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NON-HERMITIAN SOLUTIONS OF ALGEBRAIC RICCATI EQUATIONS

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ABSTRACT. Non-hermitian solutions of algebraic matrix Riccati equations (of the continuous and discrete types) are studied. Existence is proved of non-hermitian solutions with given upper bounds of the ranks of the skew-hermitian parts, under the sign controllability hypothesis.

1. **Introduction.** In this paper we study the continuous algebraic Riccati equations (CARE)

$$XDX + XA + A^*X - C = 0, \tag{1}$$

and the discrete algebraic Riccati equations (DARE)

$$X = A^*XA + Q - (A^*XB^*)(R + B^*XB)^{-1}B^*XA. \tag{2}$$

Here, $A, B, C = C^*$, $D = D^*$, $Q = Q^*$ and $R = R^*$ are given matrices with the entries in F (where either $F = \mathbf{R}$, the field of real numbers, or $F = \mathbf{C}$, the field of complex numbers) of appropriate sizes, and X is the unknown square size matrix (real or complex, as the case may be).

The CARE and DARE, as well as their more general versions, are ubiquitous in many applications, in particular, in engineering control systems. There is voluminous literature on the CARE and DARE, especially in engineering journals; we mention here only three books [LR1], [Me], [BLW] dedicated mainly to these equations. In almost all of the existing literature, the solution X of (1.1) and (1.2) is assumed to be hermitian (= real symmetric, if $F = \mathbf{R}$); non-hermitian solutions X of (1.1) such that

$$(X^* - X)(A + DX) \leq 0 \tag{3}$$

(*i.e.*, the left-hand side of (1.3) is negative semidefinite hermitian) have been studied in [LR1] (see also Section 2.2 in [LR2]).

Here we specifically study solutions of CARE and DARE that are not assumed to be hermitian. In this case existence of solutions is guaranteed under relatively mild assumptions on the coefficients of these equations. In particular, we give explicit upper bounds

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on the rank of the skew-hermitian part of a solution X of (1.1), under a weaker hypothesis (sign controllability) then the controllability hypothesis used in [LR1]. Both real and complex cases will be considered, as well as the analogous results for DARE (1.2). The obtained results generalize several well-known results in the literature concerning existence of hermitian solutions.

Our main results are expressed in terms of the neutrality index of the hamiltonian matrix.

Throughout the paper the following notation is used: The set of eigenvalues of a matrix Z (including complex conjugate pairs of nonreal eigenvalues of a real matrix) is denoted $\sigma(Z)$. The restriction of a matrix Z (considered as a linear transformation in the standard basis in $\mathbf{C}$ or in $\mathbf{R}$, as appropriate) to its invariant subspace $\mathcal{M}$ is denoted $Z|\mathcal{M}$. Finally, we write I_m for the $m \times m$ identity matrix.

2**. The CARE: main results.** In this section we state our main results concerning CARE (1.1). Let $n \times n$ be the size of matrices A, C, D, X. We consider the complex case first.

Let

$$M = i\begin{bmatrix} A & D \\ C & -A^* \end{bmatrix}, \quad \hat{H} = i\begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}.$$

Then clearly $\hat{H}$ is hermitian and $\hat{H}M = M^*\hat{H}$, in other words, M is self-adjoint with respect to the indefinite scalar product in $\mathbf{C}^{2n}$ defined by $\hat{H}$. Denote by $\gamma(M, \hat{H})$ the *neutrality index* of the pair $(M, \hat{H})$, *i.e.*, $\gamma(M, \hat{H})$ is the maximal dimension of an M-invariant, $\hat{H}$-neutral subspace (recall that a subspace $\mathcal{L} \subseteq \mathbf{C}^{2n}$ is called $\hat{H}$*-neutral* if $x^*\hat{H}y = 0$ for all $x, y \in \mathcal{L}$). In the general framework of finite dimensional spaces with an indefinite metric the notion of the neutrality index was introduced in [LMY] to study low rank perturbations of matrices which are selfadjoint with respect to an indefinite inner product, and used subsequently in [LR3], [LR4] to describe symmetric factorizations of rational matrix functions.

A pair of complex matrices (A, B), where A is $n \times n$ and B is $n \times m$, is called *sign-controllable* if for every $\lambda_0 \in \mathbf{C}$ at least one of the subspaces $\operatorname{Ker}(\lambda_0 I - A)^n$ and $\operatorname{Ker}(-\bar{\lambda}_0 I - A)^n$ is contained in the controllable subspace

(1) $$\mathcal{C}(A, B) = \operatorname{Range}[B, AB, \ldots, A^{n-1}B].$$

THEOREM 2.1. *($F = \mathbf{C}$) Assume $D \geq 0$, and (A, D) is sign controllable. If $2\gamma(M, \hat{H}) \geq n$, then there is a solution X of (1.1) for which*

(2) $$\operatorname{rank}\big(i(X - X^*)\big) \leq 2\big(n - \gamma(M, \hat{H})\big).$$

In particular, if $\gamma(M, \hat{H}) = n$, then there exists a hermitian solution of (1.1), recovering the complex analogue of a result of [F]; see also Chapter 7 in [LR2].

The proof of Theorem 2.1 reveals additional spectral properties of the solutions X whose existence is asserted in Theorem 2.1. Namely, let $\mathcal{S}$ be a subset in the complex plane with the following properties: (1) $\mathrm{Ker}(\lambda I - A)^n \subseteq \mathcal{C}(A, D)$ for $\lambda \in \mathcal{S}$; (2) $\lambda \in \mathcal{S} \Rightarrow -\overline{\lambda} \notin \mathcal{S}$; (3) $\mathcal{S}$ is maximal with respect to the properties (1) and (2): if $\mathcal{T} \supseteq \mathcal{S}$ has the properties (1) and (2), then necessarily $\mathcal{T} = \mathcal{S}$. Then (under the hypotheses of Theorem 2.1) there exists a solution X of (1.1) with the property (2.1), and such that the set of eigenvalues of $A + DX$ with nonzero real parts is contained in $\{-\overline{\lambda} \mid \lambda \in \mathcal{S}\}$.

The proof of Theorem 2.1 will be given in Section 4.

