Glasgow Math. J. **41** (1999) 1–11. © Glasgow Mathematical Journal Trust 1999. Printed in the United Kingdom

LOW DIMENSIONAL HOMOTOPY FOR MONOIDS II: GROUPS

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(Received 28 January, 1997)

Introduction. Consider a group presentation

$$\hat{\mathcal{P}} = <\mathbf{x}; \boldsymbol{r}> . \tag{1}$$

Here $\mathbf{x}$ is a set and $\mathbf{r}$ is a set of non-empty, cyclically reduced words on the alphabet $\mathbf{x} \cup \mathbf{x}^{-1}$ (where $\mathbf{x}^{-1}$ is a set in one-to-one correspondence $x \leftrightarrow x^{-1}$ with $\mathbf{x}$). We assume throughout that $\hat{\mathcal{P}}$ is finite. Let $\hat{F}$ be the free group on $\mathbf{x}$ (thus $\hat{F}$ consists of free equivalence classes $[W]$ of word on $\mathbf{x} \cup \mathbf{x}^{-1}$), and let N be the normal closure of $\{[R] : R \in \mathbf{r}\}$ in $\hat{F}$. Then the group $G = G(\hat{\mathcal{P}})$ defined by $\hat{\mathcal{P}}$ is $\hat{F}/N$. We will write $W_1 =_G W_2$ if $[W_1]N = [W_2]N$.

Associated with $\hat{\mathcal{P}}$ is a certain crossed module $(\Sigma, \hat{F}, \partial)$. This can be described in several different (but equivalent) ways:

(a) topologically as the relative second homotopy group $\pi_2(\mathcal{K}, \mathcal{K}^{(1)})$ where $\mathcal{K}$ is the standard 2-complex modelled on $\hat{\mathcal{P}}$ and $\mathcal{K}^{(1)}$ is its 1-skeleton;
(b) algebraically in terms of sequences;
(c) geometrically in terms of pictures.

Also, there is the (absolute) second homotopy group $\pi_2(\hat{\mathcal{P}}) = \text{Ker}\ \partial$, which is a ZG-module. Elements of this can be represented algebraically by identity sequences, or geometrically by spherical pictures. See [**1**], [**3**], [**10**] for details. We will use the second description (b), and refer the reader to [**10**] for basic terminology and results concerning identity sequences. (However, for the reader's convenience we give a brief account of this material in §1 below.)

Now $\hat{\mathcal{P}}$ gives rise to a *monoid* presentation $\mathcal{P}$ for G, where

$$\mathcal{P} = [\mathbf{x}, \mathbf{x}^{-1}; R = 1 (R \in \boldsymbol{r}), x^{\varepsilon} x^{-\varepsilon} = 1 (x \in \mathbf{x}, \varepsilon = \pm 1)].$$

The monoid defined by $\mathcal{P}$ is the quotient of the free monoid F on $\mathbf{x} \cup \mathbf{x}^{-1}$ by the smallest congruence ρ generated by the relations. A typical element of this monoid is a congruence class $W\rho$ ($W \in F$), and we have an isomorphism from this monoid to G, given by

$$W\rho \mapsto [W]N \ (W \in F).$$

We will often identify $W\rho$ and $[W]N$ (if no confusion can arise) and will denote this element by $\bar{W}$.

Now in [**12**] (see also [**11**]) we associated with any monoid presentation $\mathcal{Q}$ a 2-complex $\mathcal{D}(\mathcal{Q})$ ("the 2-complex of monoid pictures") and we showed that the first homology group $H_1(\mathcal{D}(\mathcal{Q}))$ has considerable significance. The fundamental groups of $\mathcal{D}(\mathcal{Q})$ are also of considerable interest and have been investigated by Guba and Sapir [**7**], and Kilibarda [**8**].

For our presentation $\mathcal{P}$ above, the 2-complex $\mathcal{D}(\mathcal{P})$ has underlying graph as follows. The vertex set is F and the edge set consists of all the atomic monoid pictures (U,T,ε,V) $(U,V\in F, T\in \mathbf{r} \cup \{xx^{-1}, x^{-1}x : x\in \mathbf{x}\},\ \varepsilon = \pm 1)$ (Figure 1). The initial, terminal and inversion functions $\iota, \tau, ^{-1}$ are given by

$$\iota(U, T, 1, V) = \tau(U, T, -1, V) = UTV,$$
$$\iota(U, T, -1, V) = \tau(U, T, 1, V) = UV,$$
$$(U, T, \varepsilon, V)^{-1} = (U, T, -\varepsilon, V).$$

There are obvious (compatible) left and right actions of F on this graph. Paths in this graph are called *(monoid) pictures*. The left and right actions of F extend to actions on pictures. The defining paths of $\mathcal{D}(\mathcal{P})$ are the paths

$$[\mathrm{A}, \mathrm{B}] = (\mathrm{A} \cdot \iota(\mathrm{B}))(\tau(\mathrm{A}) \cdot \mathrm{B})(\mathrm{A}^{-1} \cdot \tau(\mathrm{B}))(\iota(\mathrm{A}) \cdot \mathrm{B}^{-1}). \tag{2}$$

(A,B are edges of the graph.) See [**11**], [**12**] for further details.

