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GORENSTEIN PROJECTIVE, INJECTIVE AND FLAT MODULES

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Abstract

In basic homological algebra, projective, injective and flat modules play an important and fundamental role. In this paper, we discuss some properties of Gorenstein projective, injective and flat modules and study some connections between Gorenstein injective and Gorenstein flat modules. We also investigate some connections between Gorenstein projective, injective and flat modules under change of rings.

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1. Introduction

Unless stated otherwise, throughout this paper all rings are associative with identity, and all modules are unitary. Let R be a ring. We denote by R-Mod and Mod-R the categories of left and right R-modules, respectively. For any R-module M, $\mathrm{pd}_R M$, $\mathrm{id}_R M$, and $\mathrm{fd}_R M$ denote the projective, injective, and flat dimensions respectively. The character module $\mathrm{Hom}_Z(M, Q/Z)$ is denoted by M^+.

When R is two-sided Noetherian, Auslander and Bridger [2] introduced the G-dimension, $\text{G-dim}_R M$, for every finitely generated R-module M. They proved that $\text{G-dim}_R M \leq \mathrm{pd}_R M$ with equality $\text{G-dim}_R M = \mathrm{pd}_R M$ when $\mathrm{pd}_R M$ is finite. Over a general ring R, Enochs and Jenda defined in [6] a homological dimension, namely the Gorenstein projective dimension $\mathrm{Gpd}_R(-)$, for arbitrary (nonfinite) modules. It is defined via resolution with (so-called) Gorenstein projective modules. Avramov, Bachweitz, Martsinkovsky and Reiten proved that a finite module over a Noetherian ring is Gorenstein projective if and only if $\text{G-dim}_R M = 0$ (see the remark

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following [3, Theorem 4.2.6]). Holm [9] gave homological descriptions of the Gorenstein dimensions over arbitrary rings. He proved that these dimensions are similar to the classical homological dimensions; that is, the projective, injective and flat dimensions.

In Section 2, we discuss some properties of Gorenstein projective, injective and flat modules and we also discuss connections between Gorenstein injective and flat modules. In Section 3, we investigate some connections between Gorenstein projective, injective and flat modules under change of rings. We shall then be concerned with what happens when certain modifications are made to a ring.

We first recall some concepts. Let $\mathcal{X}$ be any class of R-modules, M an R-module. A left $\mathcal{X}$-resolution of M is an exact sequence $\mathbb{X} : \cdots \to X_1 \to X_0 \to M \to 0$ with $X_i \in \mathcal{X}$ for all $i \geq 0$. A right $\mathcal{X}$-resolution of M is an exact sequence $\mathbb{X} : 0 \to M \to X^0 \to X^1 \to \cdots$ with $X^i \in \mathcal{X}$ for all $i \geq 0$. Now let $\mathbb{X}$ be any (left or right) $\mathcal{X}$-resolution of M. We say that $\mathbb{X}$ is proper or co-proper when the sequence $\mathrm{Hom}_R(Y, \mathbb{X})$ or $\mathrm{Hom}_R(\mathbb{X}, Y)$ is exact for all $Y \in \mathcal{X}$ respectively.

A complete projective resolution is an exact sequence of projective modules

$$\mathbb{P} : \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow P^0 \longrightarrow P^1 \longrightarrow \cdots$$

such that $\mathrm{Hom}_R(\mathbb{P}, Q)$ is exact for every projective R-module Q. An R-module M is called Gorenstein projective (G-projective for short) if there exists a complete projective resolution $\mathbb{P}$ with $M \cong \mathrm{Im}(P_0 \to P^0)$. Every projective module is Gorenstein projective. The class of all Gorenstein projective R-modules is denoted by $\mathcal{GP}(R)$. Holm [9] proved that the class $\mathcal{GP}(R)$ is closed under arbitrary direct sums and under direct summands. Gorenstein injective (G-injective for short) modules are defined dually and every injective module is Gorenstein injective. The class of all such modules is denoted by $\mathcal{GI}(R)$. Holm [9] proved that the class $\mathcal{GI}(R)$ is closed under any direct products and under direct summands. A complete flat resolution is an exact sequence of flat (left) modules

$$\mathbb{F} : \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow F^0 \longrightarrow F^1 \longrightarrow \cdots$$

such that $I \otimes_R \mathbb{F}$ is exact for every injective right R-module I. An R-module M is called Gorenstein flat (G-flat for short) if there exists a complete flat resolution $\mathbb{F}$ with $M \cong \mathrm{Im}(F_0 \to F^0)$. Every flat module is Gorenstein flat. The class of all Gorenstein flat R-modules is denoted by $\mathcal{GF}(R)$. Holm [9] proved that the class $\mathcal{GF}(R)$ is closed under arbitrary direct sums.

2. Properties of Gorenstein modules

It is well known that R is a perfect ring if and only if any direct limit of projective R-modules is projective by [16, Theorem 1.2.13].

THEOREM 2.1. *Let R be a left perfect, right coherent ring. If $M_0 \to M_1 \to M_2 \to \cdots$ is a sequence of G-projective left R-modules, then the direct limit $\varinjlim M_n$ is again G-projective.*

