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MINIMAL LINKAGE AND THE GORENSTEIN LOCUS OF AN IDEAL

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Introduction

Let I be a Cohen-Macaulay ideal of grade $g > 0$ in a local Gorenstein ring (R, m) with residue class field k. An R-ideal J is said to be linked to I with respect to the regular sequence $\underline{\alpha} = \alpha_1, \cdots, \alpha_g \subset I \cap J$ if $J = (\underline{\alpha})\colon I$ and $I = (\underline{\alpha})\colon J$ ([6]). In this paper we are concerned with the following question: how big is $\dim_k((\underline{\alpha}, mJ)/mJ)$? Obviously this dimension is at most g, but it could be as small as 0. If it is g then the link from J to I is called a minimal link, which is in most respects the desired type of link. The only general result known in this direction is that if I is Gorenstein, then $\dim_k((\underline{\alpha}, mJ)/mJ) = g$ unless both I and J are complete intersections (see [1], Proposition 5.2). We are able to generalize this fact to the case where $(R/I)_p$ is Gorenstein for all prime ideals p in R/I with $\dim(R/I)_p \leq 4$; however we have to assume that I is generically a complete intersection ideal, and that R is a complete intersection (Theorem 2.3). Without the assumption on R we prove that if I is generically a complete intersection, and if for a fixed integer r the type of $(R/I)_p$ is at most r for all prime ideals p in R/I with $\dim(R/I)_p \leq (r+1)^2$, then $\dim_k((\underline{\alpha}, mJ/mJ)) \geq g - r$ (Proposition 2.1). If $r = 1$, i.e. if R/I is Gorenstein in codimension 4, then this estimate shows the dimension is at least $g - 1$. Theorem 2.3 can also be interpreted to yield a strong upper bound for the codimension of the non-Gorenstein-locus of certain perfect ideals: Let R be a regular local ring. Let I be an R-ideal which is generically a complete intersection, and assume that I is in the even linkage class of a Gorenstein ideal (i.e., there exists a sequence of links $I \sim I_1 \sim I_2 \sim \cdots \sim I_{2n}$ with I_{2n} a Gorenstein ideal); then I is a Gorenstein ideal provided that $(R/I)_p$ is Gorenstein for all prime ideals p of R/I with $\dim(R/I)_p \leq 4$ (Corollary 3.1).

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§1. General facts about linkage

In this section, we fix the notations we will be using throughout the paper and review some definitions and results from [4].

Let (R, m) be a local Noetherian ring, let I be an R-ideal, and M a finitely generated R-module. By $\nu(M)$ we denote the minimal number of generators of M, $\operatorname{ht}(I)$ is the height of I and $r(R) = \dim_{R/m}(\operatorname{Ext}_R^d(R/m, R))$ stands for the type of R (if R is Cohen-Macaulay of dimension d). We say that I is Cohen-Macaulay or Gorenstein if the ring R/I has any of these properties. The ideal I is a complete intersection if I is generated by a regular sequence, I is called generically a complete intersection if I is unmixed and I_p is a complete intersection for all $p \in \operatorname{Ass}(R/I)$, and I is an almost complete intersection if $\nu(I) \leq \operatorname{grade}(I) + 1$. We say that R is a complete intersection if $\hat{R}$ is a regular local ring modulo a complete intersection ideal. For an integer k, R satisfies (R_k) if R_p is regular for all $p \in \operatorname{Spec}(R)$ with $\dim R_p \leq k$, R is (G_k) if R_p is Gorenstein for all $p \in \operatorname{Spec}(R)$ with $\dim R_p \leq k$, and I satisfies (CI_k) if I_p is a complete intersection for all $p \in \operatorname{Spec}(R/I)$ with $\dim(R/I)_p \leq k$. For a matrix A with entries in R, $I_t(A)$ is the R-ideal generated by all $t \times t$ minors of A, and for a set of elements $\underline{f} = f_1, \cdots, f_n \subset R$ we will denote by $(\underline{f})$ the R-ideal generated by $f_1, \cdots, f_n$ whereas $(\underline{f})^t$ stands for the transpose of the matrix $(f_1 \cdots f_n)$. If X is a finite set of indeterminates we set $R(X) = R[X]_{mR[X]}$.

