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SEMIPERFECT RINGS AND NAKAYAMA PERMUTATIONS

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Abstract. We study the conditions which force a semiperfect ring to admit a Nakayama permutation of its basic idempotents. We also give a few necessary and sufficient conditions for a semiperfect ring R, which cogenerates every 2-generated right R-module, to be right pseudo-Frobenius.

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0. Introduction. Throughout R is an associative ring with identity and modules are unitary. The right and left annihilators of subset X of a ring R are denoted by $r_R(X)$ and $l_R(X)$ respectively. We write $J = J(R)$ for the Jacobson radical of a ring R and $Soc(M)$ for the socle of a module M. Right and left singular ideals of a ring R will be denoted by $Z(R_R)$ and $Z({}_RR)$ respectively. By $N \trianglelefteq M$ we shall mean that N is an essential submodule of a module M.

A ring R is called *right mininjective* (*right principally injective*) if every R-homomorphism from a minimal (principal) right ideal of R into R is given by left multiplication by an element of R. Mininjective rings were introduced by Harada [**7**] who studied them in Artinian case. Recently Nicholson and Yousif [**13**] studied arbitrary mininjective rings. Principally injective rings have been studied in [**3**, **12**, **16**, **17**]. A ring R is called *right Kasch* if R contains a copy of each simple right R-module. An idempotent e of a ring R is called *local* if eRe is a local ring; equivalently if eJ is the unique maximal submodule of eR. Nakayama [**10**] called a left and right Artinian ring R with basic set of idempotents $e_1, \ldots, e_n$ *quasi-Frobenius* if there exists a permutation π of $\{1, \ldots, n\}$ such that

$$Soc(Re_{\pi(i)}) \cong Re_i/Je_i \text{ and } Soc(e_iR) \cong e_{\pi(i)}R/e_{\pi(i)}J.$$

Let R be a semiperfect ring with basic set of idempotents $e_1, \ldots, e_n$. In this paper, following Nicholson and Yousif [**12**], we call a permutation σ of $\{1, \ldots, n\}$ a *Nakayama permutation* if there exists a set $k_1, \ldots, k_n$ of elements of R such that for each i

(1) $Rk_i \subseteq Re_{\sigma(i)}$ and $k_iR \subseteq e_iR$;

(2) $Rk_i \cong Re_i/Je_i$ and $k_iR \cong e_{\sigma(i)}R/e_{\sigma(i)}J$.

In particular, $\{k_1R, \ldots, k_nR\}$ and $\{Rk_1, \ldots, Rk_n\}$ are complete irredundant sets of representatives of isomorphism classes of simple right and simple left R-modules respectively and so R is left and right Kasch.

If a ring is right self injective and right cogenerator, it is called *right pseudo-Frobenius* (*PF*). Extending some well known results on PF and quasi-Frobenius (QF) rings, Nicholson and Yousif proved that a right minfull ring (that is a semiperfect right mininjective ring R with $Soc(eR) \neq 0$ for every local idempotent e [**13**]) admits a Nakayama permutation of its basic idempotents and $Soc(eR)$ is homogeneous for every local idempotent e. Moreover, its two socles are equal if every simple left ideal is a left annihilator [**13**, Theorem 3.7].

For a semiperfect ring R admitting a Nakayama permutation of its basic idempotents, $Soc(eR) \neq 0$ for every local idempotent e. Moreover, $Soc(eR)$ is homogeneous for every local idempotent e if and only if R satisfies the following condition:

$(*)$ *For every local idempotents e and f of a ring R if eR and fR contain isomorphic simple submodules then* $eR \cong fR$.

Every right mininjective ring satisfies $(*)$ [**13**, Lemma 3.4]. A ring R is called *right minsymmetric* [**13**] if kR simple implies that Rk is simple, $k \in R$. A right mininjective ring is right minsymmetric [**13**, Theorem 1.14].