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PROOF. For each n, there exists a co-proper right projective resolution $\mathbb{P}_n : 0 \to M_n \to P_n^0 \to P_n^1 \to \cdots$. Consider the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M_0 & \longrightarrow & P_0^0 & \longrightarrow & C_0^1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M_1 & \longrightarrow & P_1^0 & \longrightarrow & C_1^1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

where $C_n^1 = \mathrm{Coker}(M_n \to P_n^0)$ for each n. Then each column in the above diagram is again a direct system. Thus $0 \to \varinjlim M_n \to \varinjlim P_n^0 \to \varinjlim C_n^1 \to 0$ is exact by [7, Theorem 1.5.6], and $\varinjlim P_n^0$ is projective. Let P be a projective left R-module. Then $\mathrm{Ext}_R^1(C_n^1, P) = 0$, and so C_n^1 is G-projective by [9, Corollary 2.11] for each n. Let Q be a projective left R-module. Then Q is pure-injective by [16, Lemma 3.1.6]. Hence

$$\mathrm{Ext}_R^i(\varinjlim M_n, Q) \cong \varprojlim \mathrm{Ext}_R^i(M_n, Q) = 0 \quad \text{and}$$
$$\mathrm{Ext}_R^i(\varinjlim C_n^1, Q) \cong \varprojlim \mathrm{Ext}_R^i(C_n^1, Q) = 0$$

by [15] for all $i \geq 1$, and so

$$0 \to \mathrm{Hom}_R(\varinjlim C_n^1, Q) \to \mathrm{Hom}_R(\varinjlim P_n^0, Q) \to \mathrm{Hom}_R(\varinjlim M_n, Q) \to 0$$

is exact. Continuing this procedure yields that $\mathrm{Hom}_R(\varinjlim \mathbb{P}_n, Q)$ is exact. It follows that $\varinjlim M_n$ is G-projective. □

It is well known that R is a Noetherian ring if and only if any direct limit of injective R-modules is injective by [7, Theorem 3.1.17].

THEOREM 2.2. *Let R be left Artinian and let the injective envelope of every simple left R-module be finitely generated. If $M_0 \to M_1 \to M_2 \to \cdots$ is a sequence of G-injective left R-modules, then the direct limit $\varinjlim M_n$ is again G-injective.*

PROOF. For each n, there exists a proper left injective resolution $\mathbb{E}_n : \cdots \to E_n^1 \to E_n^0 \to M_n \to 0$. Then

$$\varinjlim \mathbb{E}_n : \ \cdots \longrightarrow \varinjlim E_n^1 \longrightarrow \varinjlim E_n^0 \longrightarrow \varinjlim M_n \longrightarrow 0$$

is exact by analogy with the proof of Theorem 2.1, and $\varinjlim E_n^k$ is injective for $k = 0, 1, \ldots$. Let J be any injective left R-module. Then $J = \bigoplus_\Lambda J_\alpha$, where J_α is an injective envelope of some simple left R-module by [10, Theorem 6.6.4]. So

$$\mathrm{Hom}_R(J, \varinjlim \mathbb{E}_n) \cong \varinjlim \prod_\Lambda \mathrm{Hom}_R(J_\alpha, \mathbb{E}_n)$$

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is exact by [7, Theorem 1.5.6] and

$$\mathrm{Ext}^i_R(J, \varinjlim M_n) \cong \varinjlim \prod_{\Lambda} \mathrm{Ext}^i_R(J_\alpha, M_n) = 0$$

by [7, Theorem 3.1.16] for all $i \geq 1$. Therefore $\varinjlim M_n$ is G-injective. $\square$

Holm in [9, Theorem 3.6] proved that if R is right coherent, then M is a G-flat left R-module if and only if M^+ is a G-injective right R-module.

THEOREM 2.3. *Let R be left Artinian and let the injective envelope of every simple left R-module be finitely generated. Then the following are equivalent.*

(1) *M is a G-injective left R-module.*
(2) *M^+ is a G-flat right R-module.*

PROOF. We show first that (1) implies (2). There exists a complete injective resolution $\mathbb{E} : \cdots \to E_1 \to E_0 \to E^0 \to E^1 \to \cdots$ with $M \cong \mathrm{Ker}(E^0 \to E^1)$. Then

$$\mathbb{E}^+ : \cdots \longrightarrow E^{1+} \longrightarrow E^{0+} \longrightarrow E_0^+ \longrightarrow E_1^+ \longrightarrow \cdots$$

is exact such that $M^+ \cong \mathrm{Coker}(E^{1+} \to E^{0+})$ and E_i^+, E^{i+} are flat for $i = 0, 1, \ldots$. Let J be any injective left R-module. Then $J = \bigoplus_\Lambda J_\alpha$, where J_α is an injective envelope of some simple left R-module by [10, Theorem 6.6.4], and so

$$\mathbb{E}^+ \otimes_R J \cong \bigoplus_{\Lambda} (\mathbb{E}^+ \otimes_R J_\alpha) \cong \bigoplus_{\Lambda} \mathrm{Hom}_R(J_\alpha, \mathbb{E})^+$$

is exact by [7, Theorem 3.2.11]. It follows that M^+ is a G-flat right R-module.

Now we prove that (2) implies (1). There exists a complete flat resolution $\mathbb{F} : \cdots \to F_1 \to F_0 \to F^0 \to F^1 \to \cdots$ such that $\mathbb{F}_l : \cdots \to F_1 \to F_0 \to M^+ \to 0$ and $\mathbb{F}_r : 0 \to M^+ \to F^0 \to F^1 \to \cdots$ are exact. Then

$$\mathbb{F}_r^+ : \cdots \longrightarrow F^{1+} \longrightarrow F^{0+} \longrightarrow M^{++} \longrightarrow 0 \quad \text{and}$$
$$\mathbb{F}_l^+ : 0 \longrightarrow M^{++} \longrightarrow F_0^+ \longrightarrow F_1^+ \longrightarrow \cdots$$

are exact. We successively pick injective left R-modules $E_0, E_1, \ldots$ and $E^0, E^1, \ldots$ such that

$$F_0^+ \oplus E_0 \cong F_0^{+++}, \quad F_i^+ \oplus E_{i-1} \oplus E_i \cong (F_i^+ \oplus E_{i-1})^{++}$$
$$F^{0+} \oplus E^0 \cong F^{0+++}, \quad F^{i+} \oplus E^{i-1} \oplus E^i \cong (F^{i+} \oplus E^{i-1})^{++}$$

for $i = 1, 2, \ldots$. By adding $0 \to E_i \to E_i \to 0$ to the sequence $\mathbb{F}_l^+$ in degree $i+1$, $i+2$ and adding $0 \to E^i \to E^i \to 0$ to the sequence $\mathbb{F}_r^+$ in degree $i+2$, $i+1$ for $i = 0, 1, \ldots$, we obtain the exact sequence

$$\cdots \longrightarrow (F^{1+} \oplus E^0)^{++} \longrightarrow F^{0+++} \longrightarrow F_0^{+++} \longrightarrow (F_1^+ \oplus E_0)^{++} \longrightarrow \cdots.$$