In the real case, an analogous result holds (under some additional hypotheses). To state the result, we use the real $2n \times 2n$ matrices

$$M_r = \begin{bmatrix} A & D \\ C & -A^T \end{bmatrix}, \quad \hat{H}_r = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}. \tag{3}$$

The *neutrality index* $\gamma_r(M_r, \hat{H}_r)$ of the pair $(M_r, \hat{H}_r)$ is defined as the maximal dimension of a *real* M_r-invariant $\hat{H}_r$-neutral subspace in $\mathbf{R}^{2n}$ (a subspace $\mathcal{L} \subseteq \mathbf{R}^{2n}$ is called $\hat{H}_r$-*neutral* if $x^T \hat{H}_r y = 0$ for all $x, y \in \mathcal{L}$). We say that a pair of real matrices (A, B), where A is $n \times n$ and B is $n \times m$, is *sign controllable* if for every $\lambda_0 \in \mathbf{R}$ at least one of the subspaces $\mathrm{Ker}(\lambda_0 I - A)^n$ and $\mathrm{Ker}(-\lambda_0 I - A)^n$ is contained in $\mathcal{C}(A, B)$ (defined by (2.1)) and for every $\lambda + i\mu \in \mathbf{C}$, where λ, μ are real and $\mu \neq 0$, at least one of the two subspaces

$$\mathrm{Ker}\Big((\lambda^2 + \mu^2)I \pm 2\lambda A + A^2\Big)^n$$

is contained in $\mathcal{C}(A, B)$. It is not difficult to see that (A, B) is sign controllable if and only if (A, B) is sign controllable as a pair of complex matrices.

Easy examples (such as the scalar equation $x^2 + 1 = 0$ which has no real solutions) show that in the real case a result analogous to Theorem 2.1 is not valid without additional hypotheses. Our main result for the CARE in the real case is:

THEOREM 2.2. *($F = \mathbf{R}$) Let $D \geq 0$, and let (A, D) be sign controllable. Assume that the matrix M_r is invertible. Then the following statements are equivalent:*

(α) The equation (1.1) has a real solution.

(β) The equation (1.1) has a real solution X for which

$$\mathrm{rank}(X - X^T) \leq 2\Big(n - \gamma_r(M_r, \hat{H}_r)\Big). \tag{4}$$

(γ) The matrix M_r has a real n-dimensional invariant subspace.

(δ) Either n is even, or n is odd and M_r has a real eigenvalue.

The particular case when $\gamma_r(M_r, \hat{H}_r) = n$ yields existence of a real symmetric solution of (1.1), thereby recovering the main result of [F] (see also Chapter 8 in [LR2]), under the additional hypothesis that M_r is invertible.

Several comments on Theorem 2.2 are in order. The equivalence of (γ) and (δ) is well-known (see Section 12.1 in [GLR2], for example), and is presented here for completeness. Note that (α) clearly implies (γ) because if X is a real solution of (1.1), then

$$\begin{bmatrix} A & D \\ C & -A^T \end{bmatrix} \begin{bmatrix} I \\ X \end{bmatrix} = \begin{bmatrix} I \\ X \end{bmatrix} (A + DX), \tag{5}$$

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and so $\operatorname{Im}\begin{bmatrix} I \\ X \end{bmatrix}$ is an M_r-invariant n-dimensional subspace. Also, the inequality (2.3) provides additional information comparing with (α) only when $2\gamma_r(M_r, \hat{H}_r) \geq n$. Thus the main new message of Theorem 2.2 is the implication (γ) $\Rightarrow$ (β). We do not know if the hypothesis that M_r is invertible can be removed in Theorem 2.2.

A sufficient condition for the matrix M_r to be invertible is that A is invertible and

$$\min\Big(\|CA^{-1}D(A^T)^{-1}\|, \|(A^T)^{-1}CA^{-1}D\|\Big) < 1, \tag{6}$$

where $\|\cdot\|$ is a power multiplicative norm (see Proposition 8.1.4 in [LR2]). In this case the matrix $\begin{bmatrix} -C & A^T \\ A & D \end{bmatrix} = -\hat{H}_r M_r$ has equal number of positive and negative eigenvalues.

In connection with Theorem 2.2 we point out also the following fact:

PROPOSITION 2.3. *If the real symmetric matrix* $\begin{bmatrix} -C & A^T \\ A & D \end{bmatrix}$ *has equal number of positive and negative eigenvalues (counted with multiplicities), then the existence of a real n-dimensional M_r-invariant subspace is guaranteed.*

Observe that the additional hypotheses of Theorem 2.2 ($D \geq 0$, sign controllability of (A, D), invertibility of M_r) are not needed in Proposition 2.3. In particular, if $C \geq 0$ (a situation frequently encountered in applications) then the matrix $\begin{bmatrix} -C & A^T \\ A & D \end{bmatrix}$ has n positive and n negative eigenvalues provided it is invertible. We obtain the following corollary.

COROLLARY 2.4. *Assume the hypotheses of Theorem 2.2. Assume further that (2.6) holds or $C \geq 0$. Then (1.1) has a real solution X for which (2.4) holds.*

It is well-known (see, *e.g.*, Theorem 9.1.2 in [LR2]) that if the pair $(-A, D)$ is stabilizable, which is a more restrictive hypothesis than the sign controllability used in Theorem 2.2, and if $D \geq 0$, $C \geq 0$, then (1.1) has a real symmetric solution.

Analogously to the the complex case, additional spectral properties of the real solutions of (1.1) can be identified. Let $\mathcal{S}$ be a set in the complex plane with the following properties: (1) $\lambda_0 \in \mathcal{S} \Rightarrow \overline{\lambda_0} \in \mathcal{S}, -\lambda_0 \notin \mathcal{S}$ (in particular, $\mathcal{S}$ does not intersect the imaginary axis); (2) $\operatorname{Ker}(\lambda_0 I + A)^n \subseteq \mathcal{C}(A, D)$ for every real $\lambda \in \mathcal{S}$; (3) $\operatorname{Ker}\big((\lambda_0^2 + \mu_0^2)I + 2\lambda_0 A + A^2\big)^n \in \mathcal{C}(A, D)$ for every pair of nonreal complex conjugate numbers $\lambda_0 \pm i\mu_0 \in \mathcal{S}$; (4) $\mathcal{S}$ is a maximal set having the properties (1), (2), and (3). Then (under the hypotheses of Corollary 2.4, or of Theorem 2.2 (assuming one of the conditions (α) through (δ) in Theorem 2.2)) there exists a real solution X of (1.1) with the property that the set of eigenvalues of $A + DX$ with nonzero real parts is contained in $\mathcal{S}$ and (2.4) holds.