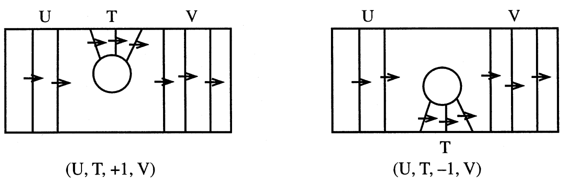

Figure 1

Now elements of the fundamental groupoid $\pi_1(\mathcal{D}(\mathcal{P}))$ are represented by monoid pictures. Consequently, in view of (c) above, it is natural to ask for our group G what is the relationship (if any) between $\pi_1(\mathcal{D}(\mathcal{P}))$ and Σ.

In fact to obtain a relationship we need to modify $\mathcal{D}(\mathcal{P})$ by adding some extra defining paths to it. For each $x\in\mathbf{x}$, $\varepsilon\ \pm 1$ we have the spherical monoid picture as in Figure 2. (This is a path of length 2 in $\mathcal{D}(\mathcal{P})$.) We let $\mathcal{D}(\mathcal{P})^*$ be the 2-complex obtained from $\mathcal{D}(\mathcal{P})$ by adding the extra defining paths

$$W \cdot \mathrm{P} \cdot V \qquad (P \text{ as in Figure 2, } W, V \in F). \tag{3}$$

Now let Σ^* be the collection of all elements of the fundamental groupoid $\pi_1(\mathcal{D}(\mathcal{P})^*)$ represented by monoid pictures which start at *freely reduced* words on $\mathbf{x} \cup \mathbf{x}^{-1}$, and end at the empty word. We show in §2 that a crossed module structure $(\Sigma^*, \hat{F}, \partial^*)$ can be imposed on Σ^*, and we prove (Theorem 1) that there is a crossed module isomorphism

$$\psi : \Sigma \to \Sigma^*.$$

By restriction, we then get a ZG-isomorphism

$$\pi_2(\hat{\mathcal{P}}) = \mathrm{Ker}\ \partial \xrightarrow{\psi} \mathrm{Ker}\ \partial^* = \pi_1(\mathcal{D}(\mathcal{P})^*, 1).$$

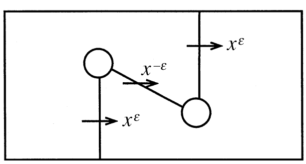

Figure 2

The notion of finite derivation type (*FDT*) was introduced by Squier in his posthumously published article [**13**]. In our terminology, a monoid presentation $\mathcal{Q}$ is *FDT* if there is a finite set X of spherical monoid pictures over $\mathcal{Q}$ such that the 2-complex $\mathcal{D}(\mathcal{Q})^X$ obtained from $\mathcal{D}(\mathcal{Q})$ by adding the defining paths

$$W \cdot \mathrm{P} \cdot V \quad (W, V \in F, \mathrm{P} \in X)$$

has trivial fundamental groups. A finitely presented monoid S is *FDT* if some (and hence, as shown by Squier [**13**], any) finite presentation of S is *FDT*. Monoids of finite derivation type have been discussed in [**4**], [**5**], [**9**], [**12**].

Now if G is a *group* then it has been shown by Cremanns and Otto [**5**] that G is *FDT* if and only if for some (and hence, in fact, any) finite *group* presentation $\hat{\mathcal{P}}$ of G, the ZG-module $\pi_2(\hat{\mathcal{P}})$ is finitely generated.

We give in §3 a simple proof of the Cremanns/Otto result mentioned above. Let $\hat{\mathcal{P}}$, $\mathcal{P}$ be as in (1), (2) respectively. We first establish the easy fact that all the fundamental groups of $\mathcal{D}(\mathcal{P})^*$ are isomorphic. Using this we prove (Theorem 2) that $\mathcal{P}$ is *FDT* if and only if the ZG-module $\pi_1(\mathcal{D}(\mathcal{P})^*,1)$ is finitely generated. Then in view of the isomorphism $\pi_2(\hat{\mathcal{P}}) \cong \pi_1(\mathcal{D}(\mathcal{P})^*,1)$ (§2), the Cremanns/Otto result follows.