So $\mathbb{E} : \cdots \to F^{1+} \oplus E^0 \to F^{0+} \to F_0^+ \to F_1^+ \oplus E_0 \to \cdots$ is exact such that $M \cong \mathrm{Ker}(F_0^+ \to F_1^+ \oplus E_0)$ and F^{0+}, $F^{i+} \oplus E^{i-1}$, F_0^+, $F_i^+ \oplus E_{i-1}$ are injective left R-modules for $i = 1, 2, \ldots$. Let J be any injective left R-module. Then

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$J = \bigoplus_{\Lambda} J_\alpha$, where J_α is an injective envelope of some simple left R-module by [10, Theorem 6.6.4]. Hence $\mathrm{Hom}_R(J_\alpha, \mathbb{F}^+) \cong (\mathbb{F} \otimes_R J_\alpha)^+$ is exact, and so

$$\mathrm{Hom}_R(J_\alpha, \mathbb{E})^{++} \cong (\mathbb{E}^+ \otimes_R J_\alpha)^+ \cong \mathrm{Hom}_R(J_\alpha, \mathbb{E}^{++})$$

is exact by [7, Theorem 3.2.11], which implies that $\mathrm{Hom}_R(J, \mathbb{E}) \cong \prod_{\Lambda} \mathrm{Hom}_R(J_\alpha, \mathbb{E})$ is exact. Thus M is a G-injective left R-module. □

COROLLARY 2.4. *Let R be left Artinian and let the injective envelope of every simple left R-module be finitely generated. Then M^+ is right G-flat for any G-injective left R-module M if and only if N^{++} is right G-flat for any G-flat right R-module N.*

LEMMA 2.5. *Let R be left Artinian and let the injective envelope of every simple left R-module be finitely generated. Then the class of all G-flat right R-modules is closed under arbitrary direct products.*

PROOF. Let $M = \prod_{i \in I} M_i$ with each M_i a G-flat right R-module. Then there is a complete flat resolution $\mathbb{F}_i : \cdots \to F_{1i} \to F_{0i} \to F_i^0 \to F_i^1 \to \cdots$ with $M_i \cong \mathrm{Ker}(F_i^0 \to F_i^1)$ for each $i \in I$. Then

$$\prod_{i \in I} \mathbb{F}_i : \cdots \longrightarrow \prod_{i \in I} F_{1i} \longrightarrow \prod_{i \in I} F_{0i} \longrightarrow \prod_{i \in I} F_i^0 \to \prod_{i \in I} F_i^1 \longrightarrow \cdots$$

is exact such that $M \cong \mathrm{Ker}(\prod_{i \in I} F_i^0 \to \prod_{i \in I} F_i^1)$ and $\prod_{i \in I} F_i^k, \prod_{i \in I} F_{ki}$ are flat for $k = 0, 1, \ldots$. Let E be any injective left R-module. Then $E = \bigoplus_{\Lambda} E_\alpha$, where E_α is an injective envelope of some simple left R-module by [10, Theorem 6.6.4], and hence $(\prod_{i \in I} \mathbb{F}_i) \otimes_R E \cong \bigoplus_{\Lambda} \prod_{i \in I} (\mathbb{F}_i \otimes_R E_\alpha)$ is exact. So M is a G-flat right R-module. □

COROLLARY 2.6. *Let R be left Artinian and let the injective envelope of every simple left R-module be finitely generated. Then the following are equivalent for an (R, S)-bimodule M.*

(1) *M is a G-injective left R-module.*
(2) *$\mathrm{Hom}_S(M, E)$ is a G-flat right R-module for all injective right S-modules E.*
(3) *$\mathrm{Hom}_S(M, E)$ is a G-flat right R-module for any injective cogenerator E for* Mod-S.
(4) *$M \otimes_S F$ is a G-injective left R-module for all flat left S-modules F.*
(5) *$M \otimes_S F$ is a G-injective left R-module for any faithfully flat left S-module F.*

PROOF. We show first that (1) implies (2). Let E be any injective right S-module. Then E is isomorphic to a summand of S^{+X} for some set X, and so $\mathrm{Hom}_S(M, E)$ is isomorphic to a summand of $\mathrm{Hom}_S(M, S^{+X}) \cong M^{+X}$. It follows that $\mathrm{Hom}_S(M, E)$ is a G-flat right R-module by Theorem 2.3 and Lemma 2.5.

That (2) implies (3) is obvious.

We prove that (3) implies (1). Since $M^+ \cong \mathrm{Hom}_S(M, S^+)$ is a G-flat right R-module, we have M is a G-injective left R-module by Theorem 2.3.

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To show that (2) implies (4), let F be any flat left S-module. Then $(M \otimes_S F)^+ \cong \mathrm{Hom}_S(M, F^+)$ is a G-flat right R-module, and so $M \otimes_S F$ is a G-injective left R-module by Theorem 2.3.

That (4) implies (5) and (5) implies (1) are obvious. $\square$

A ring R is said to be left V-ring if every simple left R-module is injective. Recall an R-module M is small projective if $\mathrm{Hom}_R(M, -)$ is exact with respect to the exact sequence $0 \to K \to L \to M \to 0$ in R-Mod with $K \ll L$.

