Bull. Aust. Math. Soc. **97** (2018), 386–395
doi:10.1017/S0004972717001198

ALEXANDER POLYNOMIALS OF COMPLEX PROJECTIVE PLANE CURVES

QUY THUONG LÊ

(Received 5 November 2017; accepted 16 November 2017; first published online 7 March 2018)

Abstract

We compute the Alexander polynomial of a nonreduced nonirreducible complex projective plane curve with mutually coprime orders of vanishing along its irreducible components in terms of certain multiplier ideals.

2010 *Mathematics subject classification*: primary 14H20; secondary 14C20, 14F18, 32S20.

Keywords and phrases: Alexander polynomial, log-resolution, multiplier ideals, local systems.

1. Introduction

Let C be a complex projective plane curve which is defined by a homogeneous polynomial $f(x, y, z)$ of degree d in $\mathbb{C}[x, y, z]$. For f reduced, Esnault [5] introduced a method to compute the Betti numbers, the rank and the signature of the intersection matrices of the singularity of f at the origin of $\mathbb{C}^3$. Esnault used mixed Hodge structures on cohomology groups of the Milnor fibre, the existence of spectral sequences converging to the cohomology groups and resolution of singularities. In follow-up work, Loeser and Vaquié [10] studied the Alexander polynomial of a reduced complex projective plane curve, generalising previous work by Libgober [8, 9]. The approaches of Libgober [9] and Loeser-Vaquié [10] to Alexander polynomials of complex projective plane curves are the starting point for the study of multiplier ideals and local systems.

In this paper, we recall the definition of the global Alexander polynomial of a reduced complex projective plane curve, Loeser–Vaquié's formula [10] and some methods for computing the Alexander polynomial by Bartolo [1]. Our main result is an extension to the Alexander polynomial of a nonreduced complex projective plane curve and a generalisation of the Loeser–Vaquié formula to certain plane curves. Let us now give a short summary of this work. Assume

$$f(x, y, z) = f_1(x, y, z)^{m_1} \cdots f_r(x, y, z)^{m_r},$$

This research is funded by Vietnam National University, Hanoi, under project number QG.16.06.

with $m_i \geq 1$, where $f_i(x, y, z)$ are distinct irreducible homogeneous polynomials of degree $d_i \geq 1$ (which define projective plane curves C_i, respectively). In [3], Budur gives an explicit description of the local system of the complement U to C in $\mathbb{P}^2$. According to [2, 3], there is an eigensheaf decomposition of the $\mathcal{O}_U$-module sheaf $\sigma_*\mathbb{C}_M$ into the unitary local systems $\mathcal{V}_k$ on U given by the eigensheaf of the geometric monodromy T with respect to the eigenvalue $e^{-2\pi ik/d}$, where σ is the canonical projection of the Milnor fibre M of f onto U. Therefore, $H^1(U, \mathcal{V}_k)$ is the eigenspace of the monodromy T on $H^1(M, \mathbb{C})$ with respect to the eigenvalue $e^{-2\pi ik/d}$, for $0 \leq k \leq d - 1$.

Inspired by Randell's theorem [12], we extend the definition of the Alexander polynomial $\Delta_C(t)$ to nonreduced complex projective plane curves. Namely, it is the characteristic polynomial of the monodromy T on $H^1(M, \mathbb{C})$, that is,

$$\triangle_C(t) = \prod_{k=0}^{d-1} (t - e^{2\pi ik/d})^{\dim_{\mathbb{C}} H^1(U, \mathcal{V}_{d-k})}.$$

The following result is the main theorem of the paper.

Theorem 1.1 (Theorem 3.1). *If $m_1, \ldots, m_r$ are mutually coprime, then*

$$\Delta_C(t) = (t - 1)^{r-1} \prod_{k=1}^{d-1} \left(t^2 - 2t \cos \frac{2k\pi}{d} + 1\right)^{\ell_k},$$

where

$$\ell_k = \dim_{\mathbb{C}} H^1\left(\mathbb{P}^2, \mathcal{J}\left(\mathbb{P}^2, \sum_{i=1}^{r} \left\{\frac{km_i}{d}\right\} C_i\right)\left(\sum_{i=1}^{r} \left\{\frac{km_i}{d}\right\} d_i - 3\right)\right).$$

Note that, as usual, we write $\mathcal{J}(\mathbb{P}^2, \sum_{i=1}^r \alpha_i C_i)$ for the multiplier ideal of $\sum_{i=1}^r \alpha_i C_i$ with α_i being given positive rational numbers (see [2, 7] for the definition), and we write $\mathcal{F}(l)$ for the twisted sheaf for a sheaf $\mathcal{F}$ and an integer l. Here, it is easy to see that $\sum_{i=1}^r \{km_i/d\} d_i$ is an integer.

To prove the theorem, we use Budur's computations on the dimension of the complex vector space $H^1(U, \mathcal{V}_{d-k})$ in terms of log-resolution of the family $\{C_1, \ldots, C_r\}$ (see [2–4]) and some arguments of Esnault and Loeser–Vaquié in [5, 10], respectively.

2. The Alexander polynomial of a complex projective plane curve

In this section, we recall the definition of the Alexander polynomial of a complex projective plane curve, Loeser–Vaquié's formula and Bartolo's computation of Alexander polynomials. We give some simple remarks and an example.

2.1. Alexander polynomials. We start with the definition of the Alexander polynomial of a projective curve. Let C be a reduced complex projective plane curve of degree d with r distinct irreducible components. Let L be a line in $\mathbb{P}^2 := \mathbb{P}^2_{\mathbb{C}}$ which is general with respect to C, that is, L intersects C at exactly d distinct points. Such a line L exists since C is reduced. Then the manifold $W := \mathbb{P}^2 \backslash (C \cup L)$ has a homotopy type of a finite CW-complex. By van Kampen's theorem [6], the natural map

$$\pi_1(L \backslash (L \cap C)) \to \pi_1(W)$$

is a surjective homomorphism and the group $\pi_1(W)$ is generated by the images of all the d standard generators of the free group $\pi_1(L \backslash (L \cap C))$. The generators of $\pi_1(W)$ are loops in L going once around a point of $L \cap C$, and if two loops respectively go around two points of $L \cap C$ belonging to the same irreducible component of C they give rise to two conjugate elements in $\pi_1(W)$. Thus,

$$H_1(W, \mathbb{Z}) \cong \pi_1(W)/[\pi_1(W), \pi_1(W)] \cong \mathbb{Z}^r,$$

and the Hurewicz morphism

$$\pi_1(W) \to H_1(W, \mathbb{Z})$$

is just the canonical projection

$$\pi_1(W) \to \pi_1(W)/[\pi_1(W), \pi_1(W)],$$

with $[\pi_1(W), \pi_1(W)]$ being the commutator subgroup of $\pi_1(W)$.

Consider the surjective homomorphism $\varphi : \pi_1(W) \to \mathbb{Z}$ which is the composition of the Hurewicz morphism and the sum function. Then there exists an infinite cyclic cover $\widetilde{W}_\varphi \to W$ with respect to φ such that $\pi_1(\widetilde{W}_\varphi) = \ker \varphi$. Let $t : \widetilde{W}_\varphi \to \widetilde{W}_\varphi$ be the canonical generator of the group of cover transformations $\mathrm{Deck}(\widetilde{W}_\varphi/W) \cong \mathbb{Z}$. Then $\mathbb{Z}$ acts naturally on $H^1(\widetilde{W}_\varphi, \mathbb{C})$ by $t \cdot c := t^*(c)$ for any class c in $H^1(\widetilde{W}_\varphi, \mathbb{C})$, from which $H^1(\widetilde{W}_\varphi, \mathbb{C})$ has a structure of a $\mathbb{C}[t, t^{-1}]$-module. Since $\mathbb{C}[t, t^{-1}]$ is a principal ideal domain, the torsion $\mathbb{C}[t, t^{-1}]$-module $H^1(\widetilde{W}_\varphi, \mathbb{C})$ admits, up to an order of summands, a unique decomposition via monic polynomials $\delta_j(t)$ in $\mathbb{C}[t] \subset \mathbb{C}[t, t^{-1}]$ with $\delta_j(0) \neq 0$, $1 \leq j \leq N$, for some N in $\mathbb{N}_{>0}$, namely,

$$H^1(\widetilde{W}_\varphi, \mathbb{C}) = \bigoplus_{j=1}^{N} \mathbb{C}[t, t^{-1}]/(\delta_j(t)).$$

Then the *(global) Alexander polynomial* $\Delta_C(t)$ of the curve C is defined by

$$\Delta_C(t) := \prod_{j=1}^{N} \delta_j(t).$$

One can prove that $\Delta_C(t)$ is independent of the choice of L provided it is general with respect to C (see [12]).

It is known that if C is irreducible and the fundamental group $\pi_1(\mathbb{P}^2 \backslash C)$ is either abelian or finite then the Alexander polynomial is trivial. One may prove easily that the multiplicity of the factor $t - 1$ in $\Delta_C(t)$ is exactly $r - 1$, where r is the number of irreducible components of C.

https://doi.org/10.1017/S0004972717001198 Published online by Cambridge University Press

Assume that p is a singular point of C. We may consider the Milnor fibre M_p and the monodromy

$$T_p : H^1(M_p, \mathbb{C}) \to H^1(M_p, \mathbb{C})$$

of the singularity (C, p). Denote by $\Delta_{C,p}(t)$ the characteristic polynomial of T_p. Let $\mathrm{Sing}(C)$ be the locus of singular points of the curve C. Then, according to Libgober [8], the Alexander polynomial $\Delta_C(t)$ divides the product $\prod_{p\in \mathrm{Sing}(C)} \Delta_{C,p}(t)$, and it also divides the so-called Alexander polynomial at infinity $(t^d - 1)^{d-2}(t - 1)$.

2.2. Loeser–Vaquié's formula. Let $f(x, y, z)$ be a homogeneous polynomial of degree d in $\mathbb{C}[x, y, z]$ that defines the reduced curve C. Assume we have the decomposition of f into distinct irreducible homogeneous polynomials,

$$f(x, y, z) = f_1(x, y, z) \cdots f_r(x, y, z),$$

with r in $\mathbb{N}_{>0}$. For $1 \le i \le r$, denote by C_i the reduced projective curve in $\mathbb{P}^2$ defined by f_i. By definition, a log-resolution of the family $\{C_1, \dots, C_r\}$ is a proper birational morphism $\pi : Y \to \mathbb{P}^2$, where Y is a smooth complex algebraic variety, such that the exceptional set

$$\mathrm{Ex}(\pi) := \{y \in Y \mid \pi \text{ is not biregular at } y\},$$

the support $\mathrm{Supp}(\det \mathrm{Jac}_\pi)$ of the determinant of the Jacobian of π, the preimages $\pi^{-1}(C_i)$ of C_i for all $1 \le i \le r$, and the union

$$\mathrm{Ex}(\pi) \cup \mathrm{Supp}(\det \mathrm{Jac}_\pi) \cup \bigcup_{i=1}^{r} \pi^{-1}(C_i)$$

are simple normal crossing divisors. The existence of such a log-resolution follows from a celebrated theorem by Hironaka. Let $K_{Y/\mathbb{P}^2}$ be the canonical divisor of π. For $\alpha = (\alpha_1, \dots, \alpha_r)$ in $\mathbb{Q}^r_{>0}$, we put

$$\mathcal{J}(\mathbb{P}^2, \alpha C) := \pi_* \mathcal{O}_Y(K_{Y/\mathbb{P}^2} - \lfloor \pi^*(\alpha C) \rfloor),$$

where $\alpha C = \sum_{i=1}^{r} \alpha_i C_i$, and $\lfloor \pi^*(\alpha C) \rfloor$ is the round-down of the coefficients of the irreducible components of the divisor $\pi^*(\alpha C)$. Clearly, $\mathcal{J}(X, \alpha C)$ is a sheaf of ideals on $\mathbb{P}^2$, which is an ideal of the sheaf $\mathcal{O}_{\mathbb{P}^2}$. An important result proved by Lazarsfeld in [7] using the Kawamata–Viehweg vanishing theorem states that, for any α in $\mathbb{Q}^r_{>0}$, the sheaf of ideals $\mathcal{J}(\mathbb{P}^2, \alpha C)$ is independent of the choice of the log-resolution π, and that $R^i \pi_* \mathcal{O}_Y(K_{Y/\mathbb{P}^2} - \lfloor \pi^*(\alpha C) \rfloor) = 0$ for all $i \ge 1$. The sheaf of ideals $\mathcal{J}(\mathbb{P}^2, \alpha C)$ is called the *multiplier ideal* of αC.

As in [10], in order to compute the Alexander polynomial $\Delta_C(t)$, it is useful to apply Randell's result in [12]. By viewing the homogeneous polynomial defining C as a germ of a singularity at the origin of $\mathbb{C}^3$ we may consider its Milnor fibre M and the monodromy T induced by

$$(x, y, z) \mapsto (e^{2\pi i/d} x, e^{2\pi i/d} y, e^{2\pi i/d} z).$$

Note that M is diffeomorphic to $\{(x, y, z) \in \mathbb{C}^3 \mid f(x, y, z) = 1\}$.

https://doi.org/10.1017/S0004972717001198 Published online by Cambridge University Press

THEOREM 2.1 (Randell [12]). *The Alexander polynomial of C is equal to the characteristic polynomial of the monodromy $T|_{H^1(M,\mathbb{C})}$.*

Applying Theorem 2.1 and the main result in Esnault [5], Loeser and Vaquié prove the following theorem, which relates the Alexander polynomial of a reduced projective plane curve to the multiplier ideals of C. For simplicity of notation, we write $\mathcal{J}_\alpha$ for $\mathcal{J}(\mathbb{P}^2, \alpha C)$, and $\mathcal{J}_\alpha(l)$ for $\mathcal{J}_\alpha \otimes \mathcal{O}_{\mathbb{P}^2}(l)$.

THEOREM 2.2 (Loeser–Vaquié [10]). *If C is a reduced complex projective plane curve of degree d with r irreducible components, then*

$$\triangle_C(t) = (t-1)^{r-1} \prod_{k=1}^{d-1} \Big(t^2 - 2t\cos\frac{2k\pi}{d} + 1\Big)^{\dim_{\mathbb{C}} H^1(\mathbb{P}^2, \mathcal{J}_{k/d}(k-3))}.$$

2.3. Computation of $\dim_{\mathbb{C}} H^1(\mathbb{P}^2, \mathcal{J}_{k/d}(k-3))$. In this paragraph, we review the work of Bartolo [1] in computing the dimension of the complex vector space $H^1(\mathbb{P}^2, \mathcal{J}_{k/d}(k-3))$. For $1 \le k \le d-1$, we denote as usual by $\mathcal{J}_{k/d,p}$ the stalk at p in C of the sheaf $\mathcal{J}_{k/d}$. It may be easily checked that, if p is nonsingular, $\mathcal{J}_{k/d,p} = \mathcal{O}_{\mathbb{P}^2,p}$. Consider a map

$$\psi_k : H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(k-3)) \to \bigoplus_{p\in C} \mathcal{O}_{\mathbb{P}^2,p}/\mathcal{J}_{k/d,p}$$

defined as follows. We may identify the vector space $H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(k-3))$ with the space of polynomials in $\mathbb{C}[x,y]$ of degree $\le k-3$. The Taylor expansion at p of each element g of $H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(k-3))$ induces a holomorphic function germ g_p at p in C. Then ψ_k is given by

$$\psi_k(g) = (g_p + \mathcal{J}_{k/d,p})_{p\in \mathrm{Sing}(C)},$$

which is a complex linear map.

LEMMA 2.3 (Bartolo [1]). $\dim_{\mathbb{C}} H^1(\mathbb{P}^2, \mathcal{J}_{k/d}(k-3)) = \dim_{\mathbb{C}} \operatorname{coker}(\psi_k)$.

By a simple computation,

$$\dim_{\mathbb{C}} H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(k-3)) = \tfrac{1}{2}(k-2)(k-1).$$

In order to compute the dimension of the target space of ψ_k, we follow [5] by using a log-resolution of the family $\{C\}$. Let $\pi : Y \to \mathbb{P}^2$ be a log-resolution of $\{C\}$, with numerical data given as follows

$$E = \pi^{-1}(C) = \sum_{i\in A} N_i E_i, \quad E_p := \sum_{i\in A, \pi(E_i)=p} N_i E_i$$

for p in C, and $K_{Y/\mathbb{P}^2} = \sum_{i\in A} a_i E_i$, where the E_i's are the irreducible components of $\pi^{-1}(C)$ and the a_i's are the discrepancies of π.

https://doi.org/10.1017/S0004972717001198 Published online by Cambridge University Press

Proposition 2.4. *With π as before,*

$$\begin{aligned}&\dim_{\mathbb{C}} H^1(\mathbb{P}^2, \mathcal{J}_{k/d}(k-3))\\&= \frac{1}{2}\sum_{p\in C}\lfloor (k/d)E_p\rfloor \cdot (K_{Y/\mathbb{P}^2} - \lfloor (k/d)E_p\rfloor) - \frac{1}{2}(k-2)(k-1) + \dim_{\mathbb{C}} \ker(\psi_k).\end{aligned}$$

Proof. From Lemma 2.3, $\dim_{\mathbb{C}} H^1(\mathbb{P}^2, \mathcal{J}_{k/d}(k-3))$ is given by

$$\sum_{p\in C}\dim_{\mathbb{C}} \mathcal{O}_{\mathbb{P}^2,p}/\mathcal{J}_{k/d,p} - \dim_{\mathbb{C}} H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(k-3)) + \dim_{\mathbb{C}} \ker(\psi_k).$$

According to [5, Remarque 11],

$$\dim_{\mathbb{C}} \mathcal{O}_{\mathbb{P}^2,p}/\mathcal{J}_{k/d,p} = \tfrac{1}{2}\lfloor (k/d)E_p\rfloor \cdot (K_{Y/\mathbb{P}^2} - \lfloor (k/d)E_p\rfloor).$$

This proves the proposition. □

In view of Proposition 2.4, computing $\dim H^1(\mathbb{P}^2, \mathcal{J}_{k/d}(d-3-k))$ reduces to computing $\dim\ker(\psi_k)$, the dimension of the vector space of complex projective plane curves of degree $k-3$ passing through all the singular points p of C with germ contained in $\mathcal{J}_{k/d,p}$. Note that, if p is a singular point of C of type A_1, the last formula in the proof of Proposition 2.4 shows that $\mathcal{J}_{k/d,p} = \mathcal{O}_{\mathbb{P}^2,p}$. Therefore, as for the case of a nonsingular point, an A_1-singularity does not contribute to $\Delta_C(t)$.

