Appendix D

Stratification and stratified Morse theory

In this final appendix we extend the Morse-theoretic decompositions of Appendix C to handle general algebraic varieties and their complements. More specifically, we cover results from *stratified* Morse theory [GM88] that characterize the topology of a stratified space X through changes in topology in the sublevel sets $X_{\leq c}$ as c passes through critical values (in a stratified sense) of a height function. We develop from scratch the notion of a Whitney stratified space, Morse functions, and stratified critical points. We discuss nonproper extensions of this material and then summarize a number of basic results of [GM88], including specific properties enjoyed by complex algebraic varieties.

D.1 Whitney stratified spaces

Ideally one would use the apparatus of manifolds, developed in the previous appendices, to do calculus on complex algebraic varieties, however many varieties that appear in interesting combinatorial problems are not manifolds. The right generalization for our purposes is the notion of a *stratified space*, which can contain *non-manifold points* whose neighborhoods in the variety are not diffeomorphic to any Euclidean space $\mathbb{R}^d$. One well-known example of such spaces are *manifolds with boundary*, and we begin with a recap of these objects. The following material on manifolds with boundary can be skipped if desired, as it is subsumed by our discussion of stratified spaces, however we include it because it is likely familiar to many readers.

A ***d*-manifold with boundary** is a subset $\mathcal{M} \subset \mathbb{R}^n$ such that every point $\boldsymbol{x} \in \mathcal{M}$ has a neighborhood in $\mathbb{R}^n$ whose intersection $\mathcal{N}$ with $\mathcal{M}$ is either diffeomorphic to $\mathbb{R}^d$ or diffeomorphic to the closed halfspace $\mathbb{R}^{d-1} \times [0, \infty)$. In

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the former case $\boldsymbol{x}$ is called a ***manifold point*** or ***interior point*** of $\mathcal{M}$, while in the latter case $\boldsymbol{x}$ is called a ***boundary point*** of $\mathcal{M}$.

Example D.1. A closed ball in any dimension d is a manifold with boundary, while a cube $[0,1]^d$ and a simplex $\{\boldsymbol{x} \in \mathbb{R}^d : x_j \geq 0 \text{ for all } j \text{ and } \sum_{j=1}^d x_j = 1\}$ are not. ◁

Example D.2. If $H = \{(x,y) \in \mathbb{R}^2 : y \geq 0\}$ is the upper half-plane then H is a manifold with boundary, the boundary points being the x-axis. Now let $K = \{(x,y) \in \mathbb{R}^2 : x, y \geq 0\}$ denote the positive quarter plane. The map $\phi(x,y) = (x^2 - y^2, 2xy)$ from K to H, constructed by taking the real and imaginary parts of $(x+iy)^2$, is analytic and one to one, so it may seem that K is diffeomorphic to H and thus a manifold with boundary. However, ϕ^{-1} is not differentiable at the origin, and in fact no neighborhood of the origin in K is diffeomorphic to a neighborhood of the origin in a half-plane. Thus, K is not a manifold with boundary. ◁

Exercise D.1. Give an example of a manifold $\mathcal{M} \subseteq \mathbb{R}^d$ with closure $\overline{\mathcal{M}}$ whose boundary $\overline{\mathcal{M}} \setminus \mathcal{M}$ is also a manifold, such that $\mathcal{M}$ is not a manifold with boundary.

A generalization of manifolds with boundary is the notion of a ***d-manifold with corners***, where every point has a neighborhood diffeomorphic to some orthant $\mathbb{R}^{d-k} \times \mathbb{R}^k_{\geq 0}$ for some $k \leq d$; see Figure D.1. Cubes and simplices are manifolds with corners, however complex algebraic varieties that are not smooth are also typically not manifolds with corners. This difficulty is why we introduce the generality of stratified spaces.

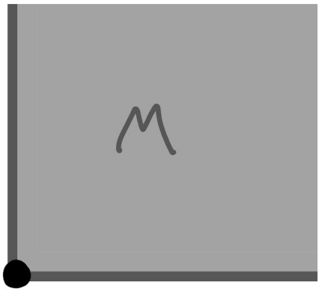

Figure D.1 The closed quadrant $\mathcal{M}$ is a manifold with corners, but not a manifold with boundary.

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Stratifications

As a first attempt to perform calculus on an algebraic variety $\mathcal{V}$, one might partition $\mathcal{V}$ into a finite disjoint union of smooth sets and then work on each piece of the partition. Although such ***smooth partitions*** can be easily understood, and easily computed using standard algebro-geometric techniques, they are not sufficient for our (and many other) purposes. The problem is that the pieces in an arbitrary smooth partition may not *fit together nicely*; among other difficulties, this means the local behavior of $\mathcal{V}$ near points in the same piece of the partition can be very different.

The issue of determining the "right" type of smooth partition to use for topological arguments was taken up by Whitney [Whi65b], who introduced what we now call (Whitney) stratifications. An ***I-decomposition*** of a space $X \subseteq \mathbb{R}^n$ is a finite disjoint union $\bigcup_{\alpha \in I} \mathcal{S}_\alpha$ of smooth manifolds of various dimensions, indexed by a partially ordered set I, such that for every $\alpha, \beta \in I$,

$$S_\alpha \cap \overline{S_\beta} \neq \emptyset \iff S_\alpha \subset \overline{S_\beta} \iff \alpha \leq \beta. \tag{D.1.1}$$

Definition D.3 (Whitney stratification)**.** Let Z be a closed subset of $\mathbb{R}^n$. A ***Whitney stratification*** of Z is an I-decomposition of Z with the additional property that whenever

- $\alpha < \beta$, and
- the sequences $\{x_i \in S_\beta\}$ and $\{y_i \in S_\alpha\}$ both converge to some $y \in S_\alpha$, and
- the lines $\ell_i = \overline{x_i\, y_i}$ converge to a line ℓ, and
- the tangent planes $T_{x_i}(S_\beta)$ converge to a plane T,

then $\ell \subseteq T$. We call Z a ***Whitney stratified space***.

Remark. In the original definition, in addition to $\ell \subseteq T$ (the so-called ***second Whitney condition***), it was required that $T_y(\mathcal{S}_\alpha) \subseteq T$ (the so-called ***first Whitney condition***). The second condition turns out to imply the first, so the first condition is usually omitted.

