4
THE MORITA THEORY

Let R and S be rings and let W be an R-S-bimodule. It may happen that the change of rings functor $- \otimes_R W$ induces an equivalence, known as a Morita equivalence, between the categories $\mathcal{M}\text{OD}_R$ and $\mathcal{M}\text{OD}_S$. The investigation of the circumstances in which this happens is called the *Morita theory*, after its instigator [Morita 1958].

It turns out that many of the module categories associated to Morita equivalent rings are also equivalent, and that the rings have many properties in common.

This chapter has only two sections. In the first, we analyse the properties that a bimodule W must have if it is to give rise to a Morita equivalence, namely, that W must be a 'projective generator', both as a left R-module and as a right S-module. In the second section, we investigate the consequences that follow from the existence of such a bimodule. Here, we also look at the Picard group of a ring. This group, which arises from the 'self-equivalences' of a ring, can be regarded as a generalization of the ideal class group of a Dedekind domain.

It is worth mentioning here two themes that are developed in the exercises. One lengthy series of computations for a special type of tiled order leads to explicit descriptions of the projective modules and generators for such orders, and hence to a computation of Picard groups. Another set of exercises outlines a Morita theory for nonunital rings that has recently been investigated in [Quillen].

4.1 PROJECTIVE GENERATORS

A ring R plays a very distinctive role in the category ${}_R\mathcal{M}\text{OD}$ of left R-modules since it enjoys two important properties as an object of that category. The first property, which we have used on many occasions, is that R is projective.

The second, which we have not formalized before, is that R is a generator: any module is the homomorphic image of ${}^{\Lambda}R$ for some index set Λ. In fact, R is an example of a left R-progenerator (that is, it is both projective and a generator).

Our aim in this section is to give a criterion for a left R-module W to be a progenerator in terms of its relationship with its dual $W^* = \mathrm{Hom}({}_RW, {}_RR)$, in preparation for our study of Morita equivalence in the next section. It turns out that a progenerator is 'invertible' in a sense to be made more precise later. As an illustration, take R to be a field $\mathcal{K}$ and W to be ${}^r\mathcal{K}$, row-space of dimension r. Then the dual can be identified as the column-space $\mathcal{K}^r$, and the phenomenon of 'invertibility' amounts to the informal identities ${}^r\mathcal{K} \cdot \mathcal{K}^r = \mathcal{K}$ and $\mathcal{K}^r \cdot {}^r\mathcal{K} = M_r(\mathcal{K})$, the $r \times r$ matrix ring.

In a departure from the usual practice in this text, the prime object of study in this section will be a left module rather than a right module. The reason is that in the Morita theory we wish to consider the effect of the functor $- \otimes_R W$ on the category $\mathcal{M}od_R$ of right modules, where R is an arbitrary ring. The notational consequences are as follows.

By our fundamental convention (1.1.4), a left R-module homomorphism from W to another left R-module X is regarded as operating on the right of W. Thus W becomes an R-$\mathrm{End}({}_RW)$-bimodule, and the additive group $\mathrm{Hom}({}_RW, {}_RX)$ of left R-module homomorphisms from W to X is then an $\mathrm{End}({}_RW)$-$\mathrm{End}({}_RX)$-bimodule by the rule

$$w(e\alpha f) = ((we)\alpha)f$$

for $\alpha \in \mathrm{Hom}({}_RW, {}_RX)$, $e \in \mathrm{End}({}_RW)$ and $f \in \mathrm{End}({}_RX)$.

Similarly, homomorphisms of right modules are written on the left. In cases where there is no special reason to write a map on one side or the other, for example, with bimodule homomorphisms or ring homomorphisms, it is put on the left. As we observed in (3.1.9), we are sometimes obliged to write tensor products of homomorphisms on the 'wrong' side as a lesser evil.

4.1.1 The dual of a module

Now let ${}_RW$ be a left R-module and write $S = \mathrm{End}({}_RW)$, so that W is an R-S-bimodule.

The *dual* of W (as a left module) is $W^* = \mathrm{Hom}({}_RW, {}_RR)$. Given $\phi \in W^*$ and $s \in S$, $r \in R$, we define $s\phi r$ by

$$w(s\phi r) = ((ws)\phi)r,$$

which makes W^* into an S-R-bimodule.

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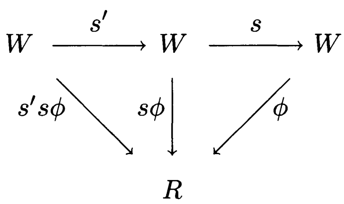
Similarly, given a right R-module V with endomorphism ring T, its dual $V^* = \mathrm{Hom}(V_R, R_R)$ is an R-T-bimodule.