Example 2.5. Consider an irreducible curve C of degree d whose singular points are either of type A_1 or of type $B_{a,b}$ (that is, the local equation is $x^a + y^b = 0$), where a and b are positive integers such that ab divides d. If p is a singular point of type $B_{a,b}$ of C, it follows that $\mathcal{J}_{1/a+1/b,p}$ is the maximal ideal of $\mathcal{O}_{\mathbb{P}^2,p}$ and, in that case, $\dim_{\mathbb{C}} H^1(\mathbb{P}^2, \mathcal{J}_{1/a+1/b}(d/a + d/b - 3))$ can be easily computed.

3. Generalisation to nonreduced complex projective plane curves

3.1. Alexander polynomial of nonreduced curves and the main theorem. Let us consider a complex projective plane curve C which is nonreduced, and define an Alexander polynomial for it. Note that the definition in the reduced case does not work now since there is no line in $\mathbb{P}^2$ which is general to the nonreduced curve C. Inspired by Theorem 2.1, however, we may define the Alexander polynomial of C in the following way. Let $f(x, y, z) \in \mathbb{C}[x, y, z]$ be a homogeneous polynomial of degree d that defines C. The polynomial f can be considered as a surface homogeneous singularity germ at the origin of $\mathbb{C}^3$. By [11, Lemma 9.4], its Milnor fibre M is diffeomorphic to $\{(x, y, z) \in \mathbb{C}^3 \mid f(x, y, z) = 1\}$. The geometric monodromy $M \to M$ is given by multiplication of elements of M by $e^{2\pi i/d}$, which induces an endomorphism T of the complex vector space $H^*(M, \mathbb{C})$. Then we define *the Alexander polynomial* $\Delta_C(t)$ *of* C to be the characteristic polynomial of the endomorphism $T|_{H^1(M,\mathbb{C})}$.

Assume

$$f(x, y, z) = f_1(x, y, z)^{m_1} \cdots f_r(x, y, z)^{m_r},$$

where $f_i(x, y, z)$ is an irreducible homogeneous polynomial of degree $d_i \geq 1$ and m_i is in $\mathbb{N}_{>0}$ for $1 \leq i \leq r$. Let C_i denote the complex projective plane curve defined by f_i for $1 \leq i \leq r$. The main result of our paper is the following theorem.

THEOREM 3.1. *If $m_1, \ldots, m_r$ are mutually coprime, then*

$$\Delta_C(t) = (t-1)^{r-1} \prod_{k=1}^{d-1} \Big(t^2 - 2t\cos\frac{2k\pi}{d} + 1\Big)^{\ell_k},$$

where

$$\ell_k = \dim_{\mathbb{C}} H^1\Big(\mathbb{P}^2, \mathcal{J}\Big(\mathbb{P}^2, \sum_{i=1}^{r}\Big\{\frac{km_i}{d}\Big\}C_i\Big)\Big(\sum_{i=1}^{r}\Big\{\frac{km_i}{d}\Big\}d_i - 3\Big)\Big).$$

From this theorem, by taking $m_1 = \cdots = m_r = 1$, one can recover Loeser–Vaquié's result on the Alexander polynomial of a reduced complex projective plane curve (Theorem 2.2).

3.2. Local systems and cyclic covers. A complex *local system* $\mathcal{V}$ on a complex manifold is a locally constant sheaf of finite dimensional complex vector spaces. The rank of a locally constant sheaf is the dimension of a stalk as a complex vector space. In [2], Budur shows that local systems of rank-one on a complex manifold U correspond to morphisms of groups $H_1(U) \to \mathbb{C}^*$. A local system is *unitary* if it corresponds to a morphism of groups $H_1(U) \to \mathbb{S}^1$. In particular, the constant sheaf $\mathbb{C}_U$ and any local system of rank-one of finite order are unitary local systems.

