ON THE PERIODICITY OF COMPOSITIONS OF ENTIRE FUNCTIONS. II

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In (**1**) the author suggested the following research problem. Does there exist a non-periodic entire function f such that ff is periodic? My aim in this note is to give a partial answer to this question and, more generally, to give a partial solution to the following problem: if f and g are entire functions and $f(g)$ is periodic, what can one say about g? These results also extend a previous result of mine; for details, see (**2**, Theorem 4). We begin with some simple lemmas.

LEMMA 1. *If g is a transcendental entire function such that*

$$(g(z+1) - g(z))(g(z+2) - g(z))$$

has at most finitely many zeros, then $g(z) = P_1(z) + Q(z)\exp(P_2(z) + C_2 z)$, where $P_i(z)$ are entire periodic functions such that $P_i(z+1) = P_i(z)$, $i = 1, 2$, C_2 is a constant, and $Q(z)$ is a polynomial.

Proof. It is clear from the hypotheses of the lemma that one can express

$$g(z+i) - g(z) = L_i(z) \exp \alpha_i(z), \tag{1}$$

where $L_i(z)$ are polynomials and $\alpha_i(z)$ are entire functions for $i = 1, 2$. One can easily verify from (1) that

$$L_1(z) \exp \alpha_1(z) + L_1(z+1) \exp \alpha_1(z+1) = L_2(z) \exp \alpha_2(z).$$

It follows from a well-known theorem of Borel and Nevanlinna (**3**) that $\alpha_1(z+1) = \alpha_1(z) + C$, so that $\alpha_1(z) = P_2(z) + C_2 z$, where $P_2(z+1) = P_2(z)$ and C_2 is a constant. Choose $Q(z)$ such that $Q(z+1)\exp C_2 - Q(z) = L_1(z)$. One can easily show that $g(z)$ has the desired form.

LEMMA 2. *Let $f(z)$, $\alpha(z)$, and $\beta(z)$ be entire functions such that*

$$f(\alpha(z)) = f(\beta(z)).$$

If for some z_0, $\alpha(z_0) = \beta(z_0)$ and $f'(\alpha(z_0)) \neq 0$, then $\alpha(z)$ is identical to $\beta(z)$.

Proof. This follows almost immediately from the fact that f is 1-1 in a neighbourhood of $\alpha(z_0)$.

THEOREM 1. *Let f and g be two entire functions such that f' and g' both have no zeros. If $f(g)$ is periodic, say with period 1, then g is either periodic or linear.*

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Proof. Assume that $g(z)$ is non-linear. From Lemmas 1 and 2 we have that $g(z) = P_1(z) + \exp(P_2(z) + C_2z)$, where P_1, P_2, and C_2 are as in Lemma 1. Using the fact that g' has no zeros, one can easily verify that for any integer n greater than 1,

$$(2) \qquad (\exp(P_2(z) + C_2z))(P_2'(z) + C_2) = \frac{\exp \alpha(z+n) - \exp \alpha(z)}{\exp(nC_2) - 1},$$

where $\alpha(z)$ is some entire function. Using the fact that the left side of (2) is independent of n, we obtain

$$(3) \quad (\exp(n+1)C_2 - 1)\exp \alpha(z+n) - (\exp(n+1)C_2 - 1)\exp \alpha(z) = (\exp nC_2 - 1)\exp \alpha(z+n+1) - (\exp nC_2 - 1)\exp \alpha(z).$$

Thus, either $\exp C_2 = 1$ or $\alpha(z+1) = \alpha(z) + C_3$, where C_3 is a constant. In the former case, g is periodic. In the latter case, we obtain, for some $k \neq 0$,

$$(4) \quad P_1' + \exp(C_2)(P_2' + C_2)\exp(P_2 + C_2z) = k(P_1' + (\exp(P_2 + C_2z))(P_2' + C_2)).$$

If $P_2' = -C_2$, then $P_2 \equiv 0$, $C_2 \equiv 0$, and g must be periodic. If $P_2' \not\equiv -C_2$, then one obtains

$$(5) \qquad (\exp C_2 - k)\exp(P_2 + C_2z) = \frac{(k-1)P_1'}{(P_2' + C_2)},$$

which implies that either $\exp C_2z$ is periodic or $\exp C_2 = 1$, and the proof is complete.

COROLLARY. *If f is entire, f' has no zeros, and ff is periodic, then f is periodic.*

More generally, we have the following theorem.

THEOREM 2. *Let f and g be two entire functions such that f' has no zeros and g' has at most finitely many. If $f(g)$ is periodic, then g is either periodic or linear.*

Proof. We write

$$(6) \qquad f'(z) = \exp \alpha(z)$$

and

$$(7) \qquad g'(z) = Q(z) \exp \beta(z),$$

where $\alpha(z)$ and $\beta(z)$ are entire functions, and $Q(z)$ is a polynomial. From (6), (7), and the hypotheses of the theorem, one obtains

$$(8) \qquad Q(z+n)\exp \gamma(z+n) = Q(z)\exp \gamma(z),$$

where $\gamma(z) = \beta(z) + \alpha(g(z))$. (8) implies that $Q(z)$ is a constant and our conclusion follows from Theorem 1.