This definition is well crafted: the conditions are easy to fulfill – for example, every algebraic variety admits a Whitney stratification, see [Whi65b, Theorem 18.11] or [Hir73, Theorem 4.8] – and the conditions have strong consequences (for instance, they are strong enough for stratified Morse theorems to hold). Stratifications of algebraic varieties are also effectively computable. A classic approach to algorithmic stratification through quantifier elimination and real algebraic geometry, relying on cylindrical algebraic decomposition, is discussed in [Ran98; MR91]. Recently, [DJ21] and [HN22] have given more

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practical algorithms[1] for the stratification of algebraic varieties using Gröbner basis computations.

Proposition D.4. *Every algebraic variety in $\mathbb{R}^d$ or $\mathbb{C}^d$ admits a Whitney stratification.* □

In examples arising from combinatorial applications, it is often possible to deduce a stratification directly from the form of the polynomials under consideration.

Example D.5. A smooth manifold is a stratified space with a single stratum. ◁

Example D.6. If X is a finite union of affine subspaces of $\mathbb{R}^n$ then a Whitney stratification of X is obtained by taking the set $\mathcal{A}$ of all intersections of the affine subspaces, and choosing the elements of $\{A \setminus B : A, B \in \mathcal{A} \text{ with } A \supsetneq B\}$ as strata. ◁

Example D.7. Let Z be a real algebraic curve $\{(x, y) \in \mathbb{R}^2 : f(x, y) = 0\}$ with f irreducible and let $Y = \{(x, y) : \nabla f(x, y) = 0\}$ be the finite set of singular points of Z. Taking $Z \setminus Y$ to be one stratum and each singleton $\{(x, y)\}$ for $(x, y) \in Y$ to be another produces a Whitney stratification of Z. The following figure shows two examples of this, the first curve $x^2 - y^3$ having a cusp at the origin and the second curve $19 - 20x - 20y + 5x^2 + 14xy + 5y^2 - 2x^2y - 2xy^2 + x^2y^2$ having a self-intersection at $(1, 1)$. ◁

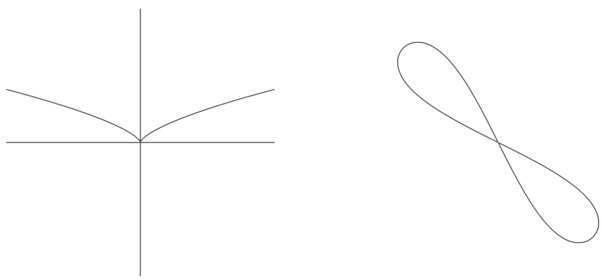

Figure D.2 Two curves, each stratified by taking one stratum consisting of a singular point and another stratum consisting of the rest of the curve.

Let $\mathcal{V}$ be any complex variety. As discussed in Chapter 8, it is possible to decompose $\mathcal{V}$ into smooth sets by determining algebraic equations for the set

[1] Helmer and Nanda [HN22] give an implementation of both of these algorithms in Macaulay2, available at http://martin-helmer.com/Software/WhitStrat/.

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Σ_0 of its singular points, letting $\mathcal{S}_0 = \mathcal{V} \setminus \Sigma_0$ encode the smooth points of $\mathcal{V}$, then recursively computing the sets Σ_{n+1} and $\mathcal{S}_{n+1}$ of smooth and singular points of Σ_n until arriving at some $\Sigma_N = \emptyset$. From the previous two examples, one might get the idea that this decomposition is always a Whitney stratification, but Exercise D.6 below shows this not to be the case. It is true, however, that any stratification must be at least this coarse.

Example D.8. Let Z be a complex algebraic hypersurface in $\mathbb{C}^3$ defined by $f(x, y, z) = 0$ and suppose ∇f vanishes along an algebraic curve γ. It is possible that $\{\gamma, Z \setminus \gamma\}$ is a Whitney stratification for Z. On the other hand, if γ is not smooth then a Whitney stratification of Z will have at least three strata, one containing singularities of γ, one containing the rest of γ, and one containing $Z \setminus \gamma$. ◁

Exercise D.2. Compute a Whitney stratification of the real variety $\mathcal{V}_Q$ where $Q(x, y, z) = z^2 - x^2 - y^2$.

The following exercise implies, with a little more work, that any manifold with corners (including any manifold with boundary) is a Whitney stratified space, with strata $\{S_j : 0 \leq j \leq d\}$ defined by the union of the open j-dimensional faces.

Exercise D.3. Let $H = \mathbb{R}^d_{\geq 0}$ be the positive orthant, let F be a (open) face of H and let $\boldsymbol{x}$ be a point of F. Prove directly that the interior $S_\beta = H^\circ$ and face $S_\alpha = F$ satisfy the Whitney condition (Definition D.3) at $\boldsymbol{x}$.

One fundamental result of stratified spaces concerns their local product structure, implying that the local behavior of a stratified space "looks the same" in neighborhoods of different points on the same stratum. The proof of this fact is long and difficult, but we sketch some of it in the next section.

Theorem D.9 (local product structure)**.** *Let $\mathbf{p}$ be a point in a k-dimensional stratum $\mathcal{S}$ of a stratified space Z. There is a topological space* N*, called the* **normal slice***, depending only on $\mathcal{S}$ and not the choice of $\mathbf{p} \in \mathcal{S}$, such that some neighborhood of $\mathbf{p}$ in Z is homeomorphic to $B^k \times N$, where B^k is a k-dimensional ball.* □

We end this section with the following concept.

Definition D.10 (stratification of a pair)**.** If $Y \subseteq X$ are closed subsets of real space then a ***stratification of the pair*** (X, Y) is defined to be a stratification of X such that intersecting each stratum with Y gives a stratification of Y and intersecting each stratum with $X \setminus Y$ gives a stratification of $X \setminus Y$.

A result of Whitney implies that if $(X \setminus Y, Y)$ is a decomposition of X into two smooth manifolds satisfying (D.1.1) then some Whitney stratification of X refines this, and is a stratification of the pair (X, Y); see, for instance, [LT10, Proposition 2.1]. Proposition D.4 extends to the following.

Proposition D.11. *If $\mathcal{V}_*$ is a complex algebraic variety in $\mathbb{C}_*^d$ with stratification $\{\mathcal{S}_\alpha : \alpha \in I\}$ then adding the stratum $\mathcal{M} = \mathbb{C}_*^d \backslash \mathcal{V}_*$ produces a stratification of the pair $(\mathbb{C}_d^*, \mathcal{V}_*)$.* □

D.2 Critical points and the fundamental lemma

We now extend the geometric concepts discussed in previous appendices to stratified spaces.