The proofs of Theorem 2.2 and Proposition 2.3 will be given in Section 5.

3. **Indefinite scalar products and neutrality indices.** In this section we develop some results concerning matrices with respect to indefinite scalar products that are needed for the proofs of Theorems 2.1 and 2.2.

Let H be an $m \times m$ hermitian invertible (complex) matrix, and let A be H-self-adjoint, *i.e.*, $HA = A^*H$. As in Section 2, we denote by $\gamma(A, H)$ the *neutrality index* of (A, H), *i.e.*, the maximal dimension of an A-invariant H-neutral subspace. By Theorem 2.1 of [LMY], an A-invariant H-neutral subspace $\mathcal{M} \subseteq \mathbf{C}^m$ is maximal in the set of all A-invariant H-neutral subspaces if and only if $\dim \mathcal{M} = \gamma(A, H)$.

Denote by $\nu_+(H)\big(\nu_-(H)\big)$ the number of positive (negative) eigenvalues of H, counted with their multiplicities. It is a well-known basic fact in the theory of operators in indefinite scalar product spaces that every H-self-adjoint matrix A has an invariant H-nonnegative subspace of dimension $\nu_+(H)$. (Recall that a subspace $\mathcal{M} \subseteq \mathbf{C}^m$ is called *H-nonnegative* if $x^*Hx \geq 0$ for all $x \in \mathcal{M}$, see, *e.g.*, Section 3.12 in [GLR1]). The following lemma is more informative; it reveals additional structural properties of such subspaces.

In the Lemmas 3.1–3.3 below, it is assumed that A is H-self-adjoint.

LEMMA 3.1. *Let $\mathcal{M}$ be A-invariant H-nonnegative (or H-nonpositive) subspace such that $A|\mathcal{M}$ has no real eigenvalues. Then, in fact, $\mathcal{M}$ is H-neutral.*

PROOF. Using the well-known canonical form (see, *e.g.*, Chapter 3 in [GLR1]) of the pair (A, H) under the transformation $(A, H) \longrightarrow (S^{-1}AS, S^*HS)$, where S is invertible, without loss of generality we can (and do) assume that one of the following two cases holds: (i) $\sigma(A) = \{\lambda_0\}$, where λ_0 is real; (ii) $\sigma(A) = \{\lambda_0 \pm i\mu_0\}$, where λ_0, μ_0 are real and $\mu_0 \neq 0$. If (i) holds, Lemma 2.1 is trivial. Assume that (ii) holds. Then there is invertible S such that

$$K := S^{-1}AS = \begin{bmatrix} J & 0 \\ 0 & J^* \end{bmatrix}; \quad Q := S^*HS = \begin{bmatrix} 0 & I_p \\ I_p & 0 \end{bmatrix},$$

where J is a Jordan matrix with $\sigma(J) = \{\lambda_0 + i\mu_0)$ (this transformation was used in [R]). The subspace $\mathcal{N} = S^{-1}\mathcal{M}$ is K-invariant and Q-nonnegative. Write

$$\mathcal{N} = \operatorname{Im} \begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix},$$

where the matrices X_1 and X_2 have full column rank and p rows (that $\mathcal{N}$ so decomposes follows because J and J^* have no common eigenvalues). We have

$$\begin{bmatrix} X_1^* & 0 \\ 0 & X_2^* \end{bmatrix} Q \begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix} = \begin{bmatrix} 0 & X_1^*X_2 \\ X_2^*X_1 & 0 \end{bmatrix},$$

which is positive semidefinite. This is possible only if $X_1^*X_2 = 0$, and then $\mathcal{N}$ is actually Q-neutral. ∎

A set $\mathcal{S}$ of non-real eigenvalues of A is called a *c-set* of A if $\lambda_0 \in \mathcal{S} \Rightarrow \overline{\lambda_0} \notin \mathcal{S}$, and the set $\mathcal{S}$ is maximal with respect to this property (in other words, if $\mathcal{T}$ is a set of non-real eigenvalues of A such that $\mathcal{S} \subseteq \mathcal{T}$, and $\lambda_0 \in \mathcal{T} \Rightarrow \overline{\lambda_0} \notin \mathcal{T}$, then necessarily $\mathcal{S} = \mathcal{T}$).

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LEMMA 3.2. *Let $\mathcal{M}$ be A-invariant H-nonnegative (resp. H-nonpositive) subspace. Then there exists A-invariant H-nonnegative (resp. H-nonpositive) subspace $\mathcal{L}$ such that $\mathcal{L} \supseteq \mathcal{M}$ and $\dim \mathcal{L} = \nu_+(H)$ (resp. $\dim \mathcal{L} = \nu_-(H)$). Moreover, if the set $\mathcal{S}$ of non-real eigenvalues of $A|\mathcal{M}$ is such that $\lambda \in \mathcal{S} \Rightarrow \bar{\lambda} \notin \mathcal{S}$, then for every c-set $\mathcal{T}$ of A containing $\mathcal{S}$ the subspace $\mathcal{L}$ can be chosen with the additional property that $\mathcal{T}$ is the set of non-real eigenvalues of $A|\mathcal{L}$.*

PROOF. Again, we use the canonical form of the pair (A, H). So it will be assumed that one of the cases (i) and (ii) of the proof of Lemma 3.1 holds. Assume (ii) holds. Then by Lemma 3.1 an A-invariant subspace is H-nonegative (or H-nonpositive) if and only if it is H-neutral. Thus the statement of Lemma 3.2 is contained in Theorem 2.1 of [LMY]. Assume (i) holds, and let $\lambda_0 = 0$ for convenience. It suffices to consider the case when $\mathcal{M}$ is H-nonnegative (then the case of H-nonpositive $\mathcal{M}$ is taken care of by replacing H with $-H$). The statement of Lemma 3.2 concerning the set of non-real eigenvalues of $A|\mathcal{L}$ is trivially empty, and we only have to prove the existence of an A-invariant H-nonegative subspace $\mathcal{L} \supseteq \mathcal{M}$ with $\dim \mathcal{L} = \nu_+(H)$. But the existence of such an $\mathcal{L}$ is a very particular case of a well-known result in the theory of linear operators on infinite dimensional Krein spaces (see, *e.g.*, Corollary 3.3.10 in [AI]) ∎

LEMMA 3.3. *For every c-set $\mathcal{S}$ there exists an A-invariant H-neutral subspace $\mathcal{L}$ of dimension $\gamma(A, H)$ such that $\mathcal{S}$ is the set of non-real eigenvalues of $A|\mathcal{L}$.*

PROOF. Without loss of generality assume that either $\sigma(A) = \{\lambda\}$, $\lambda \in \mathbf{R}$ or $\sigma(A) = \{\lambda, \bar{\lambda}\}$, $\lambda \notin \mathbf{R}$. In the first case any maximal A-invariant H-neutral subspace will do, in the second case let $\mathcal{L} = \mathrm{Ker}(A - \mu I)^m$, where $\mu \in \{\lambda, \bar{\lambda}\} \cap \mathcal{S}$. ∎

Lemmas 3.2 and 3.3 will be needed for the proof of Theorem 2.1.

