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A NOTE ON INVERSE LIMITS OF FINITE SPACES(1)

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1. **Introduction.** The following lemma, which appears as Lemma 4 in [5], was used to determine certain multicoherence properties of inverse limits of continua.

LEMMA. *Let X denote the inverse limit of an inverse system* $\{X_\lambda, f_{\lambda\mu}, \Lambda\}$ *of compact Hausdorff spaces* X_λ. *If* X_λ *has no more than k components* (*where* $k<\infty$ *is fixed*) *for each* $\lambda \in \Lambda$, *then X has no more than k components.*

In this paper we give a set theoretic analogue of this lemma and an extension which was suggested to the author by Professor F. W. Lawvere. An application to inverse limits of finite groups is then given.

2. **Results.** For the notation and terminology relating to inverse limits, see [1]. All inverse systems are taken over directed sets [4]. Throughout this paper π_λ denotes projection from the inverse limit space into the λth factor space (see [1]).

Lemma 1 below is a direct consequence of the lemma stated in the introduction (simply take the discrete topology as the topology for each set X_λ; then the bonding maps $f_{\lambda\mu}$ are continuous and the components are points). However, we include the proof for the sake of completeness and because it is somewhat shorter than the proof offered in [5] for the lemma.

LEMMA 1. *Let X denote the inverse limit of an inverse system* $\{X_\lambda, f_{\lambda\mu}, \Lambda\}$ *of sets* X_λ. *If each set* X_λ *has no more than k elements* (*where* $k<\infty$ *is fixed*), *then X has no more than k elements.*

Proof. Suppose X has $k+1$ distinct elements $x_1, x_2, \ldots, x_{k+1}$. For each $i=1,2,\ldots,k+1$ and $j=1,2,\ldots,k+1$ with $i\neq j$ let $\Lambda_{i,j}=\{\lambda\in\Lambda\colon \pi_\lambda(x_i)=\pi_\lambda(x_j)\}$. Since for each $\lambda\in\Lambda$ there exist two of the points $x_1, x_2, \ldots, x_{k+1}$ that project down into only one point of X_λ, $\bigcup\{\Lambda_{i,j}\colon 1\le i\le k+1,\ 1\le j\le k+1, \text{ and } i\neq j\}=\Lambda$. Since there are only a finite number of the sets $\Lambda_{i,j}$, at least one of them, say Λ_{i_0,j_0}, is cofinal in Λ. Now let $\lambda\in\Lambda$. There exists $\lambda'\in\Lambda_{i_0,j_0}$ such that $\lambda'\ge\lambda$. Since $f_{\lambda'\lambda}\pi_{\lambda'}=\pi_\lambda$, we have $\pi_\lambda(x_{i_0})=f_{\lambda'\lambda}\pi_{\lambda'}(x_{i_0})=f_{\lambda'\lambda}\pi_{\lambda'}(x_{j_0})=\pi_\lambda(x_{j_0})$. Hence, $x_{i_0}=x_{j_0}$ which contradicts the assumption that $x_i, x_2, \ldots, x_{k+1}$ are all distinct.

The following theorem is the extension mentioned in the introduction.

THEOREM 1. *Let X denote the inverse limit of an inverse system* $\{X_\lambda, f_{\lambda\mu}, \Lambda\}$ *of sets* X_λ. *If each set* X_λ *has no more than k elements* (*where* $k<\infty$ *is fixed*), *then there*

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is a cofinal subset Λ' *of* Λ *such that the projection* $\pi_{\lambda'}: X \to X_\lambda$, *is one-to-one for each* $\lambda' \in \Lambda'$.

Proof. By Lemma 1, X has no more than k distinct elements. Enumerate them $x_1, x_2, \ldots, x_m$ $(m \le k)$ so that $i \ne j$ implies $x_i \ne x_j$. For each $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, m$ with $i \ne j$ let $\Lambda_{i,j} = \{\lambda \in \Lambda : \pi_\lambda(x_i) = \pi_\lambda(x_j)\}$. Since $x_i \ne x_j$ for $i \ne j$, none of the sets $\Lambda_{i,j}$ is cofinal in Λ. Therefore, since there are only finitely many of the sets $\Lambda_{i,j}$, $\bigcup \Lambda_{i,j}$ is not cofinal in Λ. Hence, $\Lambda - \bigcup \Lambda_{i,j}$ is cofinal in Λ. Setting $\Lambda' = \Lambda - \bigcup \Lambda_{i,j}$, we see that if $\lambda' \in \Lambda'$ then $\pi_{\lambda'}(x_i) \ne \pi_{\lambda'}(x_j)$ for each $i \ne j$.

We now apply Theorem 1 to inverse limits of finite groups. In what follows let $\{G_\lambda, h_{\lambda\mu}, \Lambda\}$ be an inverse system of groups G_λ and homomorphisms $h_{\lambda\mu}$ and let G denote the inverse limit of $\{G_\lambda, h_{\lambda\mu}, \Lambda\}$. It is well known [3] and easy to verify that G is a subgroup of the cartesian product of the groups G_λ (with coordinatewise "addition") and that, for each $\lambda \in \Lambda$, the projection $\pi_\lambda: G \to G_\lambda$ is a homomorphism. With this in mind, the following result is an immediate consequence of Theorem 1. Recall that the *order* of a finite group is the number of elements in the group.

COROLLARY 1. *If each group* G_λ *has order less than or equal to* k (*where* $k < \infty$ *is fixed*), *then there is a cofinal subset* Λ' *of* Λ *such that, for each* $\lambda' \in \Lambda'$, G *is isomorphic to a subgroup of* $G_{\lambda'}$.

Remark. A special case of Corollary 1 above is the following: If, in addition to the hypotheses of Corollary 1, we assume that each group G_λ is cyclic, then G is also cyclic. Even this special case appears to be new and would not have been suspected if I did not know (initially from Lemma 1) that G has only a finite number of points. Note that an inverse limit of finite cyclic groups (whose orders are not bounded above) need not be cyclic. For example, the p-adic group is an inverse limit of an inverse system

$$G_1 \xleftarrow{h_1} G_2 \xleftarrow{h_2} \cdots \longleftarrow G_n \xleftarrow{h_n} \cdots$$

where G_n is a cyclic group of order p^n, but the underlying space of the p-adic group is homeomorphic to the Cantor set and, therefore, uncountable [2, p. 230].

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