It should be noted that for any group presentation $\hat{\mathcal{P}} = <\mathbf{x}; \mathbf{r}>$ there is a standard exact sequence

$$0 \to \pi_2(\hat{\mathcal{P}}) \to \oplus_{R\in\mathbf{r}} ZGe_R \to \oplus_{x\in\mathbf{x}} ZGe_x \to ZG \to Z \to 0$$

of ZG-modules (see for example [**3**], [**10**]). Using this, together with the generalised Schanuel Lemma [**2**], one easily obtains the (well-known) result that a finitely presented group G is of type FP_3 [**2**] if and only if for some (in fact any) finite presentation $\hat{\mathcal{P}}$ of G, $\pi_2(\hat{\mathcal{P}})$ is finitely generated. Thus, for finitely presented groups, *FDT* and FP_3 are equivalent. (This result is obtained in [**5**].)

1. Preliminaries. If P, P′ are paths in $\mathcal{D}(\mathcal{P})^*$ then we write $\mathrm{P} \sim \mathrm{P}'$ if P, P′ are equivalent (homotopic) in $\mathcal{D}(\mathcal{P})^*$. The equivalence class of P will be denoted by $<\mathrm{P}>$. We will assume the reader has some familiarity with the material regarding monoid pictures in [**12**, §§2,5].

An edge of $\mathcal{D}(\mathcal{P})^*$ of the form $(U, x^{\varepsilon}x^{-\varepsilon}, \pm 1, V)$ $(U,V\in F,\ x\in\mathbf{x},\ \varepsilon = \pm 1)$ will be called *trivial*, and a path will be called trivial if all its edges are trivial. Two vertices W_1, W_2 can be connected by a trivial path if and only if W_1 and W_2 are freely equivalent (the chosen path connecting W_1 to W_2 then gives a method of freely transforming W_1 to W_2). In view of the defining paths (3) of $\mathcal{D}(\mathcal{P})^*$, we have that *any two trivial paths between a given pair of vertices W_1, W_2 are homotopic in* $\mathcal{D}(\mathcal{P})^*$. This key observation allows us to replace a trivial subpath T of a given path P by any other trivial path T′ (where $\iota(\mathrm{T}') = \iota(\mathrm{T})$, $\tau(\mathrm{T}') = \tau(\mathrm{T})$) without affecting the homotopy type of P.

Suppose P is a path in $\mathcal{D}(\mathcal{P})^*$ with $\iota(\mathcal{P}) = W$, $\tau(\mathrm{P}) = Z$, and let T, $\bar{T}$ be trivial paths in $\mathcal{D}(\mathcal{P})^*$ from W_1 to W, Z_1 to Z respectively, where W_1, Z_1 are the unique reduced words freely equivalent to W, Z. Then the picture $TP\bar{T}^{-1}$ will be said to be obtained from P by *freely reducing the boundary* of P, and will be denoted by P*. Obviously this notation is ambiguous because P* depends on T, $\bar{T}$. However, since we will be working up to homotopy in $\mathcal{D}(\mathcal{P})^*$, we can, by our comment in the previous paragraph, allow ourselves to choose any trivial paths T, $\bar{T}$ that suit our purpose. This simple, but key point will be used over and over again, without further comment.

Another important point is the following.

Suppose that P_1 is obtained from P by inserting into P a pair of parallel arcs with labels x^{ε}, $x^{-\varepsilon}$ $(x \in \mathbf{x}, \varepsilon = \pm 1)$. Then $P_1^* \sim P^*$. (4)

This is because, when we freely reduce the boundary of P_1 we can begin as in Figure 3. This creates a cancelling pair of discs which can be removed.

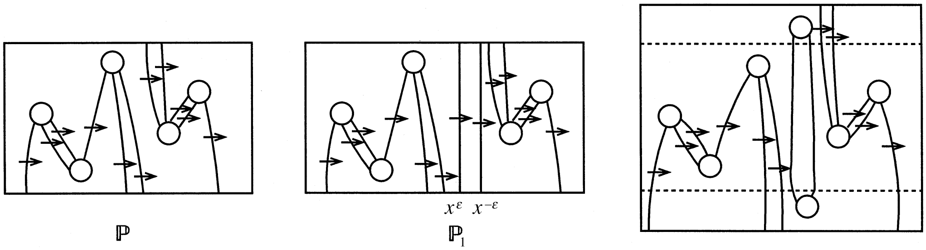

Figure 3

If P, P′ are paths in $\mathcal{D}(\mathcal{P})^*$ then we write P + P′ for the path $(\mathrm{P}\cdot\iota(\mathrm{P}'))(\tau(\mathrm{P})\cdot\mathrm{P}')$. Then for paths $\mathrm{P}_1, \mathrm{P}_2, \cdots, \mathrm{P}_n$ we define $\mathrm{P}_1 + \mathrm{P}_2 + \cdots + \mathrm{P}_n$ inductively to be $(\mathrm{P}_1 + \cdots \mathrm{P}_{n-1}) + \mathrm{P}_n$.