We prove that the mild condition of right minsymmetry ensures the existence of a Nakayama permutation of basic idempotents of a semiperfect ring R satisfying $(*)$ for which $Soc(eR) \neq 0$ for every local idempotent e. As even a commutative local ring with non-zero socle may not be minfull, this generalizes Nicholson and Yousif's result. For example,

$$S = \left\{ \begin{pmatrix} q & r \\ 0 & q \end{pmatrix} : q \in \mathbb{Q} \text{ and } r \in \mathbb{R} \right\},$$

is a commutative local ring which is not mininjective because $Soc(S) = \begin{pmatrix} 0 & \mathbb{R} \\ 0 & 0 \end{pmatrix}$ is not simple (see [**13**, Remark 1.4]).

In [**13**, Theorem 4.17] Nicholson and Yousif proved that a semiperfect right continuous ring with large right socle admits a Nakayama permutation of its basic idempotents. Also Yousif [**18**, Theorem 1] proved that a right CS ring R such that the R-dual of every simple left R-module is simple, semiperfect and admits a Nakayama permutation of its basic idempotents. Improving upon these results, we prove that these classes of semiperfect rings are right minsymmetric and satisfy $(*)$.

Osofsky [**15**] proved that a ring R is right PF if and only if it is semiperfect right self-injective with $Soc(R_R) \trianglelefteq R_R$. Recently, in their remarkable paper, Gómez Pardo and Guil Asensio [**5**, Corollary 2.8] proved that a right CS right cogenerator ring is right PF. We give some necessary and sufficient conditions for a semiperfect ring R, which cogenerates every 2-generated right R-module, to be right PF. In particular, we prove that a semiperfect ring R is right PF if and only if $J(R) \subseteq Z(R_R)$ and R cogenerates every 2-generated right R-module. As every left CS right Kasch ring is semiperfect [**6**], this extends [**14**, Theorem 2.8] where it is proved that a left CS ring R with $J(R) \subseteq Z(R_R)$ which cogenerates every 2-generated right R-module is right PF. In section 2 we study some relationships between right mininjective, right minsymmetric and left minannihilator rings (that is, rings for which every minimal left ideal is a left annihilator).

1. Nakayama permutations. A module M is called a CS module if every submodule of M is essential in a direct summand. A CS module is called *continuous* if it satisfies the following condition:

(C_2). *Every submodule of M that is isomorphic to a direct summand of M is itself a direct summand.*

A CS module is called *quasi-continuous* if it satisfies the following condition:

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(C_3). *The sum of any two direct summands of M whose intersection is zero is a direct summand.*

Continuous modules are quasi-continuous [**8**]. A ring R is said to be right CS if R_R is a CS module. Right continuous and right quasi-continuous rings are defined similarily. For a detailed study of CS, quasi-continuous and continuous modules we refer the reader to [**8**].

We begin with the following result which will be used repeatedly in this paper.

LEMMA 1.1. *Let e be a local idempotent of a ring R. Then*

(1) $J+(1-e)R$ *is the unique maximal right ideal of R containing* $(1-e)R$*;*

(2) $J+R(1-e)$ *is the unique maximal left ideal of R containing* $R(1-e)$*;*

(3) If $0 \neq K \subseteq Re$ *is an annihilator left ideal then* $Soc(R_R)e \subseteq K$*, where e is a local idempotent.*

Proof. (1) Consider the map $1 \to e+eJ$ from $R \to eR/eJ$. This is an epimorphism with kernel $J+(1-e)R$. As e is local, $J+(1-e)R$ is a maximal right ideal. If $(1-e)R \subseteq I$, where I is a maximal right ideal, then $(1-e)R+J \subseteq I+J = I$. This gives (1).

(2) This is proved similarly to (1).

(3) As $(1-e)R \subseteq r_R(K) \neq R$, by (1), $r_R(K) \subseteq J+(1-e)R$. Thus

$$\begin{aligned} K = l_Rr_R(K) &\supseteq l_R((1-e)R+J) = l_R((1-e)R) \cap l_R(J) \\ &\supseteq Re \cap Soc(R_R) = Soc(R_R)e. \end{aligned}$$ ■

The following result generalizes the analogous result proved for right minfull rings in [**13**, Theorem 3.7].