Definition 1.1 ([4]). Let (R, I) and (S, J) be pairs of Noetherian local rings R, S, and ideals $I \subset R$, $J \subset S$.

a) (S, J) is a *deformation* of (R, I) (with respect to $\underline{a}$) if there is a sequence $\underline{a} \subset S$ which is regular on S and S/J such that $(S/(\underline{a}), (J, \underline{a})/(\underline{a})) = (R, I)$.

b) (S, J) and (R, I) are *equivalent* if there are finite sets of variables X over S, and Z over R, and an isomorphism $\varphi\colon S[X] \xrightarrow{\sim} R[Z]$ such that $\varphi(JS[X]) = IR[Z]$.

Definition 1.2 ([6]). Let R be a local Cohen-Macaulay ring, and let I and J be two (proper) R-ideals, then I and J are said to be (algebraically) *linked* (with respect to $\underline{\alpha}$) (written $I \sim J$), if there exists a regular sequence $\underline{\alpha} = \alpha_1, \cdots, \alpha_g \subset I \cap J$ such that $J = (\underline{\alpha})\colon I$ and $I = (\underline{\alpha})\colon J$.

It is known that if R is a local Gorenstein ring, I an unmixed R-ideal of grade g, and $\underline{\alpha} = \alpha_1, \cdots, \alpha_g \subset I$ a regular sequence with $(\underline{\alpha}) \neq I$, then $J = (\underline{\alpha})\colon I$ is linked to I ([6]). If moreover I is Cohen-Macaulay, then J is Cohen-Macaulay, and $J/(\underline{\alpha})$ is the canonical module of R/I ([6]). Hence $\nu(J/(\underline{\alpha})) = r(R/I)$, and in particular, $\nu(J) = r(R/I) + g$ if and only if $\underline{\alpha} = \alpha_1, \cdots \alpha_g$ form part of a minimal generating set of J. In this case, we say that the link from J to I is *minimal*. Two R-ideals I and J are said to be in the same linkage class if there is a sequence of n links $I = I_0 \sim I_1 \sim \cdots \sim I_n = J$. If in addition n can be chosen to be even, then I and J are in the *same even linkage class*.

DEFINITION 1.3 ([3], [4]). Let R be a local Gorenstein ring, let I be an unmixed R-ideal of grade g, fix a generating sequence $\underline{f} = f_1, \cdots, f_n$ of I, let $X = (X_{ij})$ be a generic $g \times n$ matrix, let $S = R[X]$, $\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_g \end{pmatrix} = X \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}$. Then $\underline{\alpha} = \alpha_1, \cdots, \alpha_g \subset IS$ is an S-regular sequence, and we call $L_1(\underline{f}) = (\underline{\alpha})S \colon IS \subset S$ a *first generic link* of I.

In [4], 2.11, it is shown that up to equivalence in the sense of Definition 1.1b, the pair $(S, L_1(\underline{f}))$ only depends on I, but not on the chosen generating sequence $\underline{f}$. Hence we write $L_1(I)$ instead of $L_1(\underline{f})$. In [4], 2.13, we also remarked that if $L_1(I) \subset R[X]$ is a first generic link of I, and $p \in \operatorname{Spec}(R)$, $I \subset p$, then $L_1(I)R_p[X]$ is a first generic link of I_p. We will use the following property of generic links.