Critical points for stratified spaces

Fix a Whitney stratification $\{S_\alpha : \alpha \in I\}$ of a closed subset X of a smooth manifold $M \subseteq \mathbb{R}^n$ and let $f = h|_X$ be the restriction to X of a smooth function $h : M \to \mathbb{R}$.

Definition D.12 (stratified critical points and Morse functions)**.** Any point $\boldsymbol{p} \in X$ is contained in a unique stratum $S = S(\boldsymbol{p})$, and we say that $\boldsymbol{p}$ is a ***critical point of the height function*** h ***on the stratified space*** X if $\boldsymbol{p}$ is a critical point of $h|_{S(\boldsymbol{p})}$ (in other words, if the restriction of the differential of h to the tangent space $T_{\boldsymbol{p}}S(\boldsymbol{p})$ is zero). We call h a ***Morse function*** if

(1) the restriction $h|_{S_\alpha}$ is a Morse function for each $\alpha \in I$, meaning that its critical points are nondegenerate (i.e., its Hessian is nonsingular at each critical point), and
(2) whenever $\boldsymbol{p} \in S_\alpha$ is a critical point of $h|_{S_\alpha}$ and T is a limit of tangent planes $T_{\boldsymbol{p}_i}(S_\beta)$ as $\boldsymbol{p}_i \to \boldsymbol{p}$ in a stratum S_β with $\beta > \alpha$, then either $T = T_{\boldsymbol{p}}(S_\alpha)$ or T contains a tangent vector on which $dh(\boldsymbol{p})$ does not vanish.

This generalization of Morse functions to stratified spaces appears in [Pig79, Section 3]; see also [Laz73]. In many contexts it is assumed that Morse functions have distinct critical values – in which case we say we have a ***Morse function with distinct critical values*** – or are proper – in which case we say we have a ***proper Morse function***. Figure D.3 shows two height functions, one failing condition (2) in Definition D.12 and one satisfying it: on the left, the limit of tangent lines at the cusp is horizontal, and is therefore annihilated by

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Figure D.3 A non-Morse function (left) and a Morse function (right).

dh. A standard perturbation argument shows that coinciding critical values do not affect topology.

The stratified version of the Fundamental Morse Lemma (Lemma C.27) is the following.

Theorem D.13 (Stratified Morse Lemma [GM88, Theorem SMT part A])**.** *Let $X \subseteq \mathbb{C}_*^d$ be a stratified space with proper Morse function h and let $a < b$ be real numbers such that the interval $[a, b]$ contains no critical values of h. If $h^{-1}([a, b])$ is compact, then the inclusion $X_{\leq a} \hookrightarrow X_{\leq b}$ is a homotopy equivalence.*

□

Tangent vector fields

The argument behind Theorem D.13 is worth understanding for readers who have made it this far into the appendices. Both Theorem D.13 and Theorem D.9 will be derived from Thom's Isotopy Lemma, stated as Lemma D.16 below. For non-experts, the geometric intuition behind Theorem D.13 is not apparent, and it can be instructive to pursue a line of reasoning that sometimes fails but more closely parallels classical Morse theory.

Proposition D.14. *In the following cases, the local product structure in Theorem D.9 is induced by a diffeomorphism.*

1. *When Z is a smooth algebraic hypersurface.*
2. *When Z is the simplex or the complexification of a simplex.*
3. *When Z is a hyperplane arrangement.*
4. *When Z is the product of two spaces on which the local product is induced by a diffeomorphism.*

Proof In Cases 2 and 3, diffeomorphisms can be explicit constructed. Case 1 follows from the smooth implicit function theorem, while Case 4 follows from taking a product diffeomorphism. □

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Proof sketch of Theorem D.13 (assuming diffeomorphic product structure)
Step 1: Each stratum $\mathcal{S}$ is a smooth manifold. The nonvanishing of the gradient of $h|_{\mathcal{S}}$ implies the existence of a nonvanishing downward gradient vector field $\boldsymbol{v}_{\mathcal{S}}$ parallel to $\mathcal{S}$. More specifically, there is a smooth nonvanishing section of the tangent bundle (i.e., a map $\boldsymbol{v}_{\mathcal{S}} : \mathcal{S} \to T\mathcal{S}$) such that $dh(\boldsymbol{v}_{\mathcal{S}}) < 0$.

Step 2: By assumption of diffeomorphic local product structure, for each point $\boldsymbol{p}$ in each k-dimensional stratum $\mathcal{S}$ of X there is a $\mathbb{C}^d$-neighborhood $\mathcal{N}$ of $\boldsymbol{p}$ and a smooth change of coordinates in $\mathcal{N}$ under which $\mathcal{S} \cap \mathcal{N} = \{\boldsymbol{z} \in \mathcal{N} : z_j = 0 \text{ for } j > k\}$ and $X \cap \mathcal{N} = \{\boldsymbol{z} \in \mathcal{N} : (z_{j+1}, \ldots, z_d) \in N'\}$, where N' is the normal slice consisting of all $(d-j)$-tuples $(z_{j+1}, \ldots, z_d)$ such that $(0, \ldots, 0, z_{j+1}, \ldots, z_d) \in X$. Strata in this neighborhood are the products of strata of N' in the first k coordinates with $\mathbb{R}^{d-k}$. Vectors $\boldsymbol{v}$ tangent to $\mathcal{S}$ in this neighborhood have $v_j = 0$ for $j > k$ and are therefore tangent to all strata in the neighborhood.