For any $U \in F$, say $U = x_1 x_2 \cdots x_m$ $(x_i \in \mathbf{x} \cup \mathbf{x}^{-1}$ for $i = 1, \cdots, m)$ we denote the picture

$$\prod_{i=1}^{m} (x_1 \cdots x_{i-1}, x_i x_i^{-1}, -1, x_{i-1}^{-1} \cdots x_1^{-1})$$

(see Figure 4) by $T_{UU^{-1}}$.

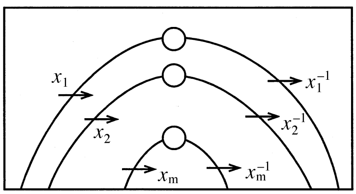

Figure 4

For $R \in \mathbf{r}$, $U \in F$, $\varepsilon \in \{-1,1\}$ we define $E_{R,U,\varepsilon}$ as follows:

$$E_{R,U,\varepsilon} = \begin{cases} (U, R, 1, U^{-1}) & \varepsilon = 1, \\ (U, R, -1, R^{-1}U^{-1}) & \varepsilon = -1. \end{cases}$$

We complete this section by giving a brief account of Σ in terms of sequences. (For further details, as well as for the elementary theory of crossed modules, see **[10]**. See also **[6]** for the theory of crossed modules.)

Let $\mathbf{r}^F$ be the set of all elements of F of the form $WR^{\varepsilon}W^{-1}$ $(W \in F, R \in \mathbf{r}, \varepsilon = \pm 1)$. We consider finite sequences $\sigma = (c_1, c_2, \cdots, c_m)$ of elements of $\mathbf{r}^F$. We define certain operations on sequences as follows.

(I) Replace some term $c_i = WR^{\varepsilon}W^{-1}$ by $c_i' = W'R^{\varepsilon}W'^{-1}$ where W' is a word freely equivalent to W.

(II) Delete two consecutive terms if one is identically equal to the inverse of the other.
(III) Replace two consecutive terms c_i, c_{i+1} by c_{i+1}, $c_{i+1}^{-1}\ c_i c_{i+1}$ or by $c_i c_{i+1}\ c_i^{-1}$, c_i.

Two sequences σ, σ′ are said to be (Peiffer) *equivalent* if one can be obtained from the other by a finite number of operations (I), (II), $(\mathrm{II})^{-1}$, (III). The equivalence class containing σ is denoted by $<\sigma>$. The set Σ of equivalence classes forms a (non-abelian) group under the binary operation

$$<\sigma_1> + <\sigma_2> = <\sigma_1\sigma_2> .$$

There is a (well-defined) action of $\hat{F}$ on Σ given by

$$[W]\cdot <\sigma> = <\sigma^W>$$

(where, if $\sigma=(c_1,\cdots,c_m)$ then $\sigma^W=(Wc_1W^{-1},\ldots, Wc_mW^{-1})$), and there is a group homomorphism

$$\partial : \Sigma \rightarrow \hat{F}, \ \ <(c_1, c_2, \cdots, c_m)> \mapsto [c_1c_2\cdots c_m].$$

The triple $(\Sigma, \hat{F}, \partial)$ then has the structure of a crossed module. A well-known result (originally proved by Whitehead [**14**]) is that this crossed module is *free*, with basis consisting of the elements $b_R = <(R)> (R \in \mathbf{r})$. By the elementary theory of crossed modules, Ker∂ is abelian and $\mathrm{Im}\partial (= N)$ acts trivially on Ker∂, so we get a well-defined action of $G=\hat{F}/N$ on Ker∂. With this action Ker∂ becomes a left ZG-module, which is the *second homotopy module* of $\hat{\mathcal{P}}$, denoted $\pi_2(\hat{\mathcal{P}})$.

2. The crossed module Σ*. We define a crossed module $(\Sigma^*, \hat{F}, \partial^*)$ as follows. The elements of Σ^* are the equivalence classes $<$ P $>$ where P is a monoid picture such that ι(P) is a *freely reduced* word on $\mathbf{x} \cup \mathbf{x}^{-1}$ and $\tau(\mathcal{P})$ is the empty word. We define a (non-commutative) operation + on Σ^* by

$$<P_1> + <P_2> = <(P_1+P_2)^*> \ \ (<P_1>, <P_2> \in \Sigma^*),$$

and an action (which is well-defined by (4)) of $\hat{F}$ on Σ^* by

$$[W]\circ <P> = <(W\cdot P\cdot W^{-1})^*> \ \ ([W]\in \hat{F}, <P> \in \Sigma^*).$$