PROPOSITION 1.4 ([4]). *Let (R, m) be a local Gorenstein ring, let I be a Cohen-Macaulay R-ideal, and let J be linked to I with respect to the regular sequence $\underline{\alpha} = \alpha_1, \cdots, \alpha_g$. Fix a generating sequence $\underline{f} = f_1, \cdots, f_n$ of I and a $g \times n$ matrix $C = (C_{ij})$ with entries in R such that $(\underline{\alpha})^t = C(\underline{f})^t$. Let $L_1(\underline{f}) \subset R[X]$ be a first generic link as defined in* 1.3, *and consider $p = (m, X_{ij} - C_{ij})R[X] \in \operatorname{Spec}(R[X])$.*

Then $(R[X]_p, L_1(\underline{f})R[X]_p)$ is a deformation of (R, J).

§ 2. Minimal linkage

For the proof of the main result (Theorem 2.3) we need two propositions which might also be of independent interest.

PROPOSITION 2.1. *Let (R, m) be a local Gorenstein ring with residue class field k, let I be a Cohen-Macaulay R-ideal of grade g which is generically a complete intersection, and assume that there is an integer r such that $r((R/I)_p) \leq r$ for all $p \in \operatorname{Spec}(R/I)$ with $\dim(R/I)_p \leq (r+1)^2$. Let J be an R-ideal linked to I with respect to the regular sequence $\underline{\alpha} = \alpha_1, \cdots, \alpha_g$.*

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Then $\dim_k((\underline{\alpha}, mJ)/mJ) \geq g - r$.

Proof. Let $L_1(I) \subset R[X]$ be a generic link of I, then by Proposition 1.4, there exists $p \in \operatorname{Spec}(R[X])$ such that $(R[X]_p, L_1(I)_p)$ is a deformation of (R, J). Set $(\tilde{R}, \tilde{J}) = (R[X]_p, L_1(I)_p)$ and let $\underline{\tilde{\alpha}} = \tilde{\alpha}_1, \cdots, \tilde{\alpha}_g$ be the $\tilde{R}$-regular sequence defining the link $I\tilde{R} \sim \tilde{J}$. The $\tilde{R}$-ideal $I\tilde{R}$ has the same properties as I, $\nu(\tilde{J}) = \nu(J)$, but since I is generically a complete intersection, and $\tilde{J}$ is the localization of a first generic link of I we also know that $\tilde{\alpha}_1, \cdots, \tilde{\alpha}_g$ generate $\tilde{J}$ generically ([3], 2.5). Moreover let $\tilde{m}$ be the maximal ideal of $\tilde{R}$, then

$$\begin{aligned}
\dim_k((\underline{\tilde{\alpha}}, \tilde{m}\tilde{J})/\tilde{m}J) &= \nu(J) - \nu(\tilde{J}/(\underline{\tilde{\alpha}})) \\
&= \nu(\tilde{J}) - r(R/IR) \\
&= \nu(J) - r(R/I) \\
&= \nu(J) - \nu(J/(\underline{\alpha})) \\
&= \dim_k((\underline{\alpha}, mJ)/mJ)\,.
\end{aligned}$$

Hence we do not change the assumptions or conclusions in the proposition if we replace $I, \underline{\alpha}, J$ by $I\tilde{R}, \underline{\tilde{\alpha}}, \tilde{J}$. However we may now assume that J is generically generated by $\alpha_1, \cdots, \alpha_g$.

Now let $t = \dim_k((\underline{\alpha}, mJ)/mJ)$. After extending the residue class field if needed and changing $\alpha_1, \cdots, \alpha_g$ by elementary transformations, we may assume that $\alpha_1, \cdots, \alpha_t$ form part of a minimal generating set of J and of J_p for all $p \in \operatorname{Ass}(R/J)$. After factoring out $\alpha_1, \cdots, \alpha_t$ we are in the following situation: (R, m) is a local Gorenstein ring, I is a Cohen-Macaulay R-ideal, $r((R/I)_p) \leq r$ for all p with $\dim (R/I)_p \leq (r+1)^2$, J is linked to I with respect to $\underline{\alpha}$, J is generically a complete intersection, but moreover $\underline{\alpha} \subset mJ$, and grade $J = g - t$. We need to prove that grade $J \leq r$, since then $t \geq g - r$. From now on we write again grade $J = g$, and we will show $g \leq r$. We may assume $g > 0$.