THEOREM 1.2. *Let R be a semiperfect ring satisfying* $(*)$ *with* $Soc(eR) \neq 0$ *for every local idempotent e. If R is right minsymmetric, then R admits a Nakayama permutation of its basic idempotents. Moreover, if every simple submodule* $K \subseteq Re$*, where e is a local idempotent, is a left annihilator, then*

(1) $\{Soc(Re_1), \ldots, Soc(Re_n)\}$ *is a complete irredundant set of representatives of simple left R-modules, where* $e_1, \ldots, e_n$ *is a basic set of idempotents of R;*

(2) $Soc(R_R) = Soc({}_RR)$*;*

(3) R is right minfull.

Proof. Let $e_1, \ldots, e_n$ be a basic set of idempotents of R. As $Soc(e_iR) \neq 0$ and R satisfies $(*)$, $Soc(e_iR)$ is homogeneous for each i. Let $K_i \subseteq Soc(e_iR)$ be simple. By $(*)$, $i \neq j$ implies $K_i \not\cong K_j$. Thus there exists a permutation σ of $\{1, \ldots, n\}$ such that $e_{\sigma(i)}R/e_{\sigma(i)}J \overset{\phi_i}{\cong} K_i$ for each i. Let $\phi_i(\overline{e_{\sigma(i)}}) = k_i$. Then $K_i = k_iR$ and as $k_ie_{\sigma(i)} = k_i$, $Rk_i \subseteq Re_{\sigma(i)}$. As $e_ik_i = k_i \neq 0$, $e_iRk_i \neq 0$ for each i. Also as R is right minsymmetric and $k_iR = K_i$ is a simple, Rk_i is simple for each i. So, by [**1**, Exercise 27.9], $Rk_i \cong Re_i/Je_i$. Thus σ is a Nakayama permutation.

Now suppose that every minimal left ideal contained in Re, where $e^2 = e$ is local, is a left annihilator. Let $K \subseteq Re$ be simple. By Lemma 1.1 $Soc(R_R)e \subseteq K$. As R is right Kasch, $Soc(R_R)e \neq 0$. So $K = Soc(R_R)e$ implying that $K = Soc(Re) = Soc(R_R)e$. Thus $Soc(R_R) = Soc({}_RR)$ and $Rk_i = Soc(Re_{\sigma(i)})$ for each i. This gives (1). Also as $Soc(R_R)e$ is a simple left R-module for every local idempotent e, by [**13**, Theorem 3.2], R is right mininjective and thus right minfull. □

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Proposition 1.3. *Let R be a semiperfect right CS ring with $Soc(R_R) \subseteq Soc({}_RR)$. Then R is right minsymmetric.*

Proof. Let kR be a minimal right ideal of R. As R is right CS, there exists an idempotent e of R such that $kR \trianglelefteq eR$. Clearly eR is uniform and so e is a local idempotent. As $kR \subseteq Soc(R_R) \subseteq Soc({}_RR)$, $R(1-e)+J \subseteq l_R(k) \neq R$. But $R(1-e)+J$ is a maximal left ideal by Lemma 1.1, so $R(1-e)+J = l_R(k)$. Thus $Rk \cong R/l_R(k)$ is a minimal left ideal. □

Remark 1.4. Proposition 1.3 also holds for right min-CS rings (that is rings whose minimal right ideals are essential in direct summands). The proof of Proposition 1.3 shows that if R is any ring with $Soc(R_R) \subseteq Soc({}_RR)$ then for any $0 \neq k \in Soc(eR)$, where e is a local idempotent, Rk is simple. This extends [**13**, Proposition 3.3(3)] where this result is proved for semiperfect right mininjective rings.

Lemma 1.5. ([**9**, Theorem 4]). *In a quasi-continuous module closures of isomorphic submodules are isomorphic. In particular, a semiperfect right quasi-continuous ring satisfies* $(*)$.

Proposition 1.6. *Let R be a semiperfect right CS ring with $Soc(R_R) \trianglelefteq R_R$. If $Soc(R_R) \subseteq Soc({}_RR)$, then R is a right minsymmetric ring satisfying $(*)$. In particular, R admits a Nakayama permutation of its basic idempotents.*

Proof. As $Soc(R_R) \subseteq Soc({}_RR)$ and $Soc(R_R) \trianglelefteq R_R$, we have $Soc({}_RR) \trianglelefteq R_R$. Thus R is left Kasch (see [**11**, Lemma 3]). By [**18**, Lemma 1], R satisfies the right C_2 condition and so R is right continuous. Now the result follows from Proposition 1.3, Lemma 1.5 and Theorem 1.2. □

The last sentence of the following result was proved in [**13**, Theorem 4.17].

