'$ is the bornological space associated with F_b'. According to Grothendieck [2], we say that F_b' satisfies the Mackey convergence condition if every null sequence in F_b' is contained in some F_n' and converges to the origin in F_n'. The quasinormable spaces were introduced by Grothendieck [2]. The Fréchet F is called quasinormable if the following condition holds:

$$(QN) \qquad \forall n \quad \exists m > n \quad \forall \varepsilon > 0 \quad \exists B \in \mathcal{B}(F) : U_m \subset B + \varepsilon U_n.$$

The positive solution to Valdivia's problem is contained in the following theorem.

THEOREM. *Let F be a Fréchet space. The following conditions are equivalent:*

(1) F is quasinormable.

(2) $\forall n \quad \exists m > n \quad \forall k > m \quad \forall \varepsilon > 0 \quad \exists \lambda > 0 : U_m \subset \lambda U_k + \varepsilon U_n$ *(cf. [6])*

(3) F_b' *satisfies the Mackey convergence condition.*

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(4) *$F'_i = \operatorname{ind} F'_n$ is a sequentially retractive inductive limit (i.e., every null sequence in F'_i is contained in some F'_n and converges to the origin in F'_n).*

PROOF. It is a direct matter to check that (1) implies (2). The fact that (1) implies (3) follows from the original definition of quasinormable Fréchet spaces (cf. [2]). Conditions (3) and (4) are equivalent since F'_b and F'_i have the same convergent sequences. Indeed, let $(x_j)_{j\in\mathbb{N}}$ be a null sequence in F'_b and let L denote the linear span of this sequence. By [7,8.2.18], F'_b and $\big(F', \beta(F', F'')\big)$ induce the same topology on L. The conclusion follows since $F'_i = \big(F', \beta(F', F'')\big)$ (see e.g. [4;29,4(2)]).

We prove now that (4) implies (2). If $F'_i = \operatorname{ind} F'_n$ is sequentially retractive, we can apply a theorem of Neus to conclude that it is even strongly boundedly retractive (see e.g. [9, p. 169] or [7,8.5.48]). This means precisely

$$\forall n \quad \exists m > n : F'_i \text{ and } F'_m \text{ induce the same topology on } U_n^\circ.$$

This implies at once

$$\forall n \quad \exists m > n \quad \forall k > m : F'_k \text{ and } F'_m \text{ induce the same topology on } U_n^\circ,$$

or equivalently

$$\forall n \quad \exists m > n \quad \forall k > m \quad \forall \alpha > 0 \quad \exists \beta > 0 : \beta U_k^\circ \cap U_n^\circ \subset \alpha U_m^\circ.$$

Taking polars in F and using the bipolar theorem, it is easy to see that this implies (2).

Now it is a direct matter to check that condition (2) is equivalent to the fact that F satisfies the property (Ω_φ) of Vogt and Wagner (see [6] and [11]) for some strictly increasing function $\varphi\colon (0,\infty) \longrightarrow (0,\infty)$. By [6, Theorem 7], this implies that F is quasinormable. The proof is already complete, but, since the proof of [6, Theorem 7] is rather involved, we present now a simple and direct proof of (2) implies (1) by use of a Mittag-Leffler procedure.

Without loss of generality, we may assume that $m = n+1$ in (2). Our assumption may be then formulated as follows

$$(*) \qquad \forall n \quad \forall k \quad \forall \varepsilon > 0 \quad \exists \lambda > 0 : U_{n+1} \subset \lambda U_k + \varepsilon U_n.$$

To prove that condition (QN) is satisfied we only do it for the first neighbourhood in the basis. For simplicity in the notation we call it $U_\circ$. We fix $n = 0$ and $\varepsilon > 0$. By (*) for "n"$= 0$, "k"$= 2$, "ε"$:= \varepsilon/2$, we have $U_1 \subset \lambda_1 U_2 + (\varepsilon/2) U_\circ$. Applying (*) to "$n$"$:= 1$, "$k$"$:= 3$, "$\varepsilon$"$:= \varepsilon/(\lambda_1 2^2)$ we get $U_2 \subset \lambda'_2 U_3 + (\varepsilon/\lambda_1 2^2) U_1$, hence $\lambda_1 U_2 \subset \lambda_2 U_3 + (\varepsilon/2^2) U_1$ with $\lambda_2 := \lambda_1 \lambda'_2$.