COROLLARY 2.7. *Let R be a left Artinian left V-ring ring. Then the following are equivalent for an (R, S)-bimodule M.*

(1) *M is a G-injective left R-module.*
(2) $\mathrm{Hom}_S(M, E)$ *is a G-flat right R-module for all injective right S-modules E.*
(3) $\mathrm{Hom}_S(M, E)$ *is a G-flat right R-module for any injective cogenerator E for* Mod-S.
(4) *$M \otimes_S F$ is a G-injective left R-module for all flat left S-modules F.*
(5) *$M \otimes_S F$ is a G-injective left R-module for any faithfully flat left S-module F.*

COROLLARY 2.8. *Let R be a commutative Artinian ring. Then the following are equivalent for an (R, S)-bimodule M.*

(1) *M is a G-injective R-module.*
(2) $\mathrm{Hom}_S(M, E)$ *is a G-flat R-module for all injective right S-modules E.*
(3) $\mathrm{Hom}_S(M, E)$ *is a G-flat R-module for any injective cogenerator E for* Mod-S.
(4) *$M \otimes_S F$ is a G-injective R-module for all flat left S-modules F.*
(5) *$M \otimes_S F$ is a G-injective R-module for any faithfully flat left S-module F.*

PROOF. If L is a simple R-module, then $E(L)$ is finitely generated by [11, Theorem 3.64]. $\square$

COROLLARY 2.9. *Let R be left Artinian. If every left R-module is small projective, then the following are equivalent for an (R, S)-bimodule M.*

(1) *M is a G-injective left R-module.*
(2) $\mathrm{Hom}_S(M, E)$ *is a G-flat right R-module for all injective right S-modules E.*
(3) $\mathrm{Hom}_S(M, E)$ *is a G-flat right R-module for any injective cogenerator E for* Mod-S.
(4) *$M \otimes_S F$ is a G-injective left R-module for all flat left S-modules F.*
(5) *$M \otimes_S F$ is a G-injective left R-module for any faithfully flat left S-module F.*

An R-module C is said to be cotorsion if $\mathrm{Ext}^1_R(F, C) = 0$ for all flat R-modules F. It is well known that every module has a cotorsion envelope and if $\varphi : M \to C$ is a cotorsion envelope of an R-module M, then $L = \mathrm{Coker}\,\varphi$ is flat. An R-module M is called FP-injective if $\mathrm{Ext}^1_R(N, M) = 0$ for all finitely presented modules N. The FP-injective dimension of M, denoted by FP-id$M$, is defined to be the least nonnegative integer n such that $\mathrm{Ext}^{n+1}_R(N, M) = 0$ for all finitely presented modules N. If no such n exists, set FP-id$M = \infty$.

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THEOREM 2.10. *Let R be two-sided coherent. Then the following are equivalent.*

(1) *R is two-sided FP-injective.*
(2) *Every R-module (left and right) is G-flat.*
(3) *Every finitely presented R-module (left and right) is G-flat.*
(4) *Every finitely presented R-module (left and right) is G-projective.*
(5) *Every cyclic R-module (left and right) is G-flat.*
(6) *Every of the quotient module R/P (P is a prime ideal) is G-flat.*
(7) *Every cotorsion R-module (left and right) is G-flat.*
(8) *Every nonzero R-module (left and right) contains a nonzero G-flat submodule.*

PROOF. The equivalences (1) if and only if (2) if and only if (3) if and only if (4) hold by [5, Theorem 6].

The implications (1) implies (5), (1) implies (6) and (1) implies (7) are trivial.

We show that (6) implies (3). We use the fact that every finitely presented R-module M can be filtered as

$$M = M_0 \supset M_1 \supset \cdots \supset M_k = 0,$$

where successive quotients are isomorphic to R/P for some prime ideal P. Hence M_{k-1} is G-flat since M_{k-2}/M_{k-1} is G-flat. Proceeding thus we can show that M is G-flat.

To show that (7) implies (1), let M be any R-module. Then there exists an exact sequence $0 \to M \to C(M) \to L \to 0$, where $C(M)$ is cotorsion and L is flat, and so M is G-flat by [9, Theorem 3.7].

To prove that (5) implies (8), let M be any nonzero left or right R-module. Then there is $0 \neq x \in M$, and so Rx or xR is a nonzero G-flat submodule of M respectively.

We prove that (8) implies (2). Let M be any nonzero R-module. Then there is a nonzero G-flat submodule N of M. Set $\mathfrak{D} = \{N \subseteq D \subseteq M \mid D \in \mathcal{GF}(R)\}$. Then $\mathfrak{D}$ is a nonempty subposet of the lattice of submodules of M and every nonempty chain in $\mathfrak{D}$ has an upper bound in $\mathfrak{D}$, namely its union. So by Zorn's Lemma, $\mathfrak{D}$ has a maximal element say L. If $L \neq M$, then $M/L \neq 0$, and so there is a nonzero G-flat submodule K/L of M/L. Hence K is G-flat by [9, Theorem 3.7] and K is strictly larger than L, which gives that $M = L$ is G-flat. □

PROPOSITION 2.11. *Let R be a commutative ring Q a flat R-module. If M is a G-flat R-module, then $M \otimes_R Q$ is a G-flat R-module.*

PROOF. Since M is G-flat, there exists a complete flat resolution $\mathbb{F} : \cdots \to F_1 \to F_0 \to F^0 \to F^1 \to \cdots$ with $M \cong \mathrm{Ker}(F^0 \to F^1)$. Then

$$\mathbb{F} \otimes_R Q : \cdots \longrightarrow F_1 \otimes_R Q \longrightarrow F_0 \otimes_R Q \longrightarrow F^0 \otimes_R Q \longrightarrow F^1 \otimes_R Q \longrightarrow \cdots$$

is exact such that $M \otimes_R Q \cong \mathrm{Ker}(F^0 \otimes_R Q \to F^1 \otimes_R Q)$ and $F_i \otimes_R Q$, $F^i \otimes_R Q$ are flat for $i = 0, 1, \ldots$. Let E be any injective R-module. Then $E \otimes_R (\mathbb{F} \otimes_R Q)$ is exact, and so $M \otimes_R Q$ is a G-flat R-module. □

PROPOSITION 2.12. *Let R be a commutative ring P a finitely generated projective R-module. If M is a G-flat R-module, then* $\mathrm{Hom}_R(P, M)$ *is a G-flat R-module.*