4. **Proof of Theorem 2.1** We continue to use the notation introduced in Sections 2 and 3.

Let

$$H = \begin{bmatrix} -C & A^* \\ A & D \end{bmatrix}$$

Replacing, if necessary, A by $A+i\alpha I$ for sufficiently large real number α (such replacement does not alter the set of solutions of (1.1)), we assume without loss of generality that H is invertible and $\nu_+(H) = \nu_-(H) = n$ (the size of the matrices A, C, D, and X). Note the equalities

$$H = \hat{H}M, \quad HM = M^*H.$$

LEMMA 4.1. *Assume $D \geq 0$, and (A, D) sign controllable. Then there exists a c-set $\mathcal{S}$ (for the matrix M) such that every n-dimensional M-invariant H-nonpositive subspace $\mathcal{M}$ with $\sigma(M \mid \mathcal{M}) \setminus \mathbf{R} = \mathcal{S}$ is a graph subspace: $\mathcal{M} = \mathrm{Im} \begin{bmatrix} I \\ X \end{bmatrix}$ for some X.*

This result is contained in the proof of Lemma 7.2.6 of [LR2].

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PROOF OF THEOREM 2.1. Let $\mathcal{S}$ be the c-set as in Lemma 4.1. By Lemma 3.3, there exists M-invariant $\hat{H}$-neutral subspace $\mathcal{L}$ of dimension $\gamma(M, \hat{H})$ such that $\mathcal{S}$ is the set of nonreal eigenvalues of $M|\mathcal{L}$. Since $H = \hat{H}M$, the subspace $\mathcal{L}$ is also H-neutral. By Lemma 3.2, there exists an n-dimensional M-invariant H-nonpositive subspace $\mathcal{M}$ such that $\mathcal{M} \supseteq \mathcal{L}$ and $\sigma(M \mid \mathcal{M}) \setminus \mathbf{R} = \mathcal{S}$. By Lemma 4.1,

$$\mathcal{M} = \operatorname{Im} \begin{bmatrix} I \\ X \end{bmatrix}$$

for some $n \times n$ matrix X. One verifies that X is a solution of (1.1). Let $f_1, \ldots, f_p \in \mathbf{C}^n$ (where $p = \gamma(M, \hat{H})$) be such that

$$\begin{bmatrix} I \\ X \end{bmatrix} f_j, \quad j = 1, \ldots, p,$$

form a basis in $\mathcal{L}$, and let $f_{p+1}, \ldots, f_n$ be such that the matrix $Y = [f_1, \ldots, f_n]$ is nonsingular. Since $\mathcal{L}$ is $\hat{H}$-neutral, we have

$$Y^*[I \quad X^*]\hat{H} \begin{bmatrix} I \\ X \end{bmatrix} Y = \begin{bmatrix} 0_p & * \\ * & * \end{bmatrix},$$

where 0_p is $p \times p$. On the other hand,

$$[I \quad X^*]\hat{H} \begin{bmatrix} I \\ X \end{bmatrix} = i(X - X^*),$$

and therefore

$$\operatorname{rank}\bigl(i(X - X^*)\bigr) = \operatorname{rank} \begin{bmatrix} 0_p & * \\ * & * \end{bmatrix} \leq 2(n - p)$$

if $2p \geq n$. (If $2p \leq n$, then $2(n-p) \geq n$, and the inequality $\operatorname{rank}\bigl(i(X - X^*)\bigr) \leq 2(n-p)$ is trivial). ∎

5. **Proofs of Theorem 2.2 and Proposition 2.3** In this section we assume that all matrices are real, and the matrices M_r and $\hat{H}_r$ are given by (2.3). As in the proof of Theorem 2.1, we start with preliminary results.

The set of eigenvalues of M_r is symmetric with respect to the imaginary axis because the matrix

$$\hat{H}_r M_r = \begin{bmatrix} C & -A^T \\ -A & -D \end{bmatrix}$$

is symmetric. Of course, this set is also symmetric with respect to the real axis because M_r is real. A set $\mathcal{S}$ of eigenvalues of M_r will be called an *r-set*, if:

(a) $\mathcal{S}$ does not intersect the imaginary axis;
(b) $\lambda_0 \in \mathcal{S} \Rightarrow \bar{\lambda}_0 \in \mathcal{S}$;
(c) $\lambda_0 \in \mathcal{S} \Rightarrow -\lambda_0 \notin \mathcal{S}$;
(d) the set $\mathcal{S}$ is a maximal (with respect to containment of sets) set of eigenvalues of M_r having the properties (a), (b), and (c).

LEMMA 5.1. *Assume that $D \geq 0$ and the pair (A, D) is sign controllable. Then there is an r-set $\mathcal{S}$ of eigenvalues of M_r such that every n-dimensional M_r-invariant $\hat{H}_r M_r$-nonnegative subspace $\mathcal{M} \subseteq \mathbf{R}^{2n}$ with $\sigma(M_r \mid \mathcal{M}) \setminus i\mathbf{R} = \mathcal{S}$ is a graph subspace: $\mathcal{M} = \operatorname{Im} \begin{bmatrix} I \\ X \end{bmatrix}$ for some X.*

PROOF. We follow the approach used to prove Lemma 7.2.6 in [LR2]. Let $\hat{\mathcal{S}}$ be a maximal set in the complex plane having the properties that $\lambda_0 \in \hat{\mathcal{S}} \Rightarrow \overline{\lambda_0} \in \hat{\mathcal{S}}, -\lambda_0 \notin \hat{\mathcal{S}}$, and $\operatorname{Ker}(\lambda_0 I + A)^n \subseteq \mathcal{C}(A, D)$ for every real $\lambda_0 \in \hat{\mathcal{S}}$ and $\operatorname{Ker}\big((\lambda_0^2 + \mu_0^2)I + 2\lambda_0 A + A^2\big)^n \subseteq \mathcal{C}(A, D)$ for every pair of complex conjugate numbers $\lambda_0 \pm i\mu_0 \in \hat{\mathcal{S}}$. We take $\mathcal{S}$ to be any r-set of eigenvalues of M_r contained in $\hat{\mathcal{S}}$.