We define

$$\partial^* : \Sigma^* \rightarrow \hat{F}$$

by

$$\partial^* <P> = [\iota(P)] \ \ (<P> \in \Sigma^*).$$

Then under the operation +, Σ^* is a group on which $\hat{F}$ acts. Clearly, for $[W]\in\hat{F}$, $<\mathrm{P}>\in\Sigma^*$ we have

$$\partial^*([W]\circ <P>) = [W]\partial^* <P> [W]^{-1}.$$

Also, as can be seen geometrically (Figure 5), for any $< \mathrm{P}_1 >$, $< \mathrm{P}_2 > \in \Sigma^*$ we have

$$< P_1 > + < P_2 > = \partial^* < P_1 > \circ < P_2 > + < P_1 > .$$

Thus $(\Sigma^*, \hat{F}, \partial^*)$ is a crossed module. Note that

$$- < P > = < (P^{-1} \cdot \iota(P)^{-1})^* > \quad (< P > \in \Sigma^*).$$

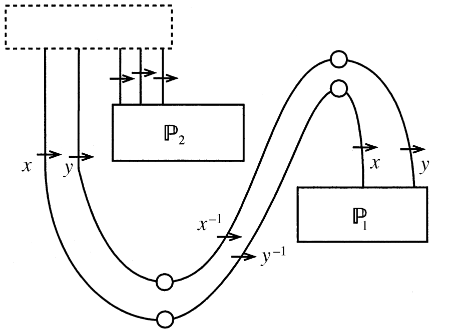

Figure 5

Let $a_R = < \mathrm{E}_{R,1,1} >$ $(R \in \mathbf{r})$.

PROPOSITION. Σ^* *is generated (as a crossed module) by the elements* a_R $(R \in \mathbf{r})$.

Proof. Let

$$\mathrm{B} = \mathrm{T}_1\mathrm{A}_1\mathrm{T}_2\mathrm{A}_2 \cdots \mathrm{T}_n\mathrm{A}_n\mathrm{T}_{n+1}$$

be a closed path in $\mathcal{D}(\mathcal{P})^*$ starting at the reduced word U and ending at the empty word 1. Here the T's are trivial paths and the A's are non-trivial edges. Write $\mathrm{A}_i = (U_i, R_i, \varepsilon_i, V_i)$ $(i = 1, \cdots, n)$. We claim that

$$< \mathrm{B} > = \varepsilon_1[U_1] \circ a_{R_1} + \cdots + \varepsilon_n[U_n] \circ a_{R_n}. \tag{5}$$

Let

$$\mathrm{P} = \mathrm{E}_1 + \mathrm{E}_2 + \cdots + \mathrm{E}_n,$$

where $\mathrm{E}_i = \mathrm{E}_{R_i, U_i, \varepsilon_i}$ $(i = 1, \cdots, n)$. Then the right hand side of (5) is $< \mathrm{P}^* >$. Now let $\bar{P}$ be the picture obtained from P by inserting immediately to the right of the ith disc a succession of parallel arcs with total label $V_i V_i^{-1}$ $(i = 1, \cdots, n)$. Then $\bar{P}^* \sim \mathrm{P}^*$ by (4). Now

$$\iota(\bar{\mathrm{P}}) = \iota(\mathrm{A}_1)\tau(\mathrm{A}_1)^{-1}\iota(\mathrm{A}_2)\tau(\mathrm{A}_2)^{-1} \cdots \iota(\mathrm{A}_n)\tau(\mathrm{A}_n)^{-1},$$
$$\tau(\bar{\mathrm{P}}) = \tau(\mathrm{A}_1)\tau(\mathrm{A}_1)^{-1}\tau(\mathrm{A}_2)\tau(\mathrm{A}_2)^{-1} \cdots \tau(\mathrm{A}_n)\tau(\mathrm{A}_n)^{-1},$$

and so we can take $\bar{P}^*$ to be $\mathrm{D}\bar{P}\mathrm{D}'$ where

$$\mathrm{D} = \mathrm{T}_1 + (\mathrm{T}_{\tau(\mathrm{A}_1)^{-1}\tau(\mathrm{A}_1)})(\tau(\mathrm{A}_1)^{-1} \cdot \mathrm{T}_2) + \cdots + (\mathrm{T}_{\tau(\mathrm{A}_n)^{-1}\tau(\mathrm{A}_n)})(\tau(\mathrm{A}_n)^{-1} \cdot \mathrm{T}_{n+1})$$
$$\mathrm{D}' = \mathrm{T}^{-1}_{\tau(\mathrm{A}_1)\tau(\mathrm{A}_1)^{-1}} + \cdots + \mathrm{T}^{-1}_{\tau(\mathrm{A}_n)\tau(\mathrm{A}_n)^{-1}}$$

(see Figure 6). Making use of the defining paths (3) of $\mathcal{D}(\mathcal{P})^*$ to eliminate the "bends" we see that $\bar{P}^* \sim \mathrm{B}$.