The within-stratum downward gradient flows $\boldsymbol{v}_{\mathcal{S}}$ can be stitched together via a partition of unity to form a single gradient-like flow $\boldsymbol{v}$ with Lipschitz constant 1. More specifically, each point $\boldsymbol{p}$ in a stratum has a neighborhood $U_{\boldsymbol{p}}$ in $\mathbb{C}^d$ that intersects only strata whose closure contains $\mathcal{S}(\boldsymbol{p})$, the stratum containing $\boldsymbol{p}$, and on which $dh(\boldsymbol{v}_{\mathcal{S}(\boldsymbol{p})}) < 0$. If $\{\psi_{U_{\boldsymbol{p}}} : \boldsymbol{p} \in E\}$ is a partition of unity subordinate to a finite subcover of $h^{-1}[a, b]$ by these neighborhoods, then

$$\boldsymbol{v} = \sum_{\boldsymbol{p} \in E} \psi_{U_{\boldsymbol{p}}} \boldsymbol{v}_{\mathcal{S}(\boldsymbol{p})} \tag{D.2.1}$$

defines the required flow. It is gradient-like because $dh(\boldsymbol{v})$ is a convex combination of values $dh(\boldsymbol{v}_{\mathcal{S}}(\boldsymbol{p}))$, which are all negative. It is tangent to each stratum because $\boldsymbol{v}(\boldsymbol{p}')$ is a convex combination of vectors $\boldsymbol{v}(\mathcal{S}(\boldsymbol{p}))$ tangent to strata $\mathcal{S}(\boldsymbol{p})$ whose tangent spaces are contained in the tangent space to $\boldsymbol{p}'$. Choosing $U_{\boldsymbol{p}}$ small enough that some constant multiple of each $\boldsymbol{v}_{\mathcal{S}(\boldsymbol{p})}$ can be chosen to have Lipschitz constant 1 on $U_{\boldsymbol{p}}$, convexity implies that $\boldsymbol{v}$ globally has Lipschitz constant 1.

Figure D.4 shows a picture of this. The left-hand picture shows that the vector field $\boldsymbol{w}(\boldsymbol{p}) = \boldsymbol{v}_{\mathcal{S}(\boldsymbol{p})}(\boldsymbol{p})$ is gradient-like but not continuous. It changes direction sharply when approaching a substratum, because $\mathcal{S}(\boldsymbol{p})$ changes discontinuously from one stratum to a substratum. The right-hand picture shows these blended by a partition of unity, so as to become smooth while remaining gradient-like.

Step 3: Let $c > 0$ be the infimum value of $|dh(\boldsymbol{v})|$ on X, and let $\Psi : X \times [0, \infty] \to X$ be the flow defined by $(d/dt)\Psi(x, t) = \boldsymbol{v}(\Psi(x, t))$, stopped when it hits $h^{-1}(c)$. Such a flow exists and is unique because $\boldsymbol{v}$ is Lipschitz, being a convex combination of locally constant vector fields (in the natural

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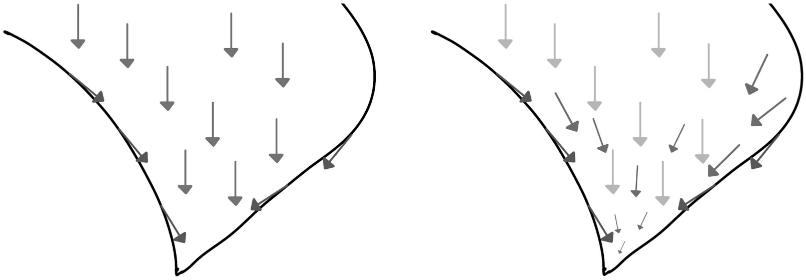

Figure D.4 *Left:* A flow in a 2D stratum that turns sharply when reaching a boundary. *Right:* A partition of unity blends the flow smoothly between strata (note that the flow smoothly becomes zero in a neighborhood of the zero-dimensional stratum).

identification of tangent spaces with subspaces of the tangent space to the ambient space $\mathbb{R}^d$). Fixing any $T \geq (b-a)/c$ the time T map defines a deformation retract of $X_{\leq b}$ onto $X_{\leq a}$, proving homotopy equivalence. □

The problem with this sketched proof is that, in general, the local product structure is not witnessed by a diffeomorphism. This is shown by Whitney's counterexample [Whi65a], reproduced in Goresky's introduction [Gor12] to Mather's cleaned up notes [Mat12] as motivation for the work that follows. Figure D.5 shows three planes and a ruled surface in $\mathbb{R}^3$, whose common intersection is the x-axis. Intersecting with a plane parallel to the yz-plane moving down the x-axis results in a configuration of four lines, the first three constant and the fourth becoming more sloped. Any coordinate system in which the first three lines remain fixed as the slice moves down the x-axis also fixes the slope at the origin of the fourth line, and therefore cannot represent the figure as a product of the x-axis with a four-line configuration.

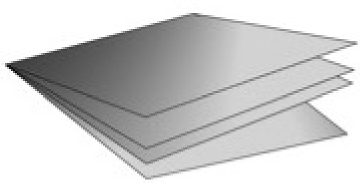

Figure D.5 Whitney's counterexample to smooth isotopy.

The trouble is that the category in which one most naturally deals with stratified spaces is *smooth within strata and continuous across strata*. Whit-

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ney's conditions do not guarantee the existence of a differential structure that is smooth across strata, even for algebraic hypersurfaces in Euclidean space. Nevertheless, it is true that there is a continuous isotopy moving the yz-plane to the right while continuously deforming a sector so that the line of intersection with the ruled surface in each slice remains identified. Working in the smooth within strata continuous across strata category, one can obtain a vector field but it will generally not be Lipschitz. The flow in Step 3 will not necessarily exist, and the argument falls apart.

Remark D.15. In the neighborhood of a hyperbolic point of a complex algebraic hypersurface, a Lipschitz vector field can be constructed explicitly from a lower-semicontinuously varying family of cones. This is carried out in [BP11] (see Lemma 5.1 there) and is based on the lengthier development in [ABG70]; the construction is summarized in Section 11.2 of this book. Thus, the three steps above prove Theorem D.13 when X is a complex algebraic hypersurface with all critical points hyperbolic, even though Proposition D.14 will not necessarily hold.

Isotopy

To repair the stratified gradient flow argument, one needs a statement of Thom's Isotopy Lemma strong enough to imply the deformation retract in Step 3 directly, as well as implying Theorem D.9. This lemma is proved by giving up on the idea that the desired vector field can be continuous, providing instead a ***controlled*** vector field satisfying a set of axioms allowing one to infer that the vector field defines a continuous flow with the desired properties. We will not go into the theory of controlled vector fields, being content to quote where they are used and referring the reader to [Mat12, Proposition 11.1] for the proof of the following results and full details of controlled vector fields for stratified spaces.