Let

$$\mathcal{M} = \operatorname{Im} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix},$$

where X_1 and X_2 are $n \times n$ matrices, be a subspace with the properties described in the lemma. Then

$$\begin{bmatrix} A & D \\ C & -A^T \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} T \tag{1}$$

for a uniquely determined matrix T. Moreover,

$$T\mathcal{K} = \mathcal{K}, \tag{2}$$

where $\mathcal{K} = \operatorname{Ker} X_1$. Indeed, using the $\hat{H}_r M_r$-nonnegativity of $\mathcal{M}$, we see that the matrix

$$-[X_1^T X_2^T]\hat{H}_r M_r \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = [X_1^T X_2^T] \begin{bmatrix} -C & A^T \\ A & D \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

is negative semidefinite, and (5.2) follows by the standard arguments (see, *e.g.*, the proof of Lemma 7.2.2 in [LR2]). Now (5.1) implies $A^T X_2 x = -X_2 T x$ for every $x \in \mathcal{K}$. Since $X_2|\mathcal{K}$ is one-to-one, we obtain that $A^T|X_2\mathcal{K}$ is similar to $-T|\mathcal{K}$; in particular, the spectrum of $A^T|X_2\mathcal{K}$ is contained in the spectrum of $-M_r|\mathcal{M}$.

Let S_0 be the union of $\hat{\mathcal{S}}$ and the set of eigenvalues of M_r with zero real parts (if any). As in the proof of Lemma 7.2.2 in [LR2], we obtain for any $y \in X_2\mathcal{K}$:

$$y \in \operatorname{Ker} \begin{bmatrix} D \\ DA^T \\ \vdots \\ D(A^T)^{n-1} \end{bmatrix} = \mathcal{C}(A, D)^{\perp} \subseteq \Big(\sum_{-\lambda \notin S_0} \mathcal{R}_{\lambda}(A)^{\perp} \Big) = \sum_{-\lambda \notin S_0} \mathcal{R}_{\lambda}(A^T), \tag{3}$$

where $\mathcal{R}_{\lambda}(Z)$ stands for the real root subspace of real $n \times n$ matrix Z corresponding to the eigenvalue λ: $\mathcal{R}_{\lambda}(Z) = \operatorname{Ker}(\lambda I - Z)^n$ if λ is real, and

$$\mathcal{R}_{\lambda}(Z) = \operatorname{Ker}\big((\nu^2 + \mu^2)I + 2\nu Z + Z^2\big)^n$$

if $\lambda = \nu + i\mu$ is nonreal (here $\nu, \mu \in \mathbf{R}$ and $\mu \neq 0$). Assuming $y \neq 0$, it follows from $\sigma(A^T \mid X_2\mathcal{K}) \subseteq \sigma(-M_r \mid \mathcal{M})$ and from (5.3) that $\sigma(M_r \mid \mathcal{M}) \setminus S_0$ is not empty. But this

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contradicts the assumption that $\sigma(M_r \mid \mathcal{M}) \setminus i\mathbf{R} = \mathcal{S}$. We must conclude that $X_2\mathcal{K} = \{0\}$, and since $\dim \mathcal{M} = n$, the subspace $\mathcal{K} = \operatorname{Ker} X_1$ must be the zero subspace. In other words, X_1 is invertible, and

$$\mathcal{M} = \operatorname{Im} \begin{bmatrix} I \\ X_2X_1^{-1} \end{bmatrix}$$

is a graph subspace. ∎

Next, we present the analogues of Lemmas 3.2 and 3.3 in the real case. Let H_r be an $m \times m$ skew-symmetric invertible matrix (in particular, m is even), and let A_r be an $m \times m$ matrix such that $H_rA_r = -A_r^T H_r$. It is easy to see that the spectrum of A_r is symmetric relative to both real and imaginary axes, *i.e.*, $\lambda_0 \in \sigma(A_r) \Rightarrow \pm\overline{\lambda}_0 \in \sigma(A_r)$. A set of eigenvalues $\mathcal{S}$ of A_r is called an *r-set* if $\mathcal{S}$ satisfies the conditions (a)–(d) stated before Lemma 5.1.

LEMMA 5.2. *Let H_r and A_r be as above, and assume that A_r is invertible. Let $\mathcal{N}$ be an A_r-invariant H_r-neutral subspace. Then there exists an A_r-invariant H_rA_r-nonnegative (resp. H_rA_r-nonpositive) subspace $\mathcal{L}$ such that $\mathcal{L} \supseteq \mathcal{N}$ and $\dim \mathcal{L} = \nu_+(H_rA_r)$ (resp. $\dim \mathcal{L} = \nu_-(H_rA_r)$). Moreover, if the set $\mathcal{S}$ of eigenvalues of $A_r|\mathcal{N}$ with nonzero real parts is such that $\lambda \in \mathcal{S} \Rightarrow -\lambda \notin \mathcal{S}$, then for every r-set $\mathcal{T}$ of A_r containing $\mathcal{S}$ the subspace $\mathcal{L}$ can be chosen with the additional property that $\mathcal{T}$ is the set of eigenvalues of $A_r|\mathcal{L}$ with nonzero real parts.*

PROOF. We proceed in two steps. Only the part concerning A_r-invariant H_rA_r-nonnegative subspaces will be proved; the corresponding result for A_r-invariant H_rA_r-nonpositive subspaces can be proved analogously.