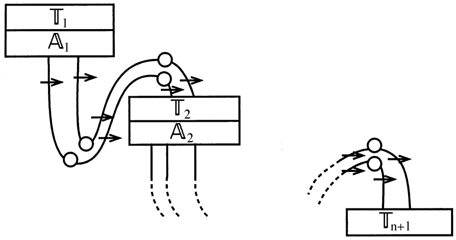

Figure 6

Now since Σ is free on the elements $b_R = <(R)>$ $(R \in \mathbf{r})$ we have a crossed module homomorphism

$$\eta : \Sigma \to \Sigma^*, \quad b_R \mapsto a_R.$$

THEOREM 1. *The crossed module homomorphism* η *is an isomorphism.*

Proof. We will construct the inverse of η.

Define a mapping ψ_0 from the edge set of $\mathcal{D}(\mathcal{P})^*$ to Σ as follows. Trivial edges are mapped to 0; an edge (U, R, ε, V) $(U,V \in F, R \in \mathbf{r},\ \varepsilon = \pm 1)$ is mapped to $< UR^{\varepsilon}U^{-1} >$. Then for any edge A

$$\partial\psi_0(\mathrm{A}) = [\iota(\mathrm{A})\tau(\mathrm{A})^{-1}]. \tag{6}$$

Now ψ_0 extends to a mapping on paths and it follows from (6) that for any path P

$$\partial\psi_0(\mathrm{P}) = [\iota(\mathrm{P})\tau(\mathrm{P})^{-1}]. \tag{7}$$

The image of each defining path of $\mathcal{D}(\mathcal{P})^*$ is 0. This is clear for paths of the form (3), and for a path as in (2) we have

$$\begin{aligned}\psi_0[\mathrm{A},\mathrm{B}] &= \psi_0(\mathrm{A}) + [\tau(\mathrm{A})]\cdot\psi_0(\mathrm{B}) - \psi_0(\mathrm{A}) - [\iota(\mathrm{A})]\cdot\psi_0(\mathrm{B}) \\ &= \partial(\psi_0(\mathrm{A}))\cdot([\tau(\mathrm{A})]\cdot\psi_0(\mathrm{B})) - [\iota(\mathrm{A})]\cdot\psi_0(\mathrm{B}) \\ &\qquad \text{(using the crossed module structure on } \Sigma) \\ &= 0 \text{ (using (6)).}\end{aligned}$$

We thus get a well-defined mapping of equivalence classes

$$< \mathrm{P} > \overset{\psi}{\mapsto} \psi_0(\mathrm{P}),$$

and in particular, we get a function

$$\psi : \Sigma^* \to \Sigma.$$

Now ψ is a group homomorphism, since for any $< \mathrm{P}_1 >, < \mathrm{P}_2 > \in \Sigma^*$ we have

$$\begin{aligned}\psi(< \mathrm{P}_1 > + < \mathrm{P}_2 >) &= \psi_0(((\mathrm{P}_1 \cdot \iota(\mathrm{P}_2))\mathrm{P}_2)^*)\\ &= \psi_0((\mathrm{P}_1 \cdot \iota(\mathrm{P}_2))\mathrm{P}_2)\\ &= \psi_0(\mathrm{P}_1 \cdot \iota(\mathrm{P}_2)) + \psi_0(\mathrm{P}_2)\\ &= \psi_0(\mathrm{P}_1) + \psi_0(\mathrm{P}_2)\\ &= \psi < \mathrm{P}_1 > + \psi < \mathrm{P}_2 > .\end{aligned}$$

Also, it is easily checked that ψ respects the $\hat{F}$-action, and it follows from (7) that $\partial\psi = \partial^*$. Hence ψ is a crossed module homomorphism.

Since $\psi\eta$ agrees with the identity on the generating set a_R ($R \in \mathbf{r}$) of Σ^*, $\psi\eta = 1$. Similarly $\eta\psi = 1$.

This proves the theorem.

Note that, by restriction, we get a mutually inverse pair of isomorphisms

$$\pi_2(\hat{\mathcal{P}}) = \mathrm{Ker}\ \partial \underset{\psi}{\overset{\eta}{\rightleftarrows}} \ker \partial^* = \pi_1(\mathcal{D}(\mathcal{P})^*, 1).$$

The G-action on $\pi_2(\hat{\mathcal{P}})$ induces a G-action on $\pi_1(\mathcal{D}(\mathcal{P})^*, 1)$ by the rule

$$\bar{W} \circ < \mathrm{P} > = < (W \cdot \mathrm{P} \cdot W^{-1})^* > \quad (\bar{W} \in G, < \mathrm{P} > \in \pi_1(\mathcal{D}(\mathcal{P})^*, 1)),$$

and η, ψ are then ZG-isomorphisms.