Let $\underline{f} = f_1, \cdots, f_n$ be a generating set of J. Since $\underline{\alpha} \subset mJ$, there exists a $g \times n$ matrix A with entries in m such that $(\underline{\alpha})^t = A(\underline{f})^t$. Let X be a generic $g \times n$ matrix, set $(\underline{\tilde{\alpha}}) = X(\underline{f})^t$, consider the first generic link $L_1(J) = L_1(\underline{f}) = (\underline{\tilde{\alpha}})R[X]: JR[X]$, and write $T = R[X]_{(m,X)}$. Because the entries of A are in m, it follows from Proposition 1.4 that $(T, L_1(\underline{f})T)$ is a deformation of (R, I). Since R/I has the property that $r((R/I)_p) \leq r$ for all prime ideals p with $\dim (R/I)_p \leq (r+1)^2$, any deformation of R/I, in particular $T/L_1(\underline{f})T$, has the same property (cf. [4], 2.3). But because the locus $\{p \mid p \in \operatorname{Spec}(R[X]),\ r(R[X]/L_1(\underline{f})_p) \geq r+1\} = \{p \mid p \in \operatorname{Spec}(R[X]),\ \nu(JR[X]/(\underline{\tilde{\alpha}})_p) \geq r+1\}$ is defined by a homogeneous ideal in $R[X]$, it even follows that $r((R[X]/L_1(\underline{f}))_p) \leq r$ for all $p \in \operatorname{Spec}(R[X]/L_1(\underline{f}))$ with $\dim (R[X]/L_1(\underline{f}))_p \leq (r+1)^2$.

For $q \in \operatorname{Ass}(R/J)$ let $\underline{h} = h_1, \cdots, h_g$ be a minimal generating set of J_q, let Y be a generic $g \times g$ matrix, set $(\underline{\beta})^t = Y(\underline{h})^t$, and consider $L_1(J_q) = L_1(\underline{h}) \subset R_q[Y]$. Then by [4], 2.13.b, $(R_q[Y], L_1(\underline{h}))$ is equivalent to the pair $(R_q[X], L_1(\underline{f})R_q[X])$, and hence also $R_q[Y]/L_1(\underline{h})$ has the property that $r((R_q[Y]/L_1(\underline{h}))_p) \leq r$ for all prime ideals p with $\dim (R_q[Y]/L_1(\underline{h}))_p \leq (r+1)^2$. Instead of J_q and R_q we write again J and R. We have to show that $g \leq r$.

Suppose that $g > r$. Then $p = (m, I_{g-r}(Y)) \in \operatorname{Spec}(R[Y])$, with $p \supset (\underline{\beta}, \det(Y)) = L_1(\underline{h})$, and $\dim (R[Y]/L_1(\underline{h}))_p = (r+1)^2$. However, $r(R[Y]/L_1(\underline{h}))_p = \nu((JR[Y]/(\underline{\beta}))_p) = r+1$, which is impossible by our assumptions. Therefore, $g \leq r$. $\square$

Proposition 2.2. *Let R be a Noetherian local ring which is a complete intersection, let I be an unmixed R-ideal of height one, and assume that I_p is principal for all $p \in \operatorname{Spec}(R)$ with $\dim R_p \leq 3$.*

Then I is a principal ideal.

Proof. By [2], Theorem 3.13, Exp. XI, any complete intersection of dimension at least 4 is parafactorial, i.e., the Picard group of its punctured spectrum is trivial.