Lemma D.16 (Thom's Isotopy Lemma)**.** *Let Z be a Whitney stratified space Z that is a closed subset of some smooth manifold $\mathcal{M}$, and suppose that $\pi : \mathcal{M} \to P$ is a smooth proper mapping to a connected manifold P such that the restriction $\pi|_{\mathcal{S}}$ of π to each stratum $\mathcal{S}$ of Z is a submersion (surjective on tangent spaces). Then any smooth vector field V on P has a lift $\tilde{V}$ to a controlled vector field on Z. By a lift, we mean that V is a (not necessarily continuous) section of the tangent bundle of each stratum of Z such that $\pi_* \circ \tilde{V} = V \circ \pi$. Although $\tilde{V}$ is not necessarily continuous, it has a continuous flow $\tilde{\Psi}$ that projects under π to the flow Ψ defined by V on P. The fact that there is a continuous flow lifting the flow of V implies that $\pi|_Z : Z \to P$ is a locally trivial fiber bundle.* □

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Proof of Theorem D.13 Apply Thom's Isotopy Lemma with manifolds $P = \mathbb{R}$ and $\mathcal{M} = \mathbb{C}_d^*$, stratified space $Z = X \cap h^{-1}(a - \varepsilon, b + \varepsilon)$, and mapping $\pi = h$. If $h : X \to \mathbb{R}$ has no critical values in $[a, b]$ then it has no critical values in $[a - \varepsilon, b + \varepsilon]$, hence h is a submersion on each stratum of X. The conclusion of the lemma is that the level surfaces of h are fibers of a local product bundle, hence the flow $\tilde{V}$ witnesses a strong deformation retraction of $X_{\leq b}$ onto $X_{\leq a}$. □

To conclude this section, we show how Thom's Isotopy Lemma can be used to derive the local product topological structure of stratified spaces.

Proof of Theorem D.9 Let Z be a stratified space in $\mathbb{R}^d$ and let $\mathcal{S}$ be a stratum of dimension k, with $\bullet\mathcal{S}$ denoting a closed tubular neighborhood of $\mathcal{S}$ in $\mathbb{R}^d$ and $\pi : \bullet\mathcal{S} \to \mathcal{S}$ denoting the projection map. Then $\bullet\mathcal{S}$ is a manifold with boundary $\circ\mathcal{S}$ and an interior which we denote $(\bullet\mathcal{S})^\circ$. If the tubular neighborhood was chosen sufficiently small, then $X = Z \cap \bullet\mathcal{S}$ is naturally stratified with strata of the form $W \cap (\bullet\mathcal{S})^\circ$ and $W \cap \circ\mathcal{S}$, where W runs over strata whose closure contains $\mathcal{S}$.

The mapping π on X satisfies the conditions of Thom's isotopy lemma. Consequently, its normal slice $\mathtt{N} = \pi^{-1}(\mathbf{p}) \cap Z$ is stratified by its intersection with the strata of X. Taking $U_{\mathbf{p}}$ to be a small ball around $\mathbf{p}$ in the stratum $\mathcal{S}$ that contains $\mathbf{p}$, there is a stratum-preserving homeomorphism, smooth in each stratum, given by $\pi^{-1}(U_p) \cap Z \cong U_{\mathbf{p}} \times N$. Since $\pi^{-1}(U_{\mathbf{p}})$ is a neighborhood of $\mathbf{p}$ in Z, we have shown that each stratum has a neighborhood that is locally a topological product of a k-ball $U_{\mathbf{p}}$ with the normal slice. □

D.3 Description of the attachments

Let $\mathcal{V}_*$ denote the intersection $\mathcal{V} \cap \mathbb{C}_*^d$ of an affine algebraic hypersurface $\mathcal{V}$ with $\mathbb{C}_*^d$, and let $\mathcal{M} = \mathbb{C}_*^d \setminus \mathcal{V}$. We return to our plan to use Morse theory to find generators for $H_d(\mathcal{M})$. Because we may want to describe either $\mathcal{M}$ or $\mathcal{V}_*$, depending on the situation, results in the literature are often stated in two parts, so as to cover both cases, and we continue to adhere to this. For what follows we fix a Whitney stratification $\{\mathcal{S}_\alpha : \alpha \in I\}$ of the pair $(\mathbb{C}_*^d, \mathcal{V}_*)$ as in Proposition D.11, so that $\mathcal{M}$ will be the unique stratum of dimension $2d$. The function $h = h_{\hat{\mathbf{r}}}$ is assumed to be a Morse function and the space X may denote either $\mathcal{V}_*$ or $\mathcal{M}$. The point $\mathbf{p}$ denotes a stratified critical point for h in the stratum $\mathcal{S}$, and we let $N = N_{\mathbf{p}}(\mathcal{V})$ denote the complex normal space to $\mathcal{V}_*$ at $\mathbf{p}$.

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The *tangential Morse data* is defined in terms of $\boldsymbol{p}$ and $\mathcal{S}$, regardless of whether $X = \mathcal{V}_*$ or $X = \mathcal{M}$.

Definition D.17 (tangential Morse data)**.** The ***tangential Morse data*** at $\boldsymbol{p}$ is the homotopy type of the pair $(B^\lambda, \partial B^\lambda)$, where λ is the Morse index of $h|_S$ at $\boldsymbol{p}$ and B^λ denotes the ball of dimension λ. By Theorem C.28, this is the Morse data at $\boldsymbol{p}$ for the height function $h|_S$ on the smooth manifold S.

The *normal Morse data* is defined in terms of the intersection of X with a slice normal to the stratum $\mathcal{S}$, localized to the point $\boldsymbol{p}$. If D is an arbitrarily small disk in $N_{\boldsymbol{p}}(\mathcal{V})$ centered at $\boldsymbol{p}$ then the ***normal slice*** at $\boldsymbol{p}$ is $\mathtt{N}(X) := X \cap D$. To visualize this, it sometimes helps to picture the ***normal link*** $\mathcal{L}(X)$ at $\boldsymbol{p}$, defined by $\mathcal{L}(X) := X \cap \partial D$. When $X = \mathcal{V}_*$ the normal slice $\mathtt{N}(X)$ is homeomorphic to a cone over $\mathcal{L}(X)$ from the point $\boldsymbol{p}$. In particular, $\mathtt{N}(X)$ is contractible. When $X = \mathcal{M}$ the point $\boldsymbol{p}$ is absent from the normal slice, which then retracts onto $\mathcal{L}(X)$, hence $\mathtt{N}(X) \simeq \mathcal{L}(X)$.