3. The fundamental groups of $\mathcal{D}(\mathcal{P})^*$. Let $U \in F$. We have a well-defined group homomorphism

$$\phi_U : \pi_1(\mathcal{D}(\mathcal{P})^*, 1) \to \pi_1(\mathcal{D}(\mathcal{P})^*, U),$$

$$< \mathrm{B} > \ \mapsto \ < \mathrm{B} \cdot U > .$$

This is in fact an isomorphism, for consider the (well-defined) function

$$\theta_U : \pi_1(\mathcal{D}(\mathcal{P})^*, U) \to \pi_1(\mathcal{D}(\mathcal{P})^*, 1)$$

$$< \mathrm{P} > \ \mapsto \ < (\mathrm{P} \cdot U^{-1})^* > .$$

Now $\theta_U\phi_U = 1$, for if B is a spherical monoid picture with $\iota(\mathrm{B}) = 1$ then $(\mathrm{B} \cdot UU^{-1})^* \sim \mathrm{B}^* = \mathrm{B}$ by (4). Also, $\phi_U\theta_U = 1$, for if P is a spherical monoid picture with $\iota(\mathrm{P}) = U$ then (see Figure 7)

$$(\mathrm{P} \cdot U^{-1})^* \cdot U \sim \mathrm{P}.$$

Thus θ_U, ϕ_U are mutually inverse isomorphisms.

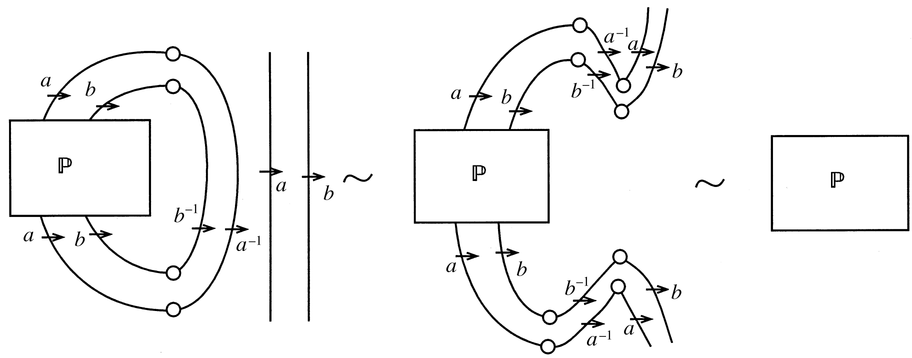

Figure 7

We will need the following result.

LEMMA. *Let* P *be a spherical monoid picture over* $\mathcal{P}$ *with* $\iota(\mathrm{P}) = U$. *Suppose* W, $V \in F$ *are such that* $WUV =_G 1$. *Let* D *be any path in* $\mathcal{D}(\mathcal{P})^*$ *from* 1 *to* WUV. *Then in* $\pi_1(\mathcal{D}(\mathcal{P})^*, 1)$ *we have*

$$< \mathrm{D}(W \cdot \mathrm{P} \cdot V)\mathrm{D}^{-1} > = \bar{W} \circ \theta_U < \mathrm{P} > .$$

This can be seen geometrically as follows. First note that $V^{-1}U^{-1}W^{-1} =_G 1$ so there is a path $\bar{D}$ in $\mathcal{D}(\mathcal{P}^*)$ from 1 to $V^{-1}U^{-1}W^{-1}$. Then we have the equivalence as in Figure 8 (where for simplicity we have taken W, U, V to each consist of a single letter).