THEOREM 3. *Let f and g be entire functions such that f', g, and g' each have at most finitely many zeros. If $f(g)$ is periodic, then g is a periodic function without zeros.*

Proof. Write f', g, and g' in the forms $Q_i(z) \exp \alpha_i(z)$, $i = 1, 2, 3$, respectively, where $Q_i(z)$ are polynomials and $\alpha_i(z)$ are entire functions. Using the periodicity of $f(g)$ and its derivative (we may assume it has period 1), we obtain, for any integer n,

$$(9) \quad Q_3(z+n) \sum_{j=0}^{k} \lambda_j Q_2(z+n)^j \exp(j\alpha_2(z+n) + \gamma(z+n)) = Q_3(z) \sum \lambda_j Q_2(z)^j \exp(j\alpha_2(z) + \gamma(z)),$$

where k is the degree of $Q_1(z)$ and $\gamma(z) = \alpha_3(z) + \alpha_1(g(z))$.

A careful analysis of (9) implies that $Q_1(z)$ or $Q_3(z)$ must be a constant and our conclusion follows from the previous theorem. It is natural to ask: what can one say about a periodic function $f(g)$ when f' and g' each have at most a finite number of zeros? Let us assume, for the sake of simplicity, that $f(g)$ has period 1. We answer this question for certain classes of entire functions g. For any complex a and any integer t, let

$$S_t(g) = \{z; g(z+t) - g(z) = 0\} \quad \text{and} \quad T_a(g) = \{z; g(z) = a\}.$$

Let

$$F = \{g; S_{ln_0}(g) \cap T_a(g) \text{ is finite for all complex numbers } a \text{ for some integer } n_0 \text{ and } l = 1, 2\}.$$

THEOREM 4. *Let f and g be entire functions such that f' and g' each have at most finitely many zeros and $g \in F$. If $f(g(z))$ is periodic of period 1, then $g(z)$ has the following form:*

$$(11) \qquad g(z) = (az+b)P_2(z) + P_1(z),$$

where $P_i(z)$ is periodic with a common integral period for $i = 1, 2$, P_2 has no zeros, and a and b are constants.

Proof. One observes from the hypotheses that

$$(11) \qquad g(z) = Q_2(z) \exp(P_2(z) + C_2 z) + P_1(z),$$

where $P_1(z)$ and $P_2(z)$ have some common integral period n, $Q_2(z)$ is a polynomial, and C_2 is a constant. Write $g'(z) = L(z) \exp \alpha(z)$, where $L(z)$ is a polynomial and $\alpha(z)$ an entire function. Denote by $D(Q_2, L, \alpha, n)$ the expression

$$((\exp(C_2 n))Q_2(z+n) - Q_2(z))(L(z+2n) \exp \alpha(z+2n) - L(z) \exp \alpha(z))$$
$$-((\exp(2C_2 n))Q_2(z+2n) - Q_2(z))(L(z+n) \exp \alpha(z+n) - L(z) \exp \alpha(z)).$$

One can easily verify that, for any period n,

$$(12) \qquad P_2'(z) + C_2 = -D(Q_2', L, \alpha, n)/D(Q_2, L, \alpha, n).$$

Using the fact that P_2 is periodic and entire, one deduces from (12) and Borel's theorem that $\alpha(z+n) = \alpha(z) + C_3$, where C_3 is a constant. This fact, together with (12), yields an expression obtained from (12) by replacing $\alpha(z+in)$ by iC_3, $i = 0, 1, 2$, respectively. This latter expression leads to the following equality:

$$(13) \quad \exp(2C_2n)Q_2(z+2n) - Q_2(z) = k(\exp(C_2n)Q(z+n) - Q_2(z))$$

for some $k \neq 0$.

This implies that

$$k = \exp C_2n + 1.$$

Repeating the above argument with ln replacing n for arbitrarily large integers l, (13) yields, for a zero, z_0, of $Q(z)$, the following:

$$\frac{\exp(C_2ln)Q_2(z_0+2ln)}{Q_2(z_0+ln)} = \exp C_2ln + 1.$$

This implies that

$$|\exp(-C_2ln)| + 1 \to 2^t$$

as $l \to \infty$, where t is the degree of Q_2. If $|\exp(-C_2)| < 1$, then it is clear that f must be a constant; thus, $\exp C_2 = 1$ and our proof is complete.

It is reasonable to conjecture that Theorem 4 remains valid without the assumption that $g \in F$. As an extension of Theorem 4 we obtain, by a similar proof, the following theorem.

THEOREM 5. *Let f and g be as in Theorem 4. Suppose, furthermore, that $P_1(z) = 0$; then $g(z) = c \cdot \exp 2\pi ikz$, where k is an integer and c is a constant.*

Using arguments as above, one can also prove the following theorem.

THEOREM 6. *Let f and g be entire functions such that f has at least one and at most finitely many zeros. If $f(g(z))$ is periodic, then the order of convergence of the zeros of g is at least one unless g has no zeros at all.*

COROLLARY. *If f is entire and has at least one zero, and if $ff(z)$ is periodic, then the order of convergence of the zeros of f is at least one.*

REFERENCES

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