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PROOF. Let Q be any flat R-module and let $0 \to A \to B$ be exact. Consider the commutative diagram

$$\begin{tikzcd} & \mathrm{Hom}_R(P, Q) \otimes_R A \arrow[r] \arrow[d, "\cong"'] & \mathrm{Hom}_R(P, Q) \otimes_R B \arrow[d, "\cong"'] \\ 0 \arrow[r] & \mathrm{Hom}_R(P, Q \otimes_R A) \arrow[r] & \mathrm{Hom}_R(P, Q \otimes_R B) \end{tikzcd}$$

with the lower row exact. Then $\mathrm{Hom}_R(P, Q)$ is flat. Since M is G-flat, there is a complete flat resolution $\mathbb{F} : \cdots \to F_1 \to F_0 \to F^0 \to F^1 \to \cdots$ with $M \cong \mathrm{Ker}(F^0 \to F^1)$. Then

$$\mathrm{Hom}_R(P, \mathbb{F}) : \cdots \longrightarrow \mathrm{Hom}_R(P, F_1) \longrightarrow \mathrm{Hom}_R(P, F_0) \longrightarrow \mathrm{Hom}_R(P, F^0) \\ \longrightarrow \mathrm{Hom}_R(P, F^1) \longrightarrow \cdots$$

is exact such that $\mathrm{Hom}_R(P, M) \cong \mathrm{Ker}(\mathrm{Hom}_R(P, F^0) \to \mathrm{Hom}_R(P, F^1))$ and all the $\mathrm{Hom}_R(P, F_i)$ and $\mathrm{Hom}_R(P, F^i)$ are flat for $i = 0, 1, \ldots$. Let I be any injective R-module. Then $\mathrm{Hom}_R(P, \mathbb{F}) \otimes_R I \cong \mathrm{Hom}_R(P, \mathbb{F} \otimes_R I)$ is exact by [1, Proposition 20.10], which means that $\mathrm{Hom}_R(P, M)$ is G-flat. □

PROPOSITION 2.13. *Let R be a commutative Artinian ring. If M is a G-injective R-module, then* $\mathrm{Hom}_R(E, M)$ *is a G-flat R-module for any injective R-module E.*

PROOF. Let E be any injective R-module. Then $E = \bigoplus_\Lambda E_\alpha$, where E_α is an injective envelope of some simple R-module by [10, Theorem 6.6.4]. Since M is G-injective, there is a complete injective resolution $\mathbb{E} : \cdots \to E_1 \to E_0 \to E^0 \to E^1 \to \cdots$ with $M \cong \mathrm{Ker}(E^0 \to E^1)$. Then

$$\mathrm{Hom}_R(E, \mathbb{E}) : \cdots \longrightarrow \mathrm{Hom}_R(E, E_1) \longrightarrow \mathrm{Hom}_R(E, E_0) \longrightarrow \mathrm{Hom}_R(E, E^0) \\ \longrightarrow \mathrm{Hom}_R(E, E^1) \longrightarrow \cdots$$

is exact such that $\mathrm{Hom}_R(E, M) \cong \mathrm{Ker}(\mathrm{Hom}_R(E, E^0) \to \mathrm{Hom}_R(E, E^1))$ and all the $\mathrm{Hom}_R(E, E_i)$ and $\mathrm{Hom}_R(E, E^i)$ are flat for $i = 0, 1, \ldots$. Let I be any injective R-module. Then $I = \bigoplus_{\Lambda'} I_\beta$, where I_β is an injective envelope of some simple R-module, and so

$$\begin{aligned} I \otimes_R \mathrm{Hom}_R(E, \mathbb{E}) &\cong \bigoplus_{\beta \in \Lambda'} \mathrm{Hom}_R\Big(\mathrm{Hom}_R\Big(I_\beta, \bigoplus_{\alpha \in \Lambda} E_\alpha\Big), \mathbb{E}\Big) \\ &\cong \bigoplus_{\beta \in \Lambda'} \prod_{\alpha \in \Lambda} \mathrm{Hom}_R(\mathrm{Hom}_R(I_\beta, E_\alpha), \mathbb{E}) \end{aligned}$$

is exact by [7, Theorem 3.2.11] since I_β and E_α are finitely generated by [11, Theorem 3.64]. It follows that $\mathrm{Hom}_R(E, M)$ is G-flat. □

3. Gorenstein modules and change of rings

Let M be an R-module. Then a sequence of central elements $a_1, \ldots, a_n$ in an ideal I of R is called an M-sequence if $(a_1, \ldots, a_n)M \neq M$ and a_i is not a zero-divisor on $M/(a_1, \ldots, a_{i-1})M$ for $1 \leq i \leq n$.

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LEMMA 3.1. *Let a be a regular element M an R-module. Then*

$$0 \longrightarrow M \xrightarrow{a} M \longrightarrow M/aM \longrightarrow 0$$

is exact if and only if

$$0 \longrightarrow M \xrightarrow{a\otimes_R 1} M \longrightarrow R/a \otimes_R M \longrightarrow 0$$

is exact.