Corollary 1.7. *A semiperfect right continuous ring R with $Soc(R_R) \trianglelefteq R_R$ is right minsymmetric and satisfies $(*)$. In particular, R admits a Nakayama permutation of its basic idempotents.*

Proof. By [**8**, Proposition 3.5] $J = Z(R_R)$ and so $Soc(R_R) \subseteq r_R(Z(R_R)) = r_R(J) = Soc({}_RR)$. Now the result follows from Proposition 1.6. □

Recently Gómez Pardo and Yousif [**6**] proved that a right CS left Kasch ring R is semiperfect right continuous with $Soc({}_RR) \trianglelefteq R_R$. Thus from Corollary 1.7 we have

Corollary 1.8. *A right CS left Kasch ring R is semiperfect. Moreover, if $Soc({}_RR) \subseteq Soc(R_R)$ then R is right minsymmetric with $(*)$ such that $Soc(R_R) \trianglelefteq R_R$.*

Corollary 1.9 ([**18**, Theorem 1]). *A right CS ring R such that the R-dual of every simple left R-module is simple is semiperfect right minsymmetric with $(*)$ such that $Soc(R_R) \trianglelefteq R_R$.*

Proof. By [**13**, Proposition 2.2] R is left mininjective and so $Soc({}_RR) \subseteq Soc(R_R)$. Thus the result follows from Corollary 1.8. □

The following result was proved by Nicholson and Yousif in [**12**, Theorem 2.3] for right generalized pseudo-Frobenius (GPF) rings (that is semiperfect right principally injective rings with large right socle). We prove this result more generally for a semiperfect right principally injective ring R with $Soc(eR) \neq 0$ for every local idempotent e. The author does not know whether such rings are right GPF.

THEOREM 1.10. *Let $e_1, ..., e_n$ be a basic set of idempotents in a semiperfect right principally injective ring with $Soc(e_iR) \neq 0$ for each i. Then there exist elements $k_1, ..., k_n$ of R and a permutation σ of $\{1, ..., n\}$ such that the following hold for each i:*

(1) $Rk_i = Soc(Re_{\sigma(i)}) \cong Re_i/Je_i$ is simple and essential in $Re_{\sigma(i)}$. In particular, Re is uniform for every local idempotent e;

(2) $Soc(e_iR)$ is homogeneous with each simple submodule isomorphic to $e_{\sigma(i)}R/e_{\sigma(i)}J$;

(3) $\{Rk_1, ..., Rk_n\}$ is a complete irredundant set of representatives of isomorphism classes of simple left R-modules;

(4) $\{k_1R, ..., k_nR\}$ is a complete irredundant set of representatives of isomorphism classes of simple right R-modules;

(5) $Soc(R_R) = Soc({}_RR) = \oplus_{i=1}^n Rk_iR$ is essential in ${}_RR$ and is finitely generated as a left R-module;

(6) Rk_iR is the homogeneous component of $Soc(R_R)$ containing k_iR and Rk_iR is the homogeneous component of $Soc({}_RR)$ containing Rk_i.

Proof. Let $k_1, ..., k_n$ and σ be as in Theorem 1.2. Then (2), (3) and (4) follow from Theorem 1.2.

(1) Let $0 \neq b \in Re_{\sigma(i)}$. As $(1 - e_{\sigma(i)})R \subseteq r_R(b)$, by Lemma 1.1, $r_R(b) \subseteq (1 - e_{\sigma(i)})R + J \subseteq r_R(k_i)$ because $Rk_i \subseteq Re_{\sigma(i)}$ is simple. Thus, by [**12**, Lemma 1.1], $Rk_i \subseteq Rb$.