Let f and f_i (respectively, C and C_i) be as before. Denote by $c_1(\mathcal{L})$ the first Chern class of a line bundle $\mathcal{L}$. We consider the group

$$\mathrm{Pic}^{\tau}(\mathbb{P}^2, C) := \{(\mathcal{L}, \alpha) \in \mathrm{Pic}(\mathbb{P}^2) \times [0,1)^r \mid c_1(\mathcal{L}) = \alpha[C] \in H^2(\mathbb{P}^2, \mathbb{R})\}$$

with the multiplication defined as follows

$$(\mathcal{L}, \alpha) \cdot (\mathcal{L}', \alpha') := (\mathcal{L} \otimes \mathcal{L}' \otimes \mathcal{O}_{\mathbb{P}^2}(-\lfloor(\alpha + \alpha')C\rfloor)), \{\alpha + \alpha'\}),$$

where $\lfloor\alpha\rfloor = (\lfloor\alpha_1\rfloor, \ldots, \lfloor\alpha_r\rfloor)$, $\{\alpha\} = \alpha - \lfloor\alpha\rfloor$, $\alpha[C] = \sum_{i=1}^{r} \alpha_i[C_i]$ and $[C_i]$ are the cohomology classes in $H^2(\mathbb{P}^2, \mathbb{R})$. By [2, Theorem 1.2], there is a canonical isomorphism $\mathrm{Pic}^{\tau}(\mathbb{P}^2, C) \cong \mathrm{Hom}(H_1(U), \mathbb{S}^1)$, which allows us to identify a unitary local system of rank-one on U with an element of $\mathrm{Pic}^{\tau}(\mathbb{P}^2, C)$.

The group μ_d of dth roots of unity is the dual group of $\mathbb{Z}/d\mathbb{Z}$ and can be considered as a subgroup of $\mathrm{Pic}^{\tau}(\mathbb{P}^2, C)$. Using this identification, $\mu_d = \{(\mathcal{L}_k, \alpha_k) \mid 0 \leq k \leq d-1\}$ which gives rise naturally to the cyclic cover

$$\phi : \tilde{X} := \mathrm{Spec}_{\mathcal{O}_{\mathbb{P}^2}}\Big(\bigoplus_{k=0}^{d-1} \mathcal{L}_k^{-1}\Big) \to \mathbb{P}^2$$

which is ramified along C. Conversely, each d-cyclic cover of $\mathbb{P}^2$ ramified along C determines a cyclic subgroup of order d of the group $\mathrm{Pic}^{\tau}(\mathbb{P}^2, C)$. Since $\mathbb{Z}/d\mathbb{Z}$ acts on $\mathcal{L}_k^{-1}$ via the character $e^{2\pi ik/d}$, it acts on the $\mathcal{O}_{\mathbb{P}^2}$-module sheaf $\phi_*\mathcal{O}_{\tilde{X}}$.

https://doi.org/10.1017/S0004972717001198 Published online by Cambridge University Press

By [2, Corollary 1.11], $\phi_*\mathcal{O}_{\widetilde{X}} = \bigoplus_{k=0}^{d-1} \mathcal{L}_k^{-1}$, with $\mathcal{L}_k^{-1}$ being the eigensheaf with respect to the eigenvalue $e^{2\pi ik/d}$ of the action of $\mathbb{Z}/d\mathbb{Z}$ on $\phi_*\mathcal{O}_{\widetilde{X}}$.

Let $U := \mathbb{P}^2\backslash C$ and M be as before. Since the action of $\mathbb{Z}/d\mathbb{Z}$ on M is free, we have a natural isomorphism $M/(\mathbb{Z}/d\mathbb{Z}) \cong U$. Consider the quotient map $\sigma : M \to U$ and write $\sigma_*\mathbb{C}_M = \bigoplus_{k=0}^{d-1} \mathcal{V}_k$, where $\mathcal{V}_k$ is the unitary local system on U given by the eigensheaf of T with respect to the eigenvalue $e^{-2\pi ik/d}$. This implies that $H^1(U, \sigma_*\mathbb{C}_M) = \bigoplus_{k=0}^{d-1} H^1(U, \mathcal{V}_k)$. Consider the Leray spectral sequence

$$E_2^{p,q} = H^q(U, R^p\sigma_*\mathbb{C}_M) \Rightarrow H^{p+q}(M, \mathbb{C}_M).$$

Since σ is finite, $R^p\sigma_*\mathbb{C}_M = 0$ for all $p \geq 1$. Hence

$$H^1(M, \mathbb{C}) = H^1(M, \mathbb{C}_M) = H^1(U, \sigma_*\mathbb{C}_M) = \bigoplus_{k=0}^{d-1} H^1(U, \mathcal{V}_k). \tag{3.1}$$

By [3, Section 4],

$$H^1(M, \mathbb{C})_{e^{-2\pi ik/d}} = H^1(U, \mathcal{V}_k) \quad \text{for } 0 \leq k \leq d-1. \tag{3.2}$$

By [3, Lemma 4.2], via the isomorphism $\mathrm{Pic}^\tau(\mathbb{P}^2, C) \cong \mathrm{Hom}(H_1(U), \mathbb{S}^1)$, the local system $\mathcal{V}_k$ corresponds to $(\mathcal{O}_{\mathbb{P}^2}(\sum_{i=1}^r \{km_i/d\}d_i), (\{km_1/d\}, \ldots, \{km_r/d\}))$. Note that $\sum_{i=1}^r \{km_i/d\}d_i$ is an integer.