Proceeding by recurrence we determine $(\lambda_k)_{k\in\mathbb{N}}$, $\lambda_\circ := 1$, such that

$$(**) \qquad \forall k \quad \lambda_{k-1} U_k \subset \lambda_k U_{k+1} + \varepsilon 2^{-k} U_{k-1}.$$

Fix $z \in U_1$. We have $z = \lambda_1 u_2 + \varepsilon 2^{-1} v_1$, where $u_2 \in U_2$ and $v_1 \in U_\circ$. If $k \in \mathbb{N}$, we have, from (**), $\lambda_{k-1} u_k = \lambda_k u_{k+1} + \varepsilon 2^{-k} v_k$, $u_{k+1} \in U_{k+1}$ and $v_k \in U_{k-1}$. Since F is a

Fréchet space and $v_k \in U_{k-1}$, the series $\sum_{k=1}^{\infty} \varepsilon 2^{-k} v_k$ converges to an element x of F which belongs to $\varepsilon U_\circ$. The set $B := \bigcap_{k\in\mathbb{N}} (\lambda_k + \varepsilon) U_k$ is bounded in F (and independent of z). We prove that $z - x \in B$. Indeed, fix $k \in \mathbb{N}$,

$$z - x = \Big(z - \sum_{j=1}^{k} \varepsilon 2^{-j} v_j\Big) - \sum_{j=k+1}^{\infty} \varepsilon 2^{-j} v_j = \lambda_k u_{k+1}$$

$$- \sum_{j=k+1}^{\infty} \varepsilon 2^{-j} v_j \in \lambda_k U_{k+1} + \varepsilon 2^{-k} U_k \subset (\lambda_k + \varepsilon) U_k.$$

Consequently, $\forall \varepsilon > 0\ \exists B \in \mathcal{B}(F) : U_1 \subset B + \varepsilon U_\circ$. The proof is complete. ∎

REMARK. Let E be a (DF)-space with a fundamental sequence of bounded sets $(B_n)_{n\in\mathbb{N}}$. We consider the following two conditions on E.

(a) $\forall n\ \exists m > n\ \forall \alpha > 0\ \exists$ a 0-neighbourhood U in $E : B_n \cap U \subset \alpha B_m$.

(b) $\forall n\ \exists m > n\ \forall k\ \forall \alpha > 0\ \exists \beta > 0 : B_n \cap \beta B_k \subset \alpha B_m$.

Property (a) is precisely the strict Mackey condition introduced by Grothendieck in [2]. Property (b) means exactly that the inductive limit ind E_{B_n} satisfies the condition (M) of Retakh (see e.g. [8, p. 164]). Clearly condition (a) implies condition (b). The converse implication holds if E is the strong dual of a Fréchet space according to our previous theorem, or if E is bornological (i.e., if $E =$ ind E_{B_n} holds topologically) by a result of Retakh (see [9, p. 164(2)]). In general (b) does not imply (a), which shows that our theorem can not be deduced from a more general result about (DF)-spaces using duality. Here is the example: let X be a Banach space such that $\big(X', \sigma(X', X)\big)$ is not separable and denote by E the linear space X endowed with the topology of uniform convergence on the countable bounded subsets of $\big(X', \sigma(X', X)\big)$. Then E is a (DF)-space which does not satisfy the strict Mackey condition (cf. [8, Prop. p. 79]). But if B is the unit ball of the Banach space X, then $(nB)_{n\in\mathbb{N}}$ is a fundamental sequence of bounded subsets of E. Property (b) is then certainly satisfied.

Our next corollary contains one of the possible extensions to Fréchet spaces of what is known as the Josefson-Nissenzweig theorem (if X is a Banach space in the dual of which all weak* convergent sequences are norm convergent, then X is finite-dimensional). The corollary is the version of [3, 11.6.3] without the assumption of separability on the Fréchet space, and constitutes the precise positive solution to Jarchow question in [3, 11.10] about the characterization of Fréchet Schwartz spaces. Our next result is obtained by combining the theorem with results of Lindström [5]. These latter results depend heavily on a version of Bourgain and Diestel of the Josefson-Nissenzweig theorem (see [5]), so that the corollary extends but not reproves the theorem.

COROLLARY. *A Fréchet space F is Schwartz if and only if every $\sigma(F', F)$-convergent sequence in F' is contained in some F'_n and converges there (i.e. converges equicontinuously).*

PROOF. Assume that every $\sigma(F', F)$-convergent sequence converges equicontinuously. This implies that F'_b satisfies the Mackey convergence condition. By our theorem F is quasinormable. Now the conclusion follows from [5, Cor. 3]. ∎

NOTES ADDED IN PROOF (7/1991). (1) The corollary in the paper was independently obtained by M. Lindstöm and T. Schlumprecht in *A Josefson-Nissenzweig theorem for Fréchet spaces,* preprint 1990.

(2) As a direct consequence of our theorem it follows that a Fréchet space F is quasinormable if and only if the space of germs $H(K)$ is strongly boundedly retractive for one (or for all) compact subset(s) $K \neq \emptyset$ of F. This is a positive answer to Problem 14 in K. D. Bierstedt, R. Meise, *Aspects of inductive limits in spaces of germs of holomorphic functions on locally convex spaces and applications to a study of* $(H(U), \tau_\omega)$, p. 111–178 in *Advances in Holomorphy,* North-Holland Math. Studies **34**, Amsterdam 1979.

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Departamento de Matemática Aplicada
E.T.S. Arquitectura
Universidad Politécnica de Valencia
E-46071 Valencia, Spain