PROOF. By analogy with the proof of [17, Lemma 3.10]. □

LEMMA 3.2. *Let M be an R-module and let I be generated by an R- and M-sequence $(a_1, \ldots, a_n)$. Then $\mathrm{Ext}^i_{R/I}(M/IM, -) \cong \mathrm{Ext}^i_R(M, -)$ for all $i \geq 1$.*

PROOF. By analogy with the proof of [17, Lemma 3.11]. □

THEOREM 3.3. *Let M be an R-module and let I be generated by an R- and M-sequence $(a_1, \ldots, a_n)$. If M is a G-projective left R-module, then M/IM is a G-projective left R/I-module.*

PROOF. Take a co-proper right projective resolution $\mathbb{P} : 0 \to M \to P^0 \to P^1 \to \cdots$. Since $0 \to R \xrightarrow{a_1} R \to R/a_1R \to 0$ is exact, $0 \to P^i \xrightarrow{a_1} P^i \to P^i/a_1P^i \to 0$ is exact for $i = 0, 1, \ldots$. Hence

$$0 \longrightarrow \mathbb{P} \longrightarrow \mathbb{P} \longrightarrow R/a_1R \otimes_R \mathbb{P} \longrightarrow 0$$

is exact, so $R/a_1R \otimes_R \mathbb{P}$ is exact. As $0 \to R/a_1R \xrightarrow{a_2} R/a_1R \to R/(a_1, a_2)R \to 0$ is exact, $0 \to P^i/a_1P^i \xrightarrow{a_2} P^i/a_1P^i \to P^i/(a_1, a_2)P^i \to 0$ is exact for $i = 0, 1, \ldots$. Thus

$$0 \longrightarrow R/a_1R \otimes_R \mathbb{P} \longrightarrow R/a_1R \otimes_R \mathbb{P} \longrightarrow R/(a_1, a_2)R \otimes_R \mathbb{P} \longrightarrow 0$$

is exact, and so $R/(a_1, a_2)R \otimes_R \mathbb{P}$ is exact. Continuing this procedure yields that

$$R/I \otimes_R \mathbb{P} : \; 0 \longrightarrow M/IM \longrightarrow P^0/IP^0 \longrightarrow P^1/IP^1 \longrightarrow \cdots$$

is exact and every P^i/IP^i is a projective left R/I-module since $\mathrm{Ext}^1_{R/I}(P/IP, -) \cong \mathrm{Ext}^1_R(P, -) = 0$ by Lemma 3.2 for any projective left R-module P. Let $\bar{Q}$ be any projective left R/I-module. Then $\mathrm{pd}_R \bar{Q} = n$ by [13, Proposition 5.8], and so $\mathrm{Hom}_{R/I}(R/I \otimes_R \mathbb{P}, \bar{Q}) \cong \mathrm{Hom}_R(\mathbb{P}, \bar{Q})$ is exact and $\mathrm{Ext}^i_{R/I}(M/IM, \bar{Q}) \cong \mathrm{Ext}^i_R(M, \bar{Q}) = 0$ by Lemma 3.2 for all $i \geq 1$. Thus M/IM is a G-projective left R/I-module. □

THEOREM 3.4. *Let M be an R-module and let I be generated by an R- and M-sequence $(a_1, \ldots, a_n)$. If M is a G-flat left R-module, then M/IM is a G-flat left R/I-module.*

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PROOF. Since M is G-flat, there exists a complete flat resolution $\mathbb{F}: \cdots \to F_1 \to F_0 \to F^0 \to F^1 \to \cdots$ with $M \cong \mathrm{Ker}(F^0 \to F^1)$. Then

$$R/I \otimes_R \mathbb{F}: \cdots \longrightarrow F_1/IF_1 \longrightarrow F_0/IF_0 \longrightarrow F^0/IF^0 \longrightarrow F^1/IF^1 \longrightarrow \cdots$$

is exact much as in the proof of Theorem 3.3 such that $M/IM \cong \mathrm{Ker}(F^0/IF^0 \to F^1/F^1)$ and F_i/IF_i, F^i/IF^i are flat left R/I-modules for $i = 0, 1, \ldots$ since

$$\mathrm{Tor}_1^{R/I}(-, F/IF)^+ \cong \mathrm{Ext}^1_{R/I}(F/IF, -^+) \cong \mathrm{Ext}^1_R(F, -^+) \cong \mathrm{Tor}_1^R(-, F)^+ = 0$$

by Lemma 3.2 for any flat left R-module F. Let $\bar{E}$ be any injective right R/I-module and let $0 \to A \to B$ be exact in Mod-R. Consider the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Hom}_{R/I}(B/BI, \bar{E}) \arrow[r] \arrow[d, "\cong"'] & \mathrm{Hom}_{R/I}(A/AI, \bar{E}) \arrow[r] \arrow[d, "\cong"'] & 0 \\
\mathrm{Hom}_R(B, \bar{E}) \arrow[r] & \mathrm{Hom}_R(A, \bar{E}) &
\end{tikzcd}$$

in which the upper row is exact. Then $\bar{E}$ is an injective right R-module. So $\bar{E} \otimes_{R/I} (R/I \otimes_R \mathbb{F}) \cong \bar{E} \otimes_R \mathbb{F}$ is exact, which implies that M/IM is a G-flat left R/I-module. $\square$

THEOREM 3.5. *Let M be an R-module and let I be generated by an R- and M-sequence $(a_1, \ldots, a_n)$. If M is a G-injective right R-module, then* $\mathrm{Hom}_R(R/I, M)$ *is a G-injective right R/I-module.*

PROOF. Since M is G-injective, there is a proper left injective resolution $\mathbb{E}: \cdots \to E_1 \to E_0 \to M \to 0$. Since $0 \to R \to^{a_1} R \to R/a_1R \to 0$ is exact and $\mathrm{pd}_R(R/a_1R) = 1$ by [11, Proposition 5.32],

$$0 \to \mathrm{Hom}_R(R/a_1R, M) \to M \to M \to 0 \quad \text{and}$$
$$0 \to \mathrm{Hom}_R(R/a_1R, E_i) \to E_i \to E_i \to 0$$

are exact by [4, Lemma 2.2] for $i = 0, 1, \ldots$. Thus

$$0 \longrightarrow \mathrm{Hom}_R(R/a_1R, \mathbb{E}) \longrightarrow \mathbb{E} \longrightarrow \mathbb{E} \longrightarrow 0$$

is exact, and so $\mathrm{Hom}_R(R/a_1R, \mathbb{E})$ is exact. Since $0 \to R/a_1R \xrightarrow{a_2} R/a_1R \to R/(a_1, a_2)R \to 0$ is exact and $\mathrm{pd}_R(R/(a_1, a_2)R) = 2$ by [11, Proposition 5.32],

$$\begin{aligned}
0 \longrightarrow \mathrm{Hom}_R(R/(a_1, a_2)R, M) &\longrightarrow \mathrm{Hom}_R(R/a_1R, M) \\
&\longrightarrow \mathrm{Hom}_R(R/a_1R, M) \longrightarrow 0 \\
0 \longrightarrow \mathrm{Hom}_R(R/(a_1, a_2)R, E_i) &\longrightarrow \mathrm{Hom}_R(R/a_1R, E_i) \\
&\longrightarrow \mathrm{Hom}_R(R/a_1R, E_i) \longrightarrow 0
\end{aligned}$$

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are exact by [4, Lemma 2.2] for $i = 0, 1, \ldots$. So

$$0 \longrightarrow \mathrm{Hom}_R(R/(a_1, a_2)R, \mathbb{E}) \longrightarrow \mathrm{Hom}_R(R/a_1R, \mathbb{E}) \longrightarrow \mathrm{Hom}_R(R/a_1R, \mathbb{E}) \longrightarrow 0$$

is exact, and hence $\mathrm{Hom}_R(R/(a_1, a_2)R, \mathbb{E})$ is exact. Continuing this procedure yields

$$\begin{aligned}\mathrm{Hom}_R(R/I, \mathbb{E}) : \cdots \longrightarrow \mathrm{Hom}_R(R/I, E_1) &\longrightarrow \mathrm{Hom}_R(R/I, E_0) \\ &\longrightarrow \mathrm{Hom}_R(R/I, M) \longrightarrow 0\end{aligned}$$

is exact and each of the $\mathrm{Hom}_R(R/I, E_i)$ is an injective right R/I-module since $\mathrm{Ext}^1_{R/I}(-, \mathrm{Hom}_R(R/I, E)) \cong \mathrm{Ext}^1_R(-, E) = 0$ by [14, p. 258, 9.21] for any injective right R-module E. Let $\bar{E}$ be any injective right R/I-module. Then $\bar{E}$ is an injective right R-module, and so $\mathrm{Hom}_{R/I}(\bar{E}, \mathrm{Hom}_R(R/I, \mathbb{E})) \cong \mathrm{Hom}_R(\bar{E}, \mathbb{E})$ is exact and $\mathrm{Ext}^i_{R/I}(\bar{E}, \mathrm{Hom}_R(R/I, M)) \cong \mathrm{Ext}^i_R(\bar{E}, M) = 0$ by [14, p. 258, 9.21] for all $i \geq 1$. It follows that $\mathrm{Hom}_R(R/I, M)$ is a G-injective right R/I-module. □

PROPOSITION 3.6. *Let (R, m) be a local Noetherian ring M a finitely generated R-module and let $(a_1, \ldots, a_s)$ be an M-regular sequence in m, $\bar{M} = M/(a_1, \ldots, a_s)M$. Then* $\mathrm{Gpd}_R \bar{M} = \mathrm{Gpd}_R M + s$.

PROOF. Let $\mathrm{Gpd}_R M = n$. We use induction on the finite number $s \geq 1$. If $s = 1$, then $\mathrm{Gpd}_R(M/a_1M) \leq \mathrm{Gpd}_R M + 1$ since $0 \to M \to^{a_1} M \to M/a_1M \to 0$ is exact. Consider the exact sequence

$$\mathrm{Ext}^n_R(M, Q) \longrightarrow \mathrm{Ext}^n_R(M, Q) \longrightarrow \mathrm{Ext}^{n+1}_R(M/a_1M, Q) \longrightarrow 0.$$

If $\mathrm{Ext}^n_R(M, Q) \neq 0$ for some projective R-module Q, then $\mathrm{Ext}^n_R(M, R) \neq 0$ since R is Noetherian and M is finitely generated, and hence $\mathrm{Ext}^{n+1}_R(M/a_1M, R) \neq 0$ by Nakayama's lemma, which implies that $\mathrm{Gpd}_R(M/a_1M) = \mathrm{Gpd}_R M + 1$. If $s > 1$, then

$$0 \longrightarrow M/(a_1, \ldots, a_{s-1})M \xrightarrow{a_s} M/(a_1, \ldots, a_{s-1})M \longrightarrow \bar{M} \longrightarrow 0$$

is exact and by induction we see that $\mathrm{Gpd}_R \bar{M} \leq \mathrm{Gpd}_R M + s$. Consider the exact sequence

$$\begin{aligned}\mathrm{Ext}^{n+s-1}_R(M/(a_1, \ldots, a_{s-1})M, Q) &\longrightarrow \mathrm{Ext}^{n+s-1}_R(M/(a_1, \ldots, a_{s-1})M, Q) \\ &\longrightarrow \mathrm{Ext}^{n+s}_R(\bar{M}, Q) \longrightarrow 0.\end{aligned}$$

If $\mathrm{Ext}^{n+s-1}_R(M/(a_1, \ldots, a_{s-1})M, Q) \neq 0$ for some projective R-module Q, then $\mathrm{Ext}^{n+s}_R(\bar{M}, R) \neq 0$ by Nakayama's lemma. Hence $\mathrm{Gpd}_R \bar{M} = \mathrm{Gpd}_R M + s$. □

By the flat and projective dimensions of a homomorphism of rings $\varphi : R \to S$ we understand the flat and projective dimensions of S over R respectively; in particular, we say that φ is (faithfully) flat if it makes S is a (faithfully) flat R-module. We call φ finite if it makes S a finite R-module.