Now fix a log-resolution $\pi : Y \to \mathbb{P}^2$ of the family $\{C_1, \ldots, C_r\}$, and define E by $E := \pi^{-1}(C_1 \cup \cdots \cup C_r)$. For $0 \leq k \leq d-1$, define

$$\mathcal{L}^{(k)} := \pi^*\mathcal{O}_{\mathbb{P}^2}\Big(\sum_{i=1}^r \Big\{\frac{km_i}{d}\Big\}d_i\Big) \otimes \mathcal{O}_Y\Big(-\Big\lfloor \sum_{i=1}^r \Big\{\frac{km_i}{d}\Big\}\pi^*C_i \Big\rfloor\Big).$$

For simplicity, we write $h^l(\mathcal{F})$ for $\dim_{\mathbb{C}} H^l(Y, \mathcal{F})$, with $\mathcal{F}$ a sheaf on Y.

Proposition 3.2. $\dim_{\mathbb{C}} H^1(U, \mathcal{V}_0) = r - 1$ *and, for* $1 \leq k \leq d-1$,

$$\dim_{\mathbb{C}} H^1(U, \mathcal{V}_{d-k}) = h^1(\mathcal{L}^{(k)^{-1}}) + h^0(\Omega_Y^1(\log E) \otimes \mathcal{L}^{(k)^{-1}}).$$

Indeed, the first identity in the proposition follows from [5, Théorème 6], and the second one is a corollary of [4, Theorem 4.6] and [3, Lemma 4.2].

3.3. Finishing the proof of Theorem 3.1. In the following, we use the hypothesis that $m_1, \ldots, m_r$ are mutually coprime, which means that the $\mathcal{V}_k$ are pairwise distinct. Then, Loeser–Vaquié's arguments in the proof of [10, Théorème 4.1] still work.

From the definition of $\Delta_C(t)$, the identifications (3.1) and (3.2) and Proposition 3.2,

$$\Delta_C(t) = (t-1)^{r-1} \prod_{k=1}^{d-1} (t - e^{2\pi ik/d})^{h^1(\mathcal{L}^{(k)^{-1}}) + h^0(\Omega_Y^1(\log E) \otimes \mathcal{L}^{(k)^{-1}})}.$$

Therefore, it suffices to prove $h^1(\mathcal{L}^{(k)^{-1}}) = \ell_k$ and $h^0(\Omega_Y^1(\log E) \otimes \mathcal{L}^{(k)^{-1}}) = \ell_{d-k}$, for $1 \leq k \leq d-1$. The first equality is a direct corollary of [4, Theorem 4.6] and Proposition 3.2. We now turn to the second.

https://doi.org/10.1017/S0004972717001198 Published online by Cambridge University Press

Consider a common $\mathbb{Z}/d\mathbb{Z}$-equivariant desingularisation $\theta : Z \to \widetilde{X}$ and $\nu : Z \to \widetilde{Y}$, such that $\pi \circ \rho \circ \nu = \phi \circ \theta =: u$, where ϕ is the cyclic cover ramified along C,

$$\phi : \widetilde{X} = \mathrm{Spec}_{O_{\mathbb{P}^2}}\Big(\bigoplus_{k=0}^{d-1} O_{\mathbb{P}^2}\Big(-\sum_{i=1}^{r}\Big\{\frac{km_i}{d}\Big\}d_i\Big)\Big) \to \mathbb{P}^2,$$

and ρ is the cyclic cover ramified along E,

$$\rho : \widetilde{Y} = \mathrm{Spec}_{O_Y}\Big(\bigoplus_{k=0}^{d-1} \mathcal{L}^{(k)^{-1}}\Big) \to Y.$$

We may choose Z such that $\Delta := Z\backslash u^{-1}(U)$ is a normal crossing divisor. Moreover, by using [5, Corollaire 4] we can prove that

$$H^0(Y, \Omega^1_Y(\log E) \otimes (\rho \circ \nu)_* O_Z) = H^0(Z, \Omega^1_Z(\log \Delta)). \tag{3.3}$$

We first compute the dimension of the complex vector space on the left-hand side of (3.3). Because $(\rho \circ \nu)_* O_Z = \rho_* O_{\widetilde{Y}} = \bigoplus_{k=0}^{d-1} \mathcal{L}^{(k)^{-1}}$, we get the decomposition

$$H^0(Y, \Omega^1_Y(\log E) \otimes (\rho \circ \nu)_* O_Z) = \bigoplus_{k=0}^{d-1} H^0(Y, \Omega^1_Y(\log E) \otimes \mathcal{L}^{(k)^{-1}}).$$

Notice that the first direct summand of the decomposition (which corresponds to $k = 0$) has complex dimension $r - 1$.