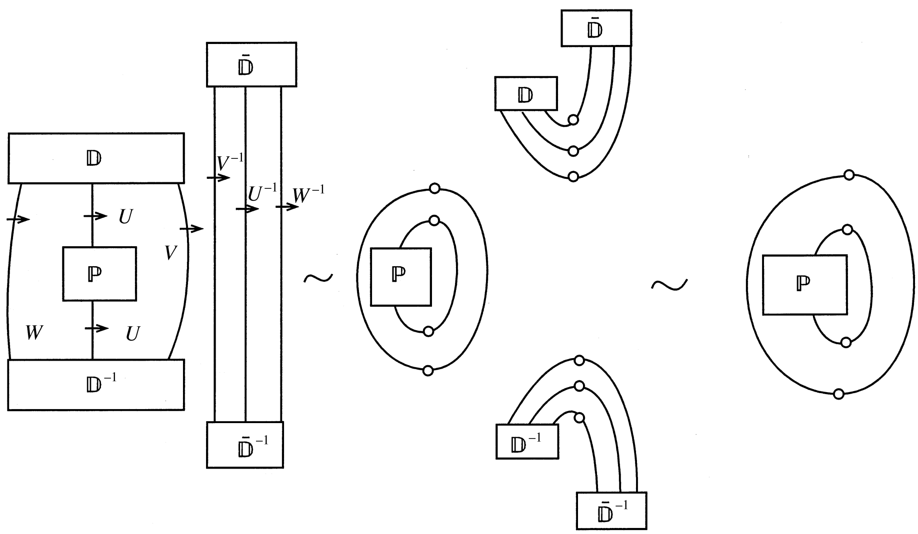

Figure 8

THEOREM 2. $\mathcal{P}$ *is of finite derivation type if and only if the left* Z*G-module* $\pi_1(\mathcal{D}(\mathcal{P})^*, 1)$ *is finitely generated.*

Proof. First suppose that $\mathcal{P}$ has finite derivation type. Then there is a finite collection X of spherical monoid pictures over $\mathcal{P}$ such that the 2-complex $\mathcal{D}(\mathcal{P})^X$ obtained from $\mathcal{D}(\mathcal{P})$ by adjoining the defining paths

$$W \cdot \mathrm{P} \cdot V \quad (W, V \in F,\ \mathrm{P} \in X)$$

has trivial fundamental groups.

Let B be any spherical monoid picture with $\iota(\mathrm{B}) = 1$. Then B is homotopic in $\mathcal{D}(\mathcal{P})$ (and hence in $\mathcal{D}(\mathcal{P})^*$) to a product of the form

$$\prod_{i=1}^{n} \mathrm{D}_i (W_i \cdot \mathrm{P}_i \cdot V_i)^{\varepsilon_i} D_i^{-1}$$

where $\mathrm{P}_i \in X$, $\varepsilon_i = \pm 1$, W_i, $V_i \in F$, D_i is some path in $\mathcal{D}(\mathcal{P})$ with $\iota(\mathrm{D}_i) = 1$, $\tau(D_i) = \iota(W_i \cdot P_i \cdot V_i)$ $(i = 1, \cdots, n)$. Hence in $\pi_1(\mathcal{D}(\mathcal{P})^*, 1)$ we have

$$\begin{aligned} < B > &= \sum_{i=1}^{n} \varepsilon_i < D_i (W_i \cdot P_i \cdot V_i) D_i^{-1} > \\ &= \sum_{i=1}^{n} \varepsilon_i \bar{W}_i \circ \theta_{\iota(P_i)} < P_i > \text{ (by the Lemma).} \end{aligned}$$

Thus the module $\pi_1(\mathcal{D}(\mathcal{P})^*, 1)$ is generated by the elements

$$\{\theta_{\iota(\mathrm{P})} < \mathrm{P} >: \mathrm{P} \in X\}.$$

Conversely, suppose there is a finite set Y of spherical monoid pictures (each starting at 1) such that the elements $< \mathrm{B} >$ $(\mathrm{B} \in Y)$ generate $\pi_1(\mathcal{D}(\mathcal{P})^*, 1)$ as a module. Let P be any spherical monoid picture, and suppose that $\iota(\mathrm{P}) = U$. Then

$$\theta_U < P >= \sum_{i=1}^{n} \varepsilon_i \bar{W}_i \circ < \mathrm{B}_i >$$

where $\mathrm{B}_i \in Y$, $W_i \in F$, $\varepsilon_i = \pm 1$ $(i = 1, \cdots, n)$. Thus in $\pi_1(\mathcal{D}(\mathcal{P})^*, U)$ we have

$$\begin{aligned} < \mathrm{P} > &= \prod_{i=1}^{n} \phi_U (\bar{W}_i \circ < \mathrm{B}_i >)^{\varepsilon_i} \\ &= \prod_{i=1}^{n} < (\mathrm{T}_{W_i W_i^{-1}} \cdot U)(W_i \cdot \mathrm{B}_i^{\varepsilon_i} \cdot W_i^{-1} U)(\mathrm{T}_{W_i W_i^{-1}} \cdot U)^{-1} > . \end{aligned}$$

Consequently, we see that if we adjoin to $\mathcal{D}(\mathcal{P})^*$ the additional defining paths

$$W \cdot \mathrm{B} \cdot V \qquad (W, V \in F, \mathrm{B} \in Y)$$

then all fundamental groups of the resulting complex are trivial. Thus if X consists of the pictures in Y together with the pictures of the form (3), then $\mathcal{D}(\mathcal{P})^X$ has trivial fundamental groups, and so $\mathcal{P}$ is of finite derivation type.

Acknowledgement. I thank Victor Guba for useful discussions relating to this paper.

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