We may now in turn take the dual of the right R-module W^*, and define the *double dual* of our original left R-module ${}_RW$ as $W^{**} = \mathrm{Hom}(W^*_R, R_R)$, which is an R-$\mathrm{End}(W^*_R)$-bimodule.

There is a ring homomorphism ι from S to $\mathrm{End}(W^*_R)$, given by

$$(\iota s)\phi = s\phi \quad \text{for } s \in S \text{ and } \phi \in W^*.$$

That ι preserves multiplication follows from commutativity of the diagram



Then W^{**} is in fact an R-S-bimodule by restriction of scalars $\iota^{\#}$. There is also a natural R-S-bimodule homomorphism $\nu : W \to W^{**}$, given by

$$(\nu w)(\phi) = w\phi \quad \text{for } w \in W \text{ and } \phi \in W^*.$$

The double dual of a right R-module is defined similarly. The following lemma is now a good exercise.

4.1.2 Lemma

There are canonical R-R-bimodule isomorphisms

$$({}_RR)^* \cong ({}_RR)^{**} \cong (R_R)^* \cong (R_R)^{**} \cong R. \qquad \square$$

4.1.3 The dual of a free module

We interpret the dual first in the important special case that $W = {}^nR$, the free left R-module of finite rank n. (Recall that we use the notation R^n for the free right R-module.) Choose a basis $\{f_1, \dots, f_n\}$ of ${}^nR$; if you wish, this can be the standard basis consisting of the row vectors $f_i = (0, \dots, 0, 1, 0, \dots)$, with 1 in the ith place. The *dual basis* for $({}^nR)^*$ is $\{f_1^*, \dots, f_n^*\}$, defined by

$$f_i \cdot f_j^* = \begin{cases} 1 & i = j, \\ 0 & i \neq j. \end{cases}$$

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To see that the dual basis is indeed a basis of $({}^nR)^*$, for any $\phi \in ({}^nR)^*$ write $f_i\phi = \phi_i \in R$. Then

$$\phi = f_1^*\phi_1 + \cdots + f_n^*\phi_n,$$

since both maps have the same effect on each f_i, and $\phi = 0$ precisely when all $\phi_i = 0$.

Thus we can identify the right R-module $({}^nR)^*$ as the standard free right module R^n. It is clear that when $\{f_1, \ldots, f_n\}$ is the standard basis, its dual is the standard basis of the column-space R^n. The effect of the action of f_j^* on f_i is given simply by the product $f_i f_j^*$ of the standard row and column vectors. More generally, for any $x \in {}^nR$ and $\phi \in ({}^nR)^*$, $x\phi$ can also be interpreted as the product of the coordinate row vector of x with respect to the basis $\{f_i\}$ and the coordinate column vector of ϕ with respect to the basis $\{f_j^*\}$.

4.1.4 Endomorphisms of a free left module

As we saw in (1.3.5), an endomorphism of a free right module can be represented by a matrix once we have chosen a basis of the module. The corresponding statement for free left modules follows by duality, but, for future applications, it will be convenient to give the details explicitly for the standard free left module ${}^nR$ of finite rank n.

Let $\{f_1, \ldots, f_n\}$ be a basis of ${}^nR$. (Although we do not assume that R has invariant basis number, it will suffice for our purposes to consider only bases of ${}^nR$ that consist of n elements.)

Define $\phi_{jk} \in \operatorname{End}({}^nR)$ for each pair of indices j, k by

$$f_i\phi_{jk} = \begin{cases} f_k & i = j, \\ 0 & i \neq j, \end{cases}$$

and define a ring homomorphism θ from R to $\operatorname{End}({}^nR)$ by

$$f_i \cdot \theta r = r f_i, \quad i = 1, \ldots, n.$$

(That this map does indeed preserve multiplication follows from the relation $r'(f_i \cdot \theta r) = (r' f_i) \cdot \theta r$.) Then $\theta r \cdot \phi_{jk} = \phi_{jk} \cdot \theta r$ in $\operatorname{End}({}^nR)$ for all r, j and k.

Given α in $\operatorname{End}({}^nR)$, write $f_i\alpha = \sum_j a_{ij} f_j$, so that $\alpha = \sum_{j,k} (\theta a_{jk})\phi_{jk}$. The map $\alpha \mapsto (a_{jk})$ is a ring isomorphism from $\operatorname{End}({}^nR)$ to the matrix ring $M_n(R)$, