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SEMIGROUP RINGS IN SEMISIMPLE VARIETIES

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We describe semigroup rings which belong to self-dual varieties generated by a finite number of finite fields.

A variety is called *semisimple* if it is generated by a finite number of finite fields. Semisimple varieties play important roles in the theory of ring varieties. They occur in solutions to several natural problems and have been investigated in [**1, 3, 4, 5, 7, 9, 10, 11, 12, 14, 16, 17, 18**], and other papers. On the other hand, considerable attention in the literature is devoted to semigroup algebras satisfying polynomial identities (see [**8**, Chapter 20]). The aim of this note is to describe semigroup algebras which belong to self-dual semisimple varieties. As a corollary, we obtain the main result of [**13**].

Denote by $GF(p^n)$ the Galois field of order p^n. The variety generated by $GF(p^n)$ will be denoted by $\operatorname{var}[GF(p^n)]$. It coincides with the variety $\operatorname{var}[px = x^{p^n} - x = 0]$ defined by identities $px = x^{p^n} - x = 0$. If n divides m, then $GF(p^m) \supseteq GF(p^n)$.

A variety of associative rings is said to be *self-dual* if the lattice of its subvarieties is self-dual. It was proved in [**14**] that a semisimple variety is self-dual if and only if it can be generated by a finite number of finite fields with pairwise distinct characteristics. We shall consider a variety

$$\operatorname{var}\left[GF(p_1^{n_1}), \dots, GF(p_m^{n_m})\right]$$

where the primes $p_1, \dots, p_m$ are pairwise distinct. These varieties also occur in [**15**].

The characteristic of a ring R will be denoted by $\operatorname{char}(R)$. A *semilattice* is a commutative semigroup satisfying the identity $x^2 = x$.

THEOREM 1. *Let $p_1, \dots, p_n$ be pairwise distinct primes, $n_1, \dots, n_m$ positive integers. A nonzero semigroup ring RS belongs to the variety*

$$V = \operatorname{var}\left[GF(p_1^{n_1}), \dots, GF(p_m^{n_m})\right]$$

if and only if $R \in V$ and either S is a semilattice or $\operatorname{char}(R) = p_i$ *and S is a commutative semigroup satisfying the identity $x^{p_i^{n_i}} = x$.*

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Note that a semigroup satisfies the identity $x^{p^n} = x$ if and only if it is a union of groups whose orders are divisors of $p^n - 1$.

Let us begin with two lemmas characterising semisimple varieties. Let $Q(p)$ be the ring defined by generator a and relations $pa = a^2 = 0$. The variety generated by $Q(p)$ will be denoted by $\operatorname{var}[Q(p)]$. It is the variety $\operatorname{var}[px = xy = 0]$.

Lemma 2. [4, Theorem 5] *A variety V is generated by a finite number of finite fields if and only if it does not contain any varieties $\operatorname{var}[Q(p)]$.*

If R is a ring, then we put

$$R_p = \{x \in R \mid p^k x = 0 \text{ for some } k > 0\}.$$

Lemma 3. *Let $p_1, \dots, p_m$ be pairwise distinct primes, $n_1, \dots, n_m$ positive integers. Then the variety*

(1) $$\operatorname{var}\left[GF(p_1^{n_1}), \dots, GF(p_m^{n_m})\right]$$

is equal to the variety

(2) $$\operatorname{var}\left[p_1 \cdots p_m x = 0,\ p_1 \cdots p_{i-1} p_{i+1} \cdots p_m \left(x^{p_i^{n_i}} - x\right) = 0,\ \text{for } i = 1, \dots, m\right]$$

and is equal to the variety

(3) $$\operatorname{var}\left[p_1 \cdots p_m x = x \prod_{i=1}^{m} \left(x^{p_i^{n_i}-1} - 1\right) = 0\right].$$

Proof: The identities of semisimple varieties were described in [**3**]. For completeness we include a short self-contained proof.

Obviously, the variety (1) satisfies all identities in the definitions of (2) and (3). Therefore (1) $\subseteq$ (2) and (1) $\subseteq$ (3).

Clearly, $Q(p)$ does not satisfy all identities of the variety (2). Lemma 2 implies that the variety (2) is generated by a finite number of finite fields. Take any of these finite fields F. Since it satisfies the identity $p_1 \cdots p_m x = 0$, we get $\operatorname{char}(F) = p_i$ for some $1 \leqslant i \leqslant m$. The identity

$$p_1 \cdots p_{i-1} p_{i+1} \cdots p_m \left(x^{p_i^{n_i}} - x\right) = 0$$

shows that F satisfies $x^{p_i^{n_i}} - x = 0$. Hence F belongs to $\operatorname{var}\left[GF(p_i^{n_i})\right] \subseteq$ (1). Thus we get (1) = (2).