Now assume I is not principal and localize at a minimal prime p such that I_p is not principal. Then R_p is a complete intersection of dimension ≥ 4 (by assumption) and I_p represents an element in $\operatorname{Pic}(U)$ where $U = \operatorname{Spec}(R_p) - \{p_p\}$. Since R_p is parafactorial this element is trivial. Hence there is an element of $a \in R$ such that $(a)_q = I_q$ for all $q_p \neq p_p$. This implies that $(a)_p : I_p$ is p-primary which is impossible or else $I_p = (a)_p$ since I is unmixed. $\square$

Theorem 2.3. *Let R be a Noetherian local ring which is a complete intersection, let I be a Cohen-Macaulay R-ideal of grade g, and assume that (R, I) has a deformation $(\tilde{R}, \tilde{I})$ where $\tilde{I}$ is generically a complete intersection and $\tilde{R}/\tilde{I}$ satisfies (G_4). Let $\underline{\alpha} = \alpha_1, \cdots, \alpha_g \subset I$ be a regular sequence with $(\underline{\alpha}) \neq I$, and set $J = (\underline{\alpha}) : I$.*

Then either $\underline{\alpha}$ form part of a minimal generating set of J, or both I and J are complete intersections.

Proof. By [4], 2.16, there exists an $\tilde{R}$-ideal $\tilde{J}$ linked to $\tilde{I}$ with respect to a regular sequence $\underline{\tilde{\alpha}}$ such that $(\tilde{R}, \tilde{J})$ is a deformation of (R, J). As in the proof of Proposition 2.1 one sees that $\underline{\alpha}$ is part of a minimal generating set of J if and only if $\underline{\tilde{\alpha}}$ is part of a minimal generating set of $\tilde{J}$. Hence we may replace $I, \underline{\alpha}, J$, by $\tilde{I}, \underline{\tilde{\alpha}}, \tilde{J}$ and thus assume that I is generically a complete intersection, and R/I satisfies (G_4).

Then we may apply Proposition 2.1 with $r = 1$, and we obtain $\dim_k((\underline{\alpha}, mJ)/mJ) \geq g - 1$. After extending the residue class field of R if needed we may assume that $\alpha_1, \cdots, \alpha_{g-1}$ form part of a minimal generating set of J. Hence by factoring out $(\alpha_1, \cdots, \alpha_{g-1})$ we do not change the assumptions and conclusion of the theorem (except possibly the assumption that I is generically a complete intersection, which is irrelevant for the remainder of this proof).

Hence from now on $g = 1$, and $\alpha_1 = \alpha$. Let m be the maximal ideal of R. Assuming that $\alpha \subset mJ$ we will show that J is principal. Then also I is principal since $g = 1$. Let $\underline{f} = f_1, \cdots, f_n$ be a generating set of J, then $\alpha = \sum_{i=1}^n C_i f_i$ with $C_i \in m$. For variables $X = X_1, \cdots, X_n$ set $\tilde{\alpha} = \sum_{i=1}^n X_i f_i \in R[X]$ and consider the first generic link $L_1(J) = L_1(\underline{f}) = \tilde{\alpha}R[X]: JR[X]$. Since $C_i \in m$, $(R[X]_{(m,X)}, L_1(\underline{f})R[X]_{(m,X)})$ is a deformation of (R, I), and it follows as in the proof of Proposition 2.1 that $R[X]/L_1(\underline{f})$ satisfies (G_4).