(6) By [**13**, Theorem 1.14(3)] the homogeneous component of $Soc(R_R)$ containing k_iR is Rk_iR. Let S_i be the homogeneous component of $Soc({}_RR)$ containing Rk_i. Clearly $Rk_iR \subseteq S_i$. Let $f_1, ..., f_m$ be a complete orthogonal set of primitive idempotents of R. By (1) above, $S_i = \oplus\{Soc(Rf_j) : Rf_j \cong Re_{\sigma(i)}\}$. Now if $Rf_j \cong Re_{\sigma(i)}$ then there exists $b \in R$ such that $Soc(Rf_j) = Soc(Re_{\sigma(i)})b = Rk_ib$. Thus $S_i \subseteq Rk_iR$.

(5) Follows from Theorem 1.2(2), and (1) and (6) above. □

COROLLARY 1.11. *Let R be a semiperfect right principally injective ring with $Soc(eR) \neq 0$ for every local idempotent e. If $S = Soc(R_R) = Soc({}_RR)$ then*

(1) $Z(R_R) = J = Z({}_RR)$;

(2) $l_R(S) = J = r_R(S)$;

(3) $l_R(J) = S = r_R(J)$.

Proof. Using Theorem 1.10, the proof follows the same lines as that of [**12**, Corollary 2.2]. □

There exists a left and right Artinian ring R such that every left ideal of R is a left annihilator, but R is not quasi-Frobenius [**2**, page 70]. Clearly R is right principally injective with large right socle. But R is not left mininjective as right Artinian, left and right mininjective rings are quasi-Frobenius [**13**, Corollary 4.8].

The next result gives several characterizations of right PF rings. Since left CS right Kasch rings are semiperfect [**6**], the implication '(4) ⇒ (1)' extends [**14**, Theorem 2.8].

Theorem 1.12. *Let R be a semiperfect ring which cogenerates every 2-generated right R-module. Then the following are equivalent:*

(1) R is right PF;
(2) $Soc(R_R) \subseteq Soc(_R R)$;
(3) $Soc(Re) \neq 0$ for every local idempotent e of R;
(4) $J(R) \subseteq Z(R_R)$.

Proof. As R cogenerates every cyclic right R-module, every right ideal is a right annihilator (see for instance [**1**, Lemma 25.2]) and so R is left principally injective. By [**12**, Theorem 1.14] $Soc(_R R) \subseteq Soc(R_R)$. Also as R is right Kasch $Soc(R_R)e \neq 0$ for every local idempotent e of R.

(1) $\Rightarrow$ (2) This is well-known.

(2) $\Rightarrow$ (3) We have $Soc(R_R) = Soc(_R R)$. Thus $Soc(Re) = Soc(_R R)e = Soc(R_R)e \neq 0$ for every local idempotent e of R.

(3) $\Rightarrow$ (4) This follows from Corollary 1.11.

(4) $\Rightarrow$ (1) As $J(R) \subseteq Z(R_R)$, $Soc(R_R) \subseteq r_R(Z(R_R)) \subseteq r_R(J(R)) = Soc(_R R)$ and so $Soc(R_R) = Soc(_R R)$. For every local idempotent e of R, $Soc(Re) = Soc(_R R)e = Soc(R_R)e \neq 0$ and so, by the proof of Theorem 1.10(1), eR is uniform. Thus $E(eR)$, the injective hull of eR, is also uniform for every local idempotent e.

Fix a local idempotent e in R. We show that $eR = E(eR)$. Suppose, on the contrary, $a \in E(eR) \backslash eR$. As $eR + aR$ is uniform with non-zero socle (as $Soc(eR) \neq 0$), it is finitely co-generated right R-module [**1**, Proposition 10.7]. So, by hypothesis, there exists an embedding $eR + aR \to R^n$ for some natural number n. As $eR + aR$ is uniform, we have an embedding $\sigma : eR + aR \to fR$ for some local idempotent f in R (see for example [**14**, Lemma 2.6]). As $a \notin eR$, $\sigma(eR)$ is a proper submodule of fR and so $\sigma(eR) \subseteq fJ \subseteq Z(R_R)$. But as $r_R(\sigma(e)) = (1 - e)R$, this is a contradiction. Thus $eR = E(eR)$ is injective for every local idempotent e of R and so R is right self-injective. As R is right Kasch, R is right PF [**1**, Proposition 18.15]. □

It is not known whether a right perfect right self-injective ring is right PF. For a detailed account of this problem we refer the reader to [**4**].