Since $Q(p)$ does not satisfy the identities of (3), Lemma 2 tells us that the variety (3) is generated by a finite number of finite fields. Take any of these fields, say F. Since it satisfies the identity $p_1 \cdots p_m x = 0$, we get $\operatorname{char}(F) = p_i$ for some $1 \leqslant i \leqslant m$. Suppose that $F = GF(p_i^n)$ for a positive integer n. Take any element x in F. If $x^{p_j^{n_j}} - x = 0$ for some $j \neq i$, then this and $x^{p_i^n} = x$ yield $x^2 = x$; whence x satisfies $x^{p_i^{n_i}} = x$. If, however,

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$x^{p_j^{n_j}} - x \neq 0$ for all $j \neq i$, then the identity $\prod_{i=1}^{m} \left(x^{p_i^{n_i}} - x\right) = 0$ implies that $x^{p_i^{n_i}} = x$, again. Thus F satisfies $x^{p_i^{n_i}} = x$, and so $n \leqslant n_i$ and F belongs to $\mathrm{var}\,[GF(p_i^{n_i})] \subseteq (1)$. Therefore $(1) = (3)$. ∎

PROOF OF THEOREM 1: Suppose that $R \in V$ and S is either a semilattice or a commutative semigroup satisfying the identity $x^{p_i^{n_i}} - x = 0$ if $p_i = \mathrm{char}\,(R)$. Lemma 3 shows that $\mathrm{char}\,(R)$ is a nonzero integer whose divisors are the p_i that appear in the definition of V. In both the cases S satisfies all identities $x^{p_i^{n_i}} = x$, for all p_i dividing $\mathrm{char}\,(R)$. Since R_p is a homomorphic image of R, $R \in V$ implies $R_p \in V$. Clearly, $R = \prod_{p|\mathrm{char}\,(R)} R_p$ and $RS \in V$ if and only if all the $R_pS \in V$, and therefore without loss of generality we may assume that $R = R_p$ for a prime $p = p_1$. Then R satisfies the identity $pp_2 \cdots p_m x = 0$. This means the additive order of a nonzero $x \in R = R_p$ divides both $pp_2 \cdots p_m$ and p^k for some $k > 0$, and this can only occur if its order equals p. Since this is true for all nonzero $x \in R$, we have $\mathrm{char}\,(R) = p$. A similar argument and the identity $p_2 \cdots p_m \left(x^{p^n} - x\right) = 0$, where $n = n_1$, of (2) in Lemma 3 imply that R satisfies the identity $x^{p^n} - x = 0$. Since R has characteristic p, it satisfies $(x+y)^{p^n} = x^{p^n} + y^{p^n}$. This and the identity $x^{p^n} = x$ of the semigroup S show that RS satisfies $x^{p^n} = x$. Hence $RS \in V$ by Lemma 3.

Conversely, assume that $RS \in V$. Obviously, $R \in V$, as it is a homomorphic image of RS. Take any $p = p_i$ dividing $\mathrm{char}\,(R)$. Then $R_p \neq 0$ and $R_pS \in V$, too. Therefore $R_p \in V$ and as above we see that R_p satisfies $px = 0$. The identities in the definition of variety (2) of Lemma 3 show that R_pS satisfies the identity $x^{p_i^{n_i}} = x$. Hence R_p satisfies the same identity, and so it contains a nonzero idempotent e.

Take any $s, t \in S$. Given that $R_pS \in V$ and V is commutative, we get $(es)(et) = (et)(es)$, whence $st = ts$. Thus S is commutative.

Further, take any s in S. Substituting es for x in the identity $x^{p_i^{n_i}} = x$ we see that S satisfies the same identity $s^{p_i^{n_i}} = s$ as a semigroup. Thus we see that if $\mathrm{char}\,(R) = p_i$, then S satisfies the identity $x^{p_i^{n_i}} - x = 0$. If, however, $\mathrm{char}\,(R)$ is not a prime, then it has two prime divisors by Lemma 3, say p_i and p_j. Then S satisfies the identities $x^{p_i^{n_i}} - x = x^{p_j^{n_j}} - x = 0$. Therefore S satisfies $x^2 = x$, and so it is a semilattice. ∎

The variety $\mathrm{var}\,[2x = x^2 - x = 0]$ of all Boolean rings is generated by the two-element field $GF(2)$. Therefore Theorem 1 yields the following corollary.

COROLLARY 4. [**13**] *A nonzero semigroup ring RS is a Boolean ring if and only if R is a Boolean ring and S is a semilattice.*

For a prime p, a ring is called a *p-ring* if it satisfies the identities $px = x^p - x = 0$ (see [**6**, Section 33]). Since these identities define the variety generated by $GF(p)$, Theorem 1 gives us the following corollary.

COROLLARY 5. *A nonzero semigroup ring RS is a p-ring if and only if R is a*

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p-ring and S is a commutative semigroup satisfying $x^p = x$.

Following [2], we say that a semigroup S *stabilises* a variety V if $R \in V$ implies $RS \in V$. Theorem 1 immediately gives the following

COROLLARY 6. *Let $p_1, \dots, p_m$ be pairwise distinct primes, $n_1, \dots, n_m$ positive integers. A semigroup S stabilises the variety $V = \operatorname{var}[GF(p_1^{n_1}), \dots, GF(p_m^{n_m})]$ if and only if either S is a semilattice or $n = 1$ and S is a commutative semigroup satisfying the identity $x^{p_1^{n_1}} = x$.*

REMARK 7. *Every semigroup ring in a variety V is semisimple if and only if V is semisimple.*

PROOF: A quasi-reqular ring does not have nonzero idempotents. Therefore the variety (2) does not contain nonzero quasiregular rings. It follows that all rings in a semisimple variety are semisimple.

Conversely, suppose that a variety V is not semisimple. By Lemma 2 V contains $\operatorname{var}[Q(p)] = \operatorname{var}[px = xy = 0]$ for a prime p. Take any semigroup S. The semigroup ring $Q(p)S$ belongs to $\operatorname{var}[Q(p)] \subseteq V$ and is not semisimple. ∎

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