Example D.18. Let $\mathcal{V}$ be the union of two complex planes in complex 3-space meeting at the line $\mathcal{S}$ and let $\boldsymbol{p}$ be a point on $\mathcal{S}$. This line is the stratum containing $\boldsymbol{p}$, and the tangent space at $\boldsymbol{p}$ or any other point on $\mathcal{S}$ is the translation of $\mathcal{S}$ to the origin. The normal space $N_{\boldsymbol{p}}(\mathcal{V})$ at $\boldsymbol{p}$ (or any other point on $\mathcal{S}$) is the complex two-space orthogonal to $\mathcal{S}$.

First consider the case $X = \mathcal{V}_*$. The intersection of X with a normal plane to $\mathcal{S}$ at $\boldsymbol{p}$ is two complex lines meeting at $\boldsymbol{p}$. The normal slice $\mathtt{N}(X)$ is the intersection of this with a ball around $\boldsymbol{p}$, and thus is two disks joined by identifying their centers. The link $\mathcal{L}(X) = X \cap \partial D$ is the union of two disjoint circles, each on one of the complex lines, and the normal slice $\mathtt{N}(X)$ is the cone over these circles.

Alternatively, if $X = \mathcal{M}$ then $\mathcal{L}(X) = X \cap \partial D$ is the complement of two intersecting lines in a small bi-disk, which is the product of two punctured disks. Each punctured disk retracts to its boundary, so the four-dimensional space $\mathtt{N}(X)$ retracts to the three-dimensional space $\mathcal{L}(X)$, which retracts to a two-dimensional torus $S^1 \times S^1$. ◁

Definition D.19 (normal Morse data)**.** Let X be $\mathcal{V}_*$ or $\mathcal{M}$. The ***normal Morse data*** for X at $\boldsymbol{p}$ is defined to be the homotopy type of the pair

$$\left(\mathtt{N}(X) \cap h^{-1}([c-\varepsilon, c+\varepsilon]), \mathtt{N}(X) \cap h^{-1}(c+\varepsilon)\right), \tag{D.3.1}$$

where the disk D in the definition of $\mathtt{N}(X)$ is sufficiently small, and ε is a sufficiently smaller positive number. It is proved in [GM88] that these homotopy types are the same for all D and ε sufficiently small.

Example D.20. Suppose that $\mathcal{V}_*$ is a smooth algebraic hypersurface near one of its points $\boldsymbol{p}$.

(1) If $X = \mathcal{V}$ then $\mathtt{N}(X)$ is the single point $\boldsymbol{p}$. Formally, the homotopy type is that of $(\{\boldsymbol{p}\}, \emptyset)$.
(2) If $X = \mathcal{M}$ then $\mathtt{N}(X)$ has the homotopy type of $(D \setminus \mathbf{0}, \boldsymbol{q})$, where D is a small disk and $\boldsymbol{q}$ is a point on the boundary of D. This is the reduced homotopy type of a circle, cyclic in dimension 1 and null in every other dimension. ◁

The following theorem is stated for the case $X = \mathcal{V}_*$ in [GM88, Theorem SMT B on page 8] and for the case $X = \mathcal{M}$ in [GM88, unnamed theorem on page 12]; the equivalent characterizations of the homotopy type are stated in [GM88, pages 7, 66–67, 120–122].

Theorem D.21 (attachments are determined by Morse data)**.** *Let X be either $\mathcal{V}_*$ or $\mathcal{M}$ with a Whitney stratification as above, and let $\mathbf{p}$ be a critical point for h in a stratum $\mathcal{S}$ with critical value $c = h(\boldsymbol{p})$.*

1. *The homotopy type of the attachment at $\mathbf{p}$ is the product, in the category of pairs, of the normal and tangential Morse data as given in Definitions D.19 and D.17.*
2. *The tangential data for a stratum of codimension k is always the reduced homology of a $(d-k)$-sphere: rank 1 in dimension $(d-k)$ and vanishing otherwise.*
3. *The normal data has the following characterizations.*

   (i) When $X = \mathcal{V}_*$, the normal data is homotopy equivalent to the pair $(Cone(\ell^-), \ell^-)$, where $Cone(Y)$ is the topological quotient $Y \times [0,1] \,/\, Y \times \{1\}$ and ℓ^- is the ***lower halflink*** *defined as the level set of $\mathtt{N}(X)$ at height $c - \varepsilon$ for sufficiently small $\varepsilon > 0$.*
   (ii) When $X = \mathcal{M}$, the normal data is homotopy equivalent to the pair $(\mathcal{L}^+(X), \partial\mathcal{L}^+(X))$, where $\mathcal{L}^+(X)$ is the part of $\mathcal{L}(X)$ at height at least c.
   (iii) When $X = \mathcal{M}$, the normal data is also homotopy equivalent to the pair $(\mathcal{L}^+(X), \mathcal{L}^0(X))$, where $\mathcal{L}^0(X)$ is the intersection of $\mathcal{L}(X)$ with the level set $\{\boldsymbol{z} \in X : h(\boldsymbol{z}) = c\}$.

□

Remark. Goresky and MacPherson have this to say [GM88, page 9]: "Theorem SMT Part B, although very natural and geometrically evident in examples, takes 100 pages to prove rigorously in this book."

Example D.22 (complement of S^2 in $\mathbb{R}^3$)**.** Let X be the complement of the unit

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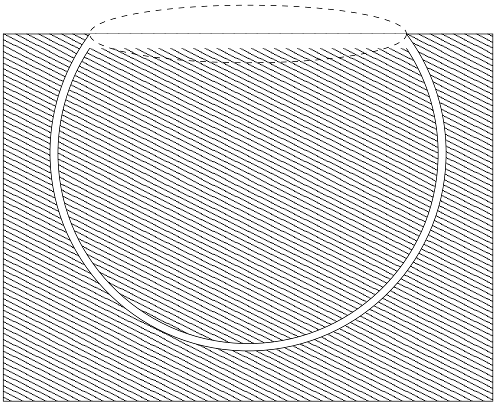

Figure D.6 The complement of the unit sphere up to height $+1/2$.

sphere $S \subseteq \mathbb{R}^3$. The function $h(x, y, z) = z$ extends to a proper height function on $\mathbb{R}^3$, which is Morse with respect to the stratification $\{S, X\}$.