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PROPOSITION 3.7. *Let R and S be rings and let $\varphi : R \to S$ be of finite projective dimension. If M is a G-projective left R-module, then $S \otimes_R M$ is a G-projective left S-module.*

PROOF. Let P be any projective left R-module. Then $\mathrm{Ext}^1_S(S \otimes_R P, -) \cong \mathrm{Hom}_R(P, \mathrm{Ext}^1_S(S, -)) = 0$ by [14, p. 258, 9.20], and so $S \otimes_R P$ is a projective left S-module. Since M is G-projective, there exists a complete projective resolution $\mathbb{P} : \cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots$ with $M \cong \mathrm{Im}(P_0 \to P^0)$. We use induction on the finite number $\mathrm{pd}_R S$. If $\mathrm{pd}_R S = 0$, then S is projective, and $S \otimes_R \mathbb{P}$ is exact. Assume $\mathrm{pd}_R S \geq 1$. Let $0 \to K \to P \to S \to 0$ be a projective resolution of S with $\mathrm{pd}_R K = \mathrm{pd}_R S - 1$. Then

$$0 \to K \otimes_R P_i \to P \otimes_R P_i \to S \otimes_R P_i \to 0$$

and

$$0 \to K \otimes_R P^i \to P \otimes_R P^i \to S \otimes_R P^i \to 0$$

are exact for $i = 0, 1, \ldots$. Thus $0 \to K \otimes_R \mathbb{P} \to P \otimes_R \mathbb{P} \to S \otimes_R \mathbb{P} \to 0$ is exact, which gives that

$$S \otimes_R \mathbb{P} : \cdots \longrightarrow S \otimes_R P_1 \longrightarrow S \otimes_R P_0 \longrightarrow S \otimes_R P^0 \longrightarrow S \otimes_R P^1 \longrightarrow \cdots$$

is exact by induction such that $S \otimes_R M \cong \mathrm{Im}(S \otimes_R P_0 \to S \otimes_R P^0)$ and $S \otimes_R P_i$, $S \otimes_R P^i$ are projective left S-modules for $i = 0, 1, \ldots$. Let $\bar{Q}$ be any projective left S-module. Then $\mathrm{pd}_R \bar{Q}$ is finite, and so $\mathrm{Hom}_S(S \otimes_R \mathbb{P}, \bar{Q}) \cong \mathrm{Hom}_R(\mathbb{P}, \bar{Q})$ is exact. Thus $S \otimes_R M$ is a G-projective left S-module. □

PROPOSITION 3.8. *Let R and S be rings and let $\varphi : R \to S$ be of finite flat dimension. If M is a G-flat right R-module, then $M \otimes_R S$ is a G-flat right S-module.*

PROOF. Since M is G-flat, there exists a complete flat resolution $\mathbb{F} : \cdots \to F_1 \to F_0 \to F^0 \to F^1 \to \cdots$ with $M \cong \mathrm{Im}(F_0 \to F^0)$. Then

$$\mathbb{F} \otimes_R S : \cdots \longrightarrow F_1 \otimes_R S \longrightarrow F_0 \otimes_R S \longrightarrow F^0 \otimes_R S \longrightarrow F^1 \otimes_R S \longrightarrow \cdots$$

is exact by an argument like the proof of Proposition 3.7 such that $M \otimes_R S \cong \mathrm{Im}(F_0 \otimes_R S \to F^0 \otimes_R S)$ and $F_i \otimes_R S$ and $F^i \otimes_R S$ are flat right S-modules by [7, p. 43, Exercise 9] for $i = 0, 1, \ldots$. Let $\bar{I}$ be any injective left S-module and let H be any left R-module and $\mathbb{P}$ be a projective resolution of H. Then $S \otimes_R \mathbb{P}$ is a projective resolution of $S \otimes_R H$ over S. Thus

$$0 = \mathrm{Ext}^1_S(S \otimes_R H, \bar{I}) = \mathrm{H}^1(\mathrm{Hom}_S(S \otimes_R \mathbb{P}, \bar{I})) \cong \mathrm{H}^1(\mathrm{Hom}_R(\mathbb{P}, \bar{I})) = \mathrm{Ext}^1_R(H, \bar{I})$$

and hence $\bar{I}$ is an injective left R-module. So $\mathbb{F} \otimes_R S \otimes_S \bar{I} \cong \mathbb{F} \otimes_R \bar{I}$ is exact, which implies that $M \otimes_R S$ is a G-flat right S-module. □

PROPOSITION 3.9. *Let R and S be rings and let $\varphi : R \to S$ be of finite projective dimension. If M is a G-injective left R-module, then $\mathrm{Hom}_R(S, M)$ is a G-injective left S-module.*

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PROOF. Let E be any injective left R-module. Then it follows from [14, p. 258, 9.21] that $\mathrm{Ext}^1_S(-, \mathrm{Hom}_R(S, E)) \cong \mathrm{Ext}^1_R(-, E) = 0$, and so $\mathrm{Hom}_R(S, E)$ is an injective left S-module. Since M is G-injective, there is a complete injective resolution $\mathbb{E} : \cdots \to E_1 \to E_0 \to E^0 \to E^1 \to \cdots$ with $M \cong \mathrm{Im}(E_0 \to E^0)$. Then

$$\mathrm{Hom}_R(S, \mathbb{E}) : \cdots \longrightarrow \mathrm{Hom}_R(S, E_1) \longrightarrow \mathrm{Hom}_R(S, E_0) \longrightarrow \mathrm{Hom}_R(S, E^0) \longrightarrow \mathrm{Hom}_R(S, E^1) \longrightarrow \cdots$$

is exact by analogy with the proof of Proposition 3.7 such that $\mathrm{Hom}_R(S, M) \cong \mathrm{Im}(\mathrm{Hom}_R(S, E_0) \to \mathrm{Hom}_R(S, E^0))$ and $\mathrm{Hom}_R(S, E_i)$, $\mathrm{Hom}_R(S, E^i)$ are injective left S-modules for each i. Let $\bar{J}$ be any injective left S-module. Then $\bar{J}$ is an injective left R-module, and so $\mathrm{Hom}_S(\bar{J}, \mathrm{Hom}_R(S, \mathbb{E})) \cong \mathrm{Hom}_R(\bar{J}, \mathbb{E})$ is exact. Thus $\mathrm{Hom}_R(S, M)$ is a G-injective left S-module. □

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