STEP 1. In addition to the hypotheses of the lemma, it will be assumed that $\mathcal{N} = \{0\}$. We use the canonical form of of the pair (A_r, H_r) (see [DPWZ], [RR], or [T], for example). Let $E = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, and for a positive real number b denote

$$J_k(\pm ib) = \begin{bmatrix} bE & I_2 & \cdots & 0 & 0 \\ 0 & bE & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & bE & I_2 \\ 0 & 0 & \cdots & 0 & bE \end{bmatrix},$$

the real Jordan block of size $2k \times 2k$ with the pure imaginary eigenvalues $\pm ib$. In view of the canonical form, and since A_r is invertible, we have to consider only two cases:

(1) $A_r = \begin{bmatrix} K & 0 \\ 0 & -K^T \end{bmatrix}$, $H_r = \begin{bmatrix} 0 & I_{m/2} \\ -I_{m/2} & 0 \end{bmatrix}$, where the matrix K has no pure imaginary or zero eigenvalues;

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(2) $A_r = J_{n_1}(\pm ib) \oplus \cdots \oplus J_{n_p}(\pm ib)$,

$$H_r = \kappa_1 \begin{bmatrix} 0 & 0 & \cdots & 0 & E^{n_1} \\ 0 & 0 & \cdots & -E^{n_1} & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ (-1)^{n_1-1}E^{n_1} & 0 & \cdots & 0 & 0 \end{bmatrix} \oplus \cdots \oplus \kappa_p \begin{bmatrix} 0 & 0 & \cdots & 0 & E^{n_p} \\ 0 & 0 & \cdots & -E^{n_p} & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ (-1)^{n_p-1}E^{n_p} & 0 & \cdots & 0 & 0 \end{bmatrix},$$

where κ_j are +1 or -1.

Theorem 6.1 in [LR3] shows that in the case (1) there exists an A_r-invariant H_r-neutral subspace $\mathcal{L}$ of dimension $m/2 = \nu_+(H_rA_r)$, and the proof of Theorem 6.1 in [LR3] shows that the additional property concerning the set of eigenvalues of $A_r|\mathcal{L}$ with nonzero real parts can be satisfied as well. In the case (2) observe that for the pair of blocks

$$\tilde{A}_r = J_{n_j}(\pm ib), \quad \tilde{H}_r = \kappa_j \begin{bmatrix} 0 & 0 & \cdots & 0 & E^{n_j} \\ 0 & 0 & \cdots & -E^{n_j} & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \pm E^{n_j} & 0 & \cdots & 0 & 0 \end{bmatrix} \tag{4}$$

an $\tilde{A}_r$-invariant $\tilde{H}_r\tilde{A}_r$-nonnegative subspace of the required dimension is given by $\operatorname{Span}\{e_1, \ldots, e_{n_j+1}\}$ if n_j is odd and $\kappa_j = -1$, by $\operatorname{Span}\{e_1, \ldots, e_{n_j-1}\}$ if n_j is odd and $\kappa_j = 1$, and by $\operatorname{Span}\{e_1, \ldots, e_{n_j}\}$ if n_j is even (we denote here by e_p the standard unit coordinate vector, having the p-th coordinate equal to 1 and all other coordinates equal to 0; the number of coordinates in e_p will be clear from the context).

STEP 2. We drop the additional assumption that $\mathcal{N} = \{0\}$.

In view of the canonical form, we need only to consider separately the cases (1) and (2) above. In the case (1), by Theorem 6.1 of [LR3] there exists an A_r-invariant H_r-neutral $m/2$-dimensional subspace $\mathcal{L} \supseteq \mathcal{N}$; but also $\nu_+(H_rA_r) = m/2$, so we are done (the proof of Theorem 6.1 in [LR3] guarantees the additional property involving the r-set $\mathcal{T}$).

Consider now the case (2). We follow the idea of the proof of Theorem 3.1 in [LR3]. Let $d = \dim \mathcal{N}$, and assume $d < \nu_+(H_rA_r)$ (if $d = \nu_+(H_rA_r)$ there is nothing to prove). Consider the subspace

$$\mathcal{N}^{[\perp]} = \{x \in \mathbf{R}^m \mid x^T H_r y = 0 \text{ for all } y \in \mathcal{N}\}$$

(the H_r-orthogonal companion of $\mathcal{N}$). As $\mathcal{N}$ is H_r-neutral, we clearly have $\mathcal{N} \subseteq \mathcal{N}^{[\perp]}$. Furthermore, $\dim \mathcal{N}^{[\perp]} = m - d$ (because $\mathcal{N}^{[\perp]}$ coincides with the (euclidean) orthogonal complement to the d-dimensional subspace $H_r\mathcal{N}$). The subspace $\mathcal{N}^{[\perp]}$ is also A_r-invariant. So, choosing (euclidean) orthonormal bases in $\mathcal{N}$, in $\mathcal{N}^{[\perp]} \ominus \mathcal{N}$, and in $\mathbf{R}^m \ominus \mathcal{N}^{[\perp]}$, we represent the matrices A_r and H_r in the following forms:

$$A'_r = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ 0 & A_{22} & A_{23} \\ 0 & 0 & A_{33} \end{bmatrix} \quad H'_r = \begin{bmatrix} 0 & 0 & H_{13} \\ 0 & H_{22} & H_{23} \\ -H_{13}^T & -H_{23}^T & H_{33} \end{bmatrix}.$$

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The matrix H_{22} is skew-symmetric and invertible (because H_r' has these properties), and $H_{22}A_{22}$ is symmetric. Since both H_r' and A_r' are invertible, it easily follows that

$$\nu_+(H_{22}A_{22}) = \nu_+(H_rA_r) - d > 0.$$

By step 1, there exists an A_{22}-invariant $H_{22}A_{22}$-nonnegative subspace $\mathcal{N}_0$ such that $\dim \mathcal{N}_0 = \nu_+(H_{22}A_{22})$. Put $\mathcal{L} = \mathcal{N} + \mathcal{N}_0$ to satisfy the requirements of Lemma 5.2. ∎

Next, we state the analogue of Lemma 3.3.

LEMMA 5.3. *Let $H_r = -H_r^T$ be a real invertible $m \times m$ matrix, and let A_r be a real matrix such that $H_rA_r = -A_r^TH_r$. Then for every r-set $\mathcal{S}$ of eigenvalues of A_r there exists an A_r-invariant H_r-neutral real subspace $\mathcal{L}$ of dimension $\gamma_r(A_r, H_r)$ such that $\mathcal{S}$ is the set of eigenvalues of $A_r|\mathcal{L}$ having nonzero real parts.*

This lemma is contained in Theorem 6.1 in [LR3] and its proof.