Proposition 1.13. *Let R be a semiperfect ring satisfying* $(*)$ *with $Soc(eR) \neq 0$ for every local idempotent e. If $Soc(R_R)$ is finitely generated then for any projective right R-modules P and Q, $Soc(P) \cong Soc(Q)$ implies that $P \cong Q$.*

Proof. Let $e_1, \ldots, e_n$ be a basic set of idempotents of R. By [**1**, Theorem 27.11] for each i there exist sets I_i and J_i such that

$$P \cong \oplus_{i=1}^n (e_i R)^{(I_i)} \text{ and } Q \cong \oplus_{i=1}^n (e_i R)^{(J_i)}.$$

Now $Soc(P) \cong Soc(Q)$ yields $\oplus_{i=1}^n Soc(e_i R)^{(I_i)} \cong \oplus_{i=1}^n Soc(e_i R)^{(J_i)}$. By hypothesis there exists a set $\{S_1, \ldots, S_n\}$ of mutually non-isomorphic simple right R-modules such that $Soc(e_i R) \cong S_i^{k_i}$ for some natural number k_i $(1 \leq i \leq n)$. Let $|S|$ denote the cardinality of set S. Then $\oplus_{i=1}^n Soc(e_i R)^{(I_i)} \cong \oplus_{i=1}^n Soc(e_i R)^{(J_i)}$ implies that $\oplus_{i=1}^n (S_i^{k_i})^{(I_i)} \cong \oplus_{i=1}^n (S_i^{k_i})^{(J_i)}$ which, in turn, gives $(S_i^{k_i})^{(I_i)} \cong (S_i^{k_i})^{(J_i)}$ for each i. Thus, $|k_i \times I_i| = |k_i \times J_i|$ and so $|I_i| = |J_i|$, proving that $P \cong Q$. □

Remark 1.14. Let R be a semiperfect ring with basic set of idempotents $\{e_1, \ldots, e_n\}$. Let there exist a set $\{S_1, \ldots, S_n\}$ of mutually non-isomorphic simple right

R-modules, such that $Soc(e_iR) \cong S_i^{k_i}$ for some natural number k_i ($1 \leq i \leq n$). Then the proof of Proposition 1.13 shows that for any projective right R-modules P and Q, $Soc(P) \cong Soc(Q)$ implies that $P \cong Q$.

The following result slightly strengthens [**13**, Theorem 3.16].

Corollary 1.15. *Let R be a left minfull ring with $r_R l_R(K) = K$ for every simple right ideal $K \subseteq eR$, where $e^2 = e$ is local. Then for any projective right R-modules P and Q, $Soc(P) \cong Soc(Q)$ implies that $P \cong Q$.*

Proof. Let $e_1, ..., e_n$ be a basic set of idempotents of R. By the proof of Theorem 1.2(1) $\{Soc(e_1R), ..., Soc(e_nR)\}$ is a complete irredundant set of representatives of simple right R-modules. Thus the result follows from Remark 1.14. □

As every principal right ideal of a left principally injective ring is a right annihilator, the following is a consequence of Corollary 1.15.

Corollary 1.16. *Let R be a semiperfect left principally injective ring with $Soc(Re) \neq 0$ for every local idempotent e of R. Then for any projective right R-modules P and Q, $Soc(P) \cong Soc(Q)$ implies that $P \cong Q$.*

2. Mininjective rings. As mentioned above every right mininjective ring is right minsymmetric. Also a left minannihilator ring is right mininjective if $Soc(R_R) \subseteq Soc(_RR)$ [**13**, Proposition 2.4] or $Soc(_RR) \trianglelefteq {_RR}$ [**13**, Corollary 2.5]. Thus right mininjective rings are closely related to right minsymmetric and left minannihilator rings. In this section we study relationships between these conditions.

Consider the following condition on a ring R:

(∗∗) *Every minimal right ideal of R is isomorphic to eR/eJ for some local idempotent e of R.*

Clearly every semiperfect ring satisfies (∗∗). A ring R satisfies (∗∗) if and only if for every minimal right ideal K of R there exists a local idempotent e of R such that $Ke \neq 0$ (see [**1**, Exercise 27.9]).