To compute the dimension of the complex vector space on the right-hand side of (3.3), we note that by [5, Lemma 7],

$$\dim_{\mathbb{C}} H^0(Z, \Omega^1_Z(\log \Delta)) = \dim_{\mathbb{C}} H^0(Z, \Omega^1_Z) + (r - 1).$$

According to [2, Corollary 1.13] and Serre duality,

$$H^0(Z, \Omega^1_Z) \cong \bigoplus_{k=1}^{d-1} H^1\Big(\mathbb{P}^2, \mathcal{J}\Big(\mathbb{P}^2, \sum_{i=1}^{r}\Big\{\frac{km_i}{d}\Big\}C_i\Big)\Big(\sum_{i=1}^{r}\Big\{\frac{km_i}{d}\Big\}d_i - 3\Big)\Big).$$

(The direct summand of $H^0(Z, \Omega^1_Z)$ corresponding to $k = 0$ in [2, Corollary 1.13] vanishes by Serre duality.)

Thus by (3.3),

$$\sum_{k=1}^{d-1} h^0(\Omega^1_Y(\log E) \otimes \mathcal{L}^{(k)^{-1}}) = \sum_{k=1}^{d-1} \ell_k.$$

Further, because $h^1(\mathcal{L}^{(k)^{-1}}) = \ell_k$, the argument in the proof of [10, Proposition 4.6] implies that $h^0(\Omega^1_Y(\log E) \otimes \mathcal{L}^{(k)^{-1}}) \geq \ell_{d-k}$, for $1 \leq k \leq d - 1$. Therefore, it follows that $h^0(\Omega^1_Y(\log E) \otimes \mathcal{L}^{(k)^{-1}}) = \ell_{d-k}$.

Remark 3.3. It seems that the above approach also works in the general case where $m_1, \ldots, m_r$ are not necessarily mutually coprime. Indeed, the maps ϕ and ρ in the proof are still cyclic covers ramified along C and E, respectively, but with order dividing d, and the above arguments may be applied.

Acknowledgement

The author thanks the Basque Center for Applied Mathematics and the Department of Mathematics – KU Leuven for warm hospitality during his visits.

References

[1] E. A. Bartolo, 'Sur les couples de Zariski', *J. Algebraic Geom.* **3** (1994), 223–247.
[2] N. Budur, 'Unitary local systems, multiplier ideals, and polynomial periodicity of Hodge numbers', *Adv. Math.* **221**(1) (2009), 217–250.
[3] N. Budur, 'Hodge spectrum of hyperplane arrangements', Preprint, 2008, arXiv:0809.3443; incorporated in N. Budur and M. Saito, 'Jumping coefficients and spectrum of a hyperplane arrangement', *Math. Ann.* **347**(3) (2010), 545–579.
[4] N. Budur, *Multiplier Ideals, Milnor Fibers, and Other Singularity Invariants*, Lecture Notes, (Luminy, 2011). https://perswww.kuleuven.be/~u0089821/LNLuminy.pdf.
[5] H. Esnault, 'Fibre de Milnor d'un cône sur une courbe plane singulilarité', *Invent. Math.* **68** (1982), 477–496.
[6] E. R. van Kampen, 'On the fundamental group of an algebraic curve', *Amer. J Math.* **55** (1933), 255–267.
[7] R. Lazarsfeld, *Positivity in Algebraic Geometry II - Positivity for Vector Bundles, and Multiplier Ideals* (Springer, Berlin, 2004).
[8] A. Libgober, 'Alexander polynomial of plane algebraic curves and cyclic multiple planes', *Duke Math. J.* **49** (1982), 833–851.
[9] A. Libgober, 'Alexander invariants of plane albebraic curves', *Proc. Sympos. Pure Math.* **40** (1983), 135–143.
[10] F. Loeser and M. Vaquié, 'Le polynôme d'Alexander d'une courbe plane projective', *Topology* **29** (1990), 163–173.
[11] J. Milnor, 'Singular points of complex hypersurfaces', in: *Ann. Math. Studies*, Vol. 61 (Princeton University Press, Princeton, 1968).
[12] R. Randell, 'Milnor fibers and Alexander polynomials of plane curves', *Proc. Sympos. Pure Math.* **40** (1983), 415–419.

QUY THUONG LÊ, Department of Mathematics,
Vietnam National University, Hanoi, 334 Nguyen Trai Street,
Thanh Xuan District, Hanoi, Vietnam
e-mail: leqthuong@gmail.com