Now the proof of Theorem 2.2 proceeds in the same way as the proof of Theorem 2.1, using the Lemmas 5.1, 5.2, and 5.3 in place of Lemmas 4.1, 3.2, and 3.3, respectively, and using $\hat{H}_r$ and M_r in place of $\hat{H}$ and M.

Finally, we turn to Proposition 2.3. It follows (by letting $\hat{H}_r = H_r$, $M_r = A_r$) from a general statement on pairs of matrices:

LEMMA 5.4. *Let $H_r = -H_r^T$ and A_r be as in Lemma 5.3. If the symmetric matrix H_rA_r has equal number of positive and of negative eigenvalues (counted with multiplicities), then A_r has an $m/2$-dimensional real invariant subspace.*

PROOF. We can assume that A_r has no real eigenvalues (otherwise the lemma is trivial). Again by the canonical form for the pair (A_r, H_r) (*cf.* the proof of Lemma 5.2), we further assume that A_r and H_r have the forms of the cases (1) and (2) of the proof of Lemma 5.2. In the case (1) the statement of Lemma 5.4 is evident. In the case (2) construct an A_r-invariant subspace $\mathcal{M}$ (as in the proof of Lemma 5.2) by selecting for each pair of blocks (5.4) the subspace given by $\mathrm{Span}\{e_1, \ldots, e_{n_j+1}\}$ if n_j is odd and $\kappa_j = -1$, by $\mathrm{Span}\{e_1, \ldots, e_{n_j-1}\}$ if n_j is odd and $\kappa_j = 1$, and by $\mathrm{Span}\{e_1, \ldots, e_{n_j}\}$ if n_j is even. Clearly,

$$\dim \mathcal{M} = \frac{m}{2} + \#\{j \mid n_j \text{ is odd}, \kappa_j = -1\} - \#\{j \mid n_j \text{ is odd}, \kappa_j = +1\}. \tag{5}$$

On the other hand, for the pair of blocks $(\tilde{A}_r, \tilde{H}_r)$ given by (2.6), an easy inspection shows that

$$\nu_\pm(\tilde{H}_r\tilde{A}_r) = \begin{cases} n_j & \text{if } n_j \text{ is even} \\ n_j \pm 1 & \text{if } n_j \text{ is odd and } \kappa_j = -1 \\ n_j \mp 1 & \text{if } n_j \text{ is odd and } \kappa_j = 1 \end{cases}$$

and by the hypothesis of the lemma the right hand side of (5.5) is equal to $m/2$. ∎

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6. **Discrete algebraic Riccati equations.** Consider the DARE,

$$X = A^*XA + Q - (A^*XB)^*(R + B^*XB)^{-1}B^*XA, \tag{1}$$

where the given complex matrices A, B, $Q = Q^*$ and $R = R^*$ are of sizes $n \times n$, $m \times n$, $n \times n$, and $m \times m$, respectively. A complex $n \times n$ matrix X (not necessarily hermitian) satisfying (6.1) is sought. In this section we will obtain results analogous to Theorems 2.1 and 2.2. The main method of proof will consist of reduction of the DARE (6.1) to the CARE (1.1). Because of the limitations of this method, it will be assumed throughout this section that A and R are invertible matrices.

Introduce the matrix

$$T = \begin{bmatrix} A + BR^{-1}B^*A^{*-1}Q & -BR^{-1}B^*A^{*-1} \\ -A^{*-1}Q & A^{*-1} \end{bmatrix}, \tag{2}$$

and the rational matrix function

$$\Psi(z) = R + [-B^*A^{*-1}Q, B^*A^{*-1}]\left(zI - \begin{bmatrix} A & 0 \\ -A^{*-1}Q & A^{*-1} \end{bmatrix}\right)^{-1} \begin{bmatrix} B \\ 0 \end{bmatrix}, \tag{3}$$

whose inverse is given by

$$\Psi(z)^{-1} = R^{-1} - [-R^{-1}B^*A^{*-1}Q, R^{-1}B^*A^{*-1}](zI - T)^{-1} \begin{bmatrix} BR^{-1} \\ 0 \end{bmatrix}.$$

The approach to study the hermitian solutions of DARE using T and $\Psi(z)$ was used first in [LRR] (this approach had many precursors, see, *e.g.*, [PLS]) and exposed with full details in Chapter 12 of [LR2]. Here we use this approach to study non-hermitian solutions of DARE.

If X is an $n \times n$ matrix, the subspace $\operatorname{Im} \begin{bmatrix} I \\ X \end{bmatrix}$ is called the *graph subspace* of X.

PROPOSITION 6.1. *The graph subspace of X is T-invariant if and only if X is a solution of (6.1).*

For the proof, simply repeat a part of the proof of Proposition 12.2.2 in [LR2].

PROPOSITION 6.2. *An $n \times n$ matrix X is a solution of (6.1) if only if X is a solution of the CARE,*

$$XDX - XC - C^*X - E = 0, \tag{4}$$

where

$$D = -2(\eta I - A)^{-1}B\Psi(\eta)^{-1}B^*(\bar{\eta}I - A^*)^{-1}, \tag{5}$$

$$\begin{aligned} C &= (A^{-1} - \bar{\eta}I)^{-1}(A^{-1} + \bar{\eta}I) \\ &\quad +2(\eta I - A)^{-1}B\Psi(\eta)^{-1}B^*\left(\bar{\eta}I - A^*\right)^{-1}QA^{-1}(\bar{\eta}I - A)^{-1})^{-1}, \end{aligned} \tag{6}$$

and E is a suitable hermitian matrix (the exact form of E is not important here). The function $\Psi(z)$ is defined in the equation (6.3), and η is a unimodular number such that the

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matrices $\Psi(\eta)$, $\eta I - A$, $\overline{\eta} I - A^{-1}$ *and* $\eta I - T$ *are invertible, where* T *is given by equation (6.2) (and the existence of such an* η *is guaranteed).*

The proof proceeds as in the proof of Proposition 12.2.2 of [LR2]. For future reference we indicate that in fact

$$\phi(T) = \begin{bmatrix} -C & D \\ E & C^* \end{bmatrix},$$

where $\phi(\lambda) = (\lambda + \eta)(\lambda - \eta)^{-1}$, and the matrices T, C, and D are given by (6.2), (6.4), and (6.5), respectively.