There are no critical points in X but there are two in S: the South pole and the North pole. In each case the normal slice is an interval minus a point, so the normal data is homotopy equivalent to (S^0, S^0_-), where S^0 is two points, one higher than the other, and S^0_- is the lower of the two points. For the South pole, which has Morse index 0, the tangential data is a point, so the attachment is (S^0, S^0_-), which is the addition of a disconnected point. Figure D.6 illustrates that for $-1 < a < 1$, the space $X_{\leq a}$ is in fact the union of two contractible components. The North pole has Morse index 2, so the tangential data at the North pole is $(D^2, \partial D^2)$, a polar cap modulo its boundary. Taking the product with the normal data gives two polar caps modulo all of the lower one and the boundary of the upper one. This is the upper polar cap sewn down along its boundary, the boundary being a point in one of the components. Thus, one component becomes a sphere and the other remains contractible. ◁

Suppose we have a closed space $Y \subset \mathbb{R}^d$ whose complement X we view as a stratified space with Morse function h. If $\boldsymbol{p}$ is a critical point for h in some stratum S then there is a local coordinatization of Y as $S \times B_{\boldsymbol{p}}$, where $B_{\boldsymbol{p}}$ is a small ball of dimension $d - k$ and k is the dimension of S. The set $B_{\boldsymbol{p}} \setminus Y$ is this ball minus the origin, so it is a cone over $L(\boldsymbol{p})$ with vertex $\boldsymbol{p}$. Any chain in $B_{\boldsymbol{p}} \setminus Y$ may be brought arbitrarily close to $\boldsymbol{p}$.

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D.4 Stratified Morse theory for complex manifolds

If X is a complex variety then the Morse data has an alternate description obeying the complex structure of X. Let S be a stratum containing a critical point $\boldsymbol{p}$, let $N(\boldsymbol{p})$ be a small ball in the normal space to S at $\boldsymbol{p}$, and define the ***complex link*** $\mathcal{L}(S)$ to be the intersection of X with a generic hyperplane $A \subseteq N(\boldsymbol{p})$ that comes sufficiently close to $\boldsymbol{p}$ but does not contain it. It is shown in [GM88, page 16] that the normal Morse data at $\boldsymbol{p} \in X$ is given in terms of $\mathcal{L}(S)$ by the pair

$$(\text{Cone}^{\mathbb{R}}(\mathcal{L}(S)), \mathcal{L}(S)), \tag{D.4.1}$$

where $\text{Cone}^{\mathbb{R}}(\mathcal{L}(S))$ denotes the real cone over $\mathcal{L}(S)$. In other words, the normal link has the homotopy type of the pair $(\mathcal{L}(S) \times [0,1] \;/\; \mathcal{L}(S) \times \{1\}\,,\; \mathcal{L}(S) \times \{0\})$, where the real cone (the first space of this pair) is defined as a quotient.

Suppose that X has dimension d, the stratum S has dimension k, and the ambient space has dimension n (all dimensions are complex). Then $N(\boldsymbol{p})$ is a complex space of dimension $n - k$, its intersection with a generic hyperplane has dimension $n - k - 1$, and thus

$$\dim_{\mathbb{C}} \mathcal{L}(S) = d - k - 1\,.$$

In fact the homeomorphism type of the complex link depends on X and S but not on the individual choice of $p \in S$, nor the ambient space, nor the choice of proper Morse function h on the stratified space X (see [GM88, Section II:2.3]).

Suppose next that X is the complement of a d-dimensional variety in $\mathbb{C}^{d+1}$. A formula for the Morse data at a point $p \notin X$ in a stratum S is given [GM88, page 18] by

$$(\mathcal{L}(S), \partial\mathcal{L}(S)) \times (B^1, \partial B^1)\,, \tag{D.4.2}$$

where B^1 is a real interval (which can be interpreted as a 1-ball).

Theorem D.23. (i) *If X is a complex analytic variety of dimension d then X has the homotopy type of a cell complex of dimension at most d.* (ii) *If X is the complement in a domain of $\mathbb{C}^n$ of a complex variety of dimension d then X has the homotopy type of a cell complex of dimension at most* $2n - d - 1$.

Remark. The proof of this result in [GM88] is somewhat difficult, mostly due to the necessity of establishing the invariance properties of the complex link. The result, however, is very useful. For example, suppose that X is the complement of the zero set of a polynomial in n variables. Then $d = n - 1$ and the homotopy dimension of X is at most n. Note that X may have strata of any complex dimension $j \le d$, and that the complement of a j-dimensional complex space in $\mathbb{C}^n$ has homotopy dimension $2n - 2j$. The theorem asserts that the complex structure prevents the dimensions of contributions at strata of dimensions $j < d$ from exceeding the dimension of the contributions from d-dimensional strata.

Proof sketch (*i*) Assume that the variety is embedded in $\mathbb{C}^n$ and that the height function h has been chosen to be the square of the distance from a generic point. We examine the homotopy type of the attachment at a point $\boldsymbol{p}$ in a stratum of dimension k. It suffices, as in the proof of Theorem C.39, to show that each attachment has the homotopy type of a cell complex of dimension at most d.

First, if $k = d$ ($\boldsymbol{p}$ is a smooth point) then, as was observed prior to stating Theorem C.39, the Morse index of h is at most d. The attachment is $(B^i, \partial B^i)$ where i is the Morse index of h, so in this case the homotopy type of the attachment is at most d.

When $k < d$, we proceed by induction on d. The tangential Morse data has the homotopy type of a cell complex of dimension at most k. The space $\mathcal{L}(S)$ is a complex analytic space, with complex dimension one less than the dimension of the normal slice, meaning it has dimension $d-k-1$. The induction hypothesis shows that the homotopy dimension of $\mathcal{L}(S)$ is at most $d - k - 1$. Taking the cone brings the dimension to at most $d - k$ and adding the dimension of the tangential data brings this up to at most d, completing the induction.

(*ii*) When X is the complement of a variety $\mathcal{V}$, still assuming h to be the square of the distance to a generic point, all critical points with respect to the pair $(X, \mathcal{V})$ are contained in $\mathcal{V}$, not in X. Again it suffices to show that the attachments all have homotopy dimension at most $2n - d - 1$, and again we start with the case $k = d$. Here $\boldsymbol{p}$ is a smooth point of $\mathcal{V}$, so the normal data is the same as for the complement of a point in $\mathbb{C}^{n-d}$, which is $S^{2(n-d)-1}$. The tangential data has homotopy dimension at most d, so the attachment has dimension at most $2n - d - 1$.