Lemma 2.1. ([**13**, Lemma 3.1]). *Let R be a ring satisfying (∗∗). Then R is right mininjective if and only if for every local idempotent e of R either $Soc(R_R)e$ is simple or zero.*

Proposition 2.2. *Let R be a ring satisfying (∗∗). Then*

(1) If for every local idempotent e of R there exists a minimal left ideal in $Soc(Re)$ which is a left annihilator then R is right mininjective;

(2) If for every local idempotent e either $Soc(Re) = 0$ or every minimal left ideal contained in Re is a left annihilator, then the following are equivalent:

(a) R is right mininjective;

(b) R is right minsymmetric;

(c) $Soc(R_R) \subseteq Soc(_RR)$.

Proof. (1) Let e be a local idempotent of R and K be a minimal left ideal in Re such that $l_R r_R(K) = K$. By Lemma 1.1 $Soc(R_R)e \subseteq K$. So either $Soc(R_R)e = 0$ or $Soc(R_R)e = K$ is simple. Thus, by Lemma 2.1, R is right mininjective.

(2) (a) $\Rightarrow$ (b) follows from [**13**, Theorem 1.14] and (b) $\Rightarrow$ (c) is clear.

(c) $\Rightarrow$ (a) Let e be a local idempotent of R. In view of Lemma 2.1 we have to show that $Soc(R_R)e$ is either zero or simple. If $Soc(Re) = 0$ then $Soc(R_R)e \subseteq Soc(_RR)e = Soc(Re) = 0$. Now suppose that $Soc(Re) \neq 0$ and let $K \subseteq Re$ be simple. By Lemma 1.1 $Soc(R_R)e \subseteq K$. Thus either $Soc(R_R)e = 0$ or $Soc(R_R)e = K$ is simple. □

LEMMA 2.3. *Let R be a right Kasch ring with every minimal left ideal $K \subseteq Re$ a left annihilator, where e is a local idempotent. Then either $Soc(Re) = 0$ or $Soc(Re) = Soc(R_R)e$ is a simple left R-module.*

Proof. Let $Soc(Re) \neq 0$ and $K \subseteq Re$ be simple. By Lemma 1.1 $Soc(R_R)e \subseteq K$. As R is right Kasch, $Soc(R_R)e \neq 0$ and so $K = Soc(R_R)e$. As K is an arbitrary simple submodule of Re, $Soc(Re) = Soc(R_R)e$ is a simple left R-module. □

The equivalence of following conditions was observed by Nicholson and Yousif in [**13**, Proposition 3.3 (4)] for semiperfect right mininjective right Kasch rings. We prove that these equivalences also hold for non-semiperfect rings.

PROPOSITION 2.4. *Let e be a local idempotent in a right mininjective right Kasch ring R. Then the following are equivalent:*

(1) $l_Rr_R(K) = K$ for every minimal left ideal $K \subseteq Re$;

(2) $Soc(Re) = Soc(R_R)e$;

(3) $Soc(Re)$ is simple.

Proof. As R is right Kasch, $Soc(R_R)e \neq 0$. Also by [**13**, Theorem 1.14] $Soc(R_R) \subseteq Soc(_RR)$.

(1) $\Rightarrow$ (2) As $0 \neq Soc(R_R)e \subseteq Soc(_RR)e = Soc(Re)$, by Lemma 2.3 we find $Soc(Re) = Soc(R_R)e$.

(2) $\Rightarrow$ (3) As R is right mininjective, by [**13**, Lemma 3.1], $Soc(R_R)e$ is zero or simple. But as $Soc(R_R)e \neq 0$, $Soc(Re) = Soc(R_R)e$ is simple.

(3) $\Rightarrow$ (1) Let $K \subseteq Re$ be simple. Then $0 \neq Soc(R_R)e \subseteq Soc(_RR)e = Soc(Re) = K$. This gives $K = Soc(R_R)e \subseteq l_R(J)e \cong Hom(\frac{eR}{eJ}, R)$. As R is right mininjective, the R-dual of every simple right R-module is either zero or a simple left R-module [**13**, Proposition 2.2]. Thus $K = l_R(J)e = l_R(J) \cap Re = l_R(J) \cap l_R((1-e)R) = l_R(J + (1-e)R)$. □

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