Suppose that J is not principal, then by Proposition 2.2 there exists a prime ideal $p \supset J$ with $\dim R_p \leq 3$ such that J_p is not principal. On the other hand, R/I being (G_4) it follows that I_p is either Gorenstein or the unit ideal, and hence $\nu(J_p) \leq g + 1 = 2$. Thus $\nu(J_p) = 2$, since J_p is not principal. Moreover, any generic link of J_p is equivalent (in the sense of Definition 1.1b) to a localization of a generic link of J, and hence also satisfies (G_4). Therefore localizing at p we may assume that $\dim R \leq 3$, and $\nu(J) = 2$. Let $J = (h_1, h_2)$, $\beta = Y_1 h_1 + Y_2 h_2 \in R[Y_1, Y_2] = S$, and $L_1(J) = L_1(h_1, h_2) = \beta S: JS$. Then $\dim S/L_1(J) \leq 4$, and since $S/L_1(J)$ is (G_4), it follows that $S/L_1(J)$ is Gorenstein. Therefore

$$\nu((JS/\beta S)_{(m,Y_1,Y_2)}) = r((S/L_1(J))_{(m,Y_1,Y_2)}) = 1$$

which is impossible, since $\beta \in (Y_1, Y_2)J$ and therefore

$$\nu((JS/\beta S)_{(m,Y_1,Y_2)}) = \nu(J) = 2 . \qquad \square$$

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§ 3. Applications

The following corollary generalizes a result from [4] which states that if I is an ideal in a regular local ring R such that I is in the linkage class of a complete intersection and R/I satisfies (G_4), then I is Gorenstein.

COROLLARY 3.1. *Let R be a regular local ring, let I be a perfect R-ideal which is generically a complete intersection, and assume that R/I satisfies (G_4).*

Then for any R-ideal J in the even linkage class of I, $r(R/J) \geq r(R/I)$. In particular if I is in the even linkage class of a Gorenstein ideal, then I is Gorenstein.

Proof. Assume that there is a sequence of links $I = I_0 \sim I_1 \sim \cdots \sim I_{2n} = J$. We will prove by induction on n that $r(R/J) \geq r(R/I)$. Let $n = 1$. We may suppose that I is not a complete intersection. Let $\underline{\alpha} = \alpha_1, \cdots, \alpha_g$ be the regular sequence defining the link $I \sim I_1$. By Theorem 2.3, $\underline{\alpha}$ is part of a minimal generating set of I_1, and hence $\nu(I_1) = \nu(I_1/(\underline{\alpha})) + g = r(R/I) + g$. Let $\underline{\beta} = \beta_1, \cdots, \beta_g$ be the regular sequence giving the link $I_1 \sim J$. Then $\nu(I_1) \leq \nu(I_1/(\underline{\beta})) + g = r(R/J) + g$. The above inequations now imply $r(R/J) \geq r(R/I)$. Now let $n \geq 2$. In [4], 2.17 we showed that in some local ring $S = R(X)$, which is obtained from R by a purely transcendental extension of the residue class field, one can find a sequence of links $IS = J_0 \sim J_1 \sim \cdots \sim J_{2n}$ such that S/J_{2n-2} is generically a complete intersection and satisfies (G_4) (since R/I has these properties), and moreover $r(S/J_{2n}) \leq r(R/J)$. Then by induction hypothesis, applied to IS and J_{2n-2}, $r(R/I) = r(S/IS) \leq r(S/J_{2n-2})$ and $r(S/J_{2n-2}) \leq r(S/J_{2n})$. Combining the above inequalities we obtain $r(R/I) \leq r(R/J)$. □

Let R be a regular local ring with residue class field k, and let I be an R-ideal. Consider the graded algebra $A_{\cdot} = \mathrm{Tor}_{\cdot}^{R}(R/I, k)$. We are interested in the condition $A_1^2 = 0$, which means that in a minimal free R-resolution of R/I, none of the Koszul relations on I can occur among the minimal generators of the first syzygy module of I. It is well-known that $A_1^2 = 0$ if I is a Gorenstein ideal of grade 3, but not a complete intersection ([1]). The next corollary generalizes this result:

COROLLARY 3.2. *Let R be a regular local ring, let I be a perfect R-ideal of grade* 3, *which is not a complete intersection, and assume that I is generically a complete intersection and R/I satisfies (G_4).*

Then $A_1^2 = 0$.