Let $J = \begin{bmatrix} 0 & -iI \\ iI & 0 \end{bmatrix}$. A straightforward calculation shows that $T^*JT = J$; in other words, T is J-unitary. We define the neutrality index $\gamma(T, J)$ of the pair (T, J) analogously to that of self-adjoint matrices with respect to an indefinite scalar product. Namely, $\gamma(T, J)$ is the dimension of a maximal T-invariant J-neutral subspace.

A pair (A, B) with $n \times n$ matrix A is called *d-sign controllable* if for every $\lambda_0 \neq 0$ at least one of the subspaces $\operatorname{Ker}(\lambda_0 I - A)^n$ and $\operatorname{Ker}(\overline{\lambda}_0^{-1} I - A)^n$ is contained in the controllable subspace $\mathcal{C}(A, B)$ (defined by (2.1)).

THEOREM 6.3. *Assume that* A *and* R *are invertible, and there exists a unimodular number* η *such that* $\Psi(\eta)$ *is positive definite. If the pair* (A, B) *is d-sign controllable, then the equation (6.1) has a solution* X *such that*

$$\operatorname{rank}\bigl(i(X - X^*)\bigr) \leq 2\bigl(n - \gamma(T, J)\bigr).$$

For the proof we need a lemma.

LEMMA 6.4. *Let* A *be* $n \times n$*, and* B *be* $n \times m$*. Then* (A, B) *is sign controllable (resp. d-sign controllable) if and only if* $(SAS^{-1} + SBK, SBL)$ *is sign controllable (resp. d-sign controllable) for any* $m \times n$ *matrix* K*, any* $m \times p$ *matrix* L *such that* $\operatorname{Im}(BL) = \operatorname{Im} B$*, and any invertible* $n \times n$ *matrix* S*.*

PROOF. The set of transformations $(A, B) \longrightarrow (SAS^{-1} + SBK, SBL)$, where K, L, S satisfy the conditions of the lemma, is easily seen to be a group. It is therefore sufficient to prove the lemma for the three particular cases: (1) $K = 0$, $L = I$; (2) $S = I$, $K = 0$; (3) $S = I$, $L = I$. In the case (1) the result follows immediately from the obvious observation that $\mathcal{C}(SAS^{-1}, SB) = S\mathcal{C}(A, B)$ and $\operatorname{Ker}(SAS^{-1} - \lambda_0 I)^n = S[\operatorname{Ker}(A - \lambda_0 I)^n]$. The case (2) is trivial because the condition $\operatorname{Im}(BL) = \operatorname{Im} B$ guarantees that $\mathcal{C}(A, B) = \mathcal{C}(A, BL)$. Finally, assume $S = I$, $L = I$, and consider the sign controllability property (the proof for the d-sign controllability is completely analogous). Write A and B as 2×2 block matrices with respect to the orthogonal decomposition $\mathbf{C}^n = \mathcal{C}(A, B) \oplus \mathcal{C}(A, B)^{\perp}$:

$$A = \begin{bmatrix} A_1 & A_{12} \\ 0 & A_2 \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}.$$

The sign controllability condition implies that for every pair of complex numbers λ, $-\overline{\lambda}$ at least one of λ and $-\overline{\lambda}$ is not an eigenvalue of A_2 (in particular, A_2 has no pure imaginary or zero eigenvalues). Indeed, for every such pair λ, $-\overline{\lambda}$ at least one of them, call it λ_0, has the property that $\mathcal{R}_{\lambda_0}(A) \subseteq \mathcal{C}(A, B)$. Therefore, $\dim \mathcal{R}_{\lambda_0}(A) = \dim\big(\mathcal{R}_{\lambda_0} \cap \mathcal{C}(A, B)\big) = \dim \mathcal{R}_{\lambda_0}\big(A|\mathcal{C}(A, B)\big) = \dim \mathcal{R}_{\lambda_0}(A_1)$. On the other hand, $\dim \mathcal{R}_{\lambda_0}(A)$ (resp. $\dim \mathcal{R}_{\lambda_0}(A_1)$) is just the multiplicity of λ_0 as a zero of $\det(\lambda I - A)$ (resp. of $\det(\lambda I - A_1)$). Since $\det(\lambda I - A) = \det(\lambda I - A_1)\det(\lambda I - A_2)$, it follows that λ_0 is not a zero of $\det(\lambda I - A_2)$.

Now $A + BK = \begin{bmatrix} A_1 + B_1K & A_{12} + B_1K \\ 0 & A_2 \end{bmatrix}$. Hence $A + BK$ also has the property that for every pair of eigenvalues λ, $-\overline{\lambda}$ of $A + BK$ at least one of them is not an eigenvalue of A_2. Reversing the above argument, we obtain that at least one of λ, $-\overline{\lambda}$ (call it λ_0) satisfies the property $\mathcal{R}_{\lambda_0}(A) \subseteq \mathcal{C}(A, B)$. Since clearly $\mathcal{C}(A + BK, B) = \mathcal{C}(A, B)$, the sign controllability of $(A + BK, B)$ follows. ■

PROOF OF THEOREM 6.3. Lemma 6.4 implies that the pair (C, D) is sign controllable, where C and D are given by (6.6) and (6.5), respectively. Now apply Proposition 6.2 and Theorem 2.1. ■

A real version of Theorem 6.3 can be obtained, under the additional hypotheses that η can be chosen to be either 1 or -1 in Theorem 6.3 and that the matrix

$$\begin{bmatrix} C & -D \\ -E & -C^T \end{bmatrix}$$

is invertible. Here C, D and E are taken from Proposition 6.2. We state the result, omitting the proof.

THEOREM 6.5. *($F = \mathbf{R}$). Assume that A and R are invertible, the pair (A, B) is d-sign controllable, and there exists $\eta_0 \in \{1, -1\}$ such that the matrices $\eta_0 I - A$, $\eta_0 I - A^{-1}$, $\eta_0 I \pm T$ are invertible and the matrix $\Psi(\eta_0)$ is positive definite. Then the equation (6.1) admits a real solution X such that*

$$\operatorname{rank}(X - X^T) \leq 2\big(n - \gamma(T, \hat{J})\big),$$

where $\hat{J} = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$.

As in Theorems 2.1 and 2.2, additional information can be given concerning the spectral properties of the solutions whose existence is asserted in Theorems 6.1 and 6.5.

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