When $k < d$, we again proceed by induction on d. The link $\mathcal{L}(S)$ is the complement of $\mathcal{V} \cap A$ in a generic hyperplane A. We have directly $\dim_{\mathbb{C}} N(\boldsymbol{p}) = n - k$ and $\dim_{\mathbb{C}}(A) = n - k - 1$, and $\dim_{\mathbb{C}}(\mathcal{V} \cap A) = d - k - 1$ because $\mathcal{V}$ has codimension $n - d$, intersects A generically, and $k \leq d - 1$. The induction hypothesis applied to the complement of $\mathcal{V} \cap A$ in A shows that $\mathcal{L}(S)$ has the homotopy type of a cell complex of dimension at most $2(n - k - 1) - (d - k - 1) - 1 = 2n - d - k - 2$. The normal Morse data is the product of this with a 1-complex, hence it has homotopy dimension at most $2n - d - k - 1$, and taking the product with the tangential Morse data brings the dimension up to at most $2n - d - 1$, completing the induction. □

It is useful for the main part of this book to summarize the results from this

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section for complements of manifolds, applying the Künneth formula to obtain a description of the attachments in terms of specific relative cycles.

Definition D.24 (quasi-local cycles). A (relative or absolute) ***local cycle*** at a point $\mathbf{p}$ is a cycle which may be deformed so as to be in an arbitrarily small neighborhood of $\mathbf{p}$. Given a stratified space with Morse function h, a ***quasi-local cycle*** at a critical point $\mathbf{p}$ of the stratification is a cycle $C_\perp \times C_\|$, where $C_\|$ is a disk in S on which h is strictly maximized at $\mathbf{p}$, $B_{\mathbf{p}}$ is a small ball around $\mathbf{p}$ in the normal slice, $C^\perp$ is a local cycle in $(B_{\mathbf{p}} \setminus Y, (B_{\mathbf{p}} \setminus Y)_{\leq h(\mathbf{p})-\varepsilon})$, and the product is taken in any local coordinatization of a neighborhood of $\mathbf{p}$ by $B_{\mathbf{p}} \times S$.

Theorem D.25. *Let X be the complement of a complex variety of dimension d in $\mathbb{C}^{d+1}$. Then X may be built by attaching spaces that are homotopy equivalent to cell complexes of dimension at most $d+1$. Consequently, $H_d(X)$ has a basis of quasi-local cycles which may be described as $\mathcal{B} = \{\sigma_{\mathbf{p},i}\}_{\mathbf{p},i}$, where $\mathbf{p}$ ranges over critical points in different strata, and each $\sigma_{\mathbf{p},i} \in X^{c,\mathbf{p}}$. For each fixed $\mathbf{p}$, the projection $\pi_* : X^{c,\mathbf{p}} \to (X^{c,\mathbf{p}}, X_{\leq c-\varepsilon}) = X^{\mathbf{p},\mathrm{loc}}$ maps the set $\{\sigma_{\mathbf{p},i}\}$ to a basis for the relative homology group $H_d(X^{\mathbf{p},\mathrm{loc}})$.* □

Notes

The idea to use Morse theory to evaluate integrals was not one of the original purposes of Morse theory. Nevertheless, the utility of Morse theory for this purpose has been known for over 50 years. Much of the history appears difficult to trace: the present authors learned it from Yuliy Baryshnikov, who related it as mathematical folklore from Arnold's seminar. The smooth Morse theory in this chapter (and some of the pictures) is borrowed from Milnor's classic text [Mil63]. Stratified Morse theory is a relatively new field, in which the seminal text is [GM88]; most of our understanding came from this text.

The result usually quoted as the description of the attachment in the stratified case (a stratified version of Theorem C.28) is an unnumbered result named "Theorem" in [GM88, Section 3.12]. This computes the change in topology of a stratified space X on which the function h is proper. When h is a continuous function on $\mathbb{C}_*^d$, this requires the subset X to be closed. We are chiefly interested in the space $X = \mathcal{V}^c$ which is not closed. Dealing with nonproper height functions requires two extra developmental steps. The first is to develop a system for keeping track of the change in topology of the complement of a closed space up to a varying height cutoff. This computation is similar to the one for the space itself. Goresky and MacPherson state the two results together in a

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later version of the “Main Theorem” of [GM88], and we have followed their example, stating the results together in Theorem D.21. The second way h can fail to be proper occurs at infinity. The results of [GM88] across the height interval $[a, b]$ can be extended to unbounded spaces when there are no critical points at infinity with heights in $[a, b]$. This was the motivation for the results on CPAI derived in [BMP22], which we use in Chapter 7.

Additional exercises

Exercise D.4 (Whitney umbrella)**.** Let $f(x, y, z) = x^2 + y^2 z$ be the polynomial whose real variety $\mathcal{V}_f$ forms the ***Whitney umbrella***. Decompose $\mathcal{V}_f$ into the union of smooth sets by computing algebraic equations for its singularities, the singularities of its singularities, and so on until no singularities remain. Either prove that this decomposition is a Whitney stratification of $\mathcal{V}_f$, or prove that it is not and find a refinement that is.

Exercise D.5. Let X be the complement in $\mathbb{C}^2$ of the smooth curve $x^2 + y^2 = 1$. Define a Morse function and use it to compute the homology of X.

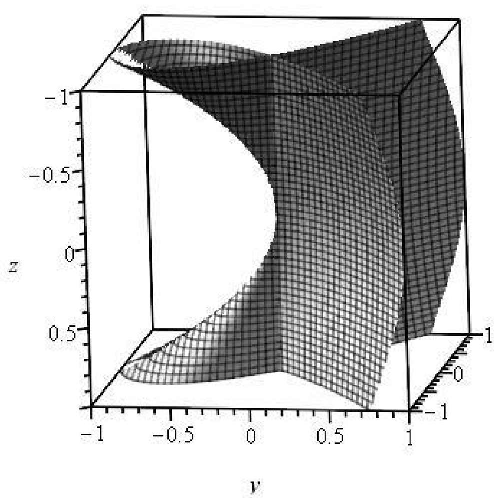

Figure D.7 The real variety in Exercise D.6.

Exercise D.6. Let $Q = x^2 - y^3 - z^2 y^2$ and let $\mathcal{V}_Q$ denote the corresponding

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real affine variety shown in Figure D.7. Compute the set S of singularities of $\mathcal{V}_Q$, and then determine whether $\{\mathcal{V}_Q \setminus S, S\}$ is a Whitney stratification of $\mathcal{V}_Q$. *Hint:* Consider points $x_n = (0, -t^2, t) \in \mathcal{V}_Q \setminus S$ and $y_n = (0, t, 0) \in S$.

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