Proof. Let m be the maximal ideal of R, and let F: $0 \to F_3 \xrightarrow{\varphi_3} F_2 \xrightarrow{\varphi_2} F_1 \xrightarrow{\varphi_1} F_0 \to R/I \to 0$ be a minimal free R-resolution of R/I. We choose bases $F_2 = \oplus Rd_i$, $F_1 = \oplus \mathrm{Re}_i$, and set $f_i = \varphi_1(e_i)$.

Suppose that $\Lambda_1^2 \neq 0$. Then we may assume that $\varphi_2(d_1) = f_2e_1 - f_1e_2$. It is clear that $\mathrm{ht}\,(f_1R + f_2R) = 2$, since otherwise $f_1 = ab_1$ and $f_2 = ab_2$ with $0 \neq a \in m$, $b_1 \in R$, $b_2 \in R$, and hence $\varphi_2(d_1) = a(b_2e_1 - b_1e_2)$ with $b_2e_1 - b_1e_2 \in \ker \varphi_1$ which is a contradiction to the minimality of $F_{\bullet}$. Because $\mathrm{ht}\,(f_1R + f_2R) = 2$, we may complete f_1, f_2 to a regular sequence $\underline{f} = f_1, f_2, f_3 \subset I$. Let $K_{\bullet} = K(\underline{f}, R) = \Lambda(Rg_1 \oplus Rg_2 \oplus Rg_3)$ be the Koszul complex, and $u_{\bullet}$: $K_{\bullet} \to F_{\bullet}$ a morphism of complexes with $u_0 = \mathrm{id}_R$. We may choose $u_2(g_1 \wedge g_2) = -d_1$.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F_3 & \xrightarrow{\varphi_3} & F_2 & \longrightarrow & F_1 & \longrightarrow & F_0 \\
 & & \uparrow{\scriptstyle u_3} & & \uparrow{\scriptstyle u_2} & & \uparrow{\scriptstyle u_1} & & \| \\
0 & \longrightarrow & K_2 & \xrightarrow{\psi_3} & K_2 & \xrightarrow{\psi_2} & K_1 & \longrightarrow & K_0
\end{array}$$

Set $J = (\underline{f})\colon I$. Since the R-dual (denoted by $-^*$) of the mapping cone of $u_{\bullet}$ yields a resolution of R/J ([6]), we obtain the following presentation of J:

$$K_1^* \oplus F_2^* \xrightarrow{\begin{pmatrix} \psi_2^* & 0 \\ u_2^* & \varphi_3^* \end{pmatrix}} K_2^* \oplus F_3^* \longrightarrow J \longrightarrow 0$$

Since $u_2(g_1 \wedge g_2) = -d_1$ and hence $u_2^*(d_1^*) \notin mK_2^*$, it follows that $\nu(J) < \mathrm{rank}\,(K_2^* \oplus F_3^*) = 3 + r(R/I)$. Thus f_1, f_2, f_3 cannot be part of a minimal generating set of J. This is a contradiction to Theorem 2.3. □

References

[1] D. Buchsbaum and D. Eisenbud, Algebra structures for finite free resolutions, and some structure theorems for ideals of codimension 3, Amer. J. Math., **99** (1977), 447–485.

[2] A. Grothendieck, Cohomologie locale des faisceaux cohérents et Théorèmes de Lefschetz locaux et globaux (SGA II), Exposé XIV, Advanced Studies in Mathematics, Vol. II (1968), North Holland Publishing Co., Amsterdam.

[3] C. Huneke and B. Ulrich, Divisor class groups and deformations, Amer. J. Math., **107** (1985), 1265–1303.

[4] ——, The structure of linkage, Ann. of Math., **126** (1987), 277–334.

[5] ——, Algebraic linkage, to appear, Duke J. Math.

[6] C. Peskine and L. Szpiro, Liaison des variétés algébriques, Invent. Math., **26** (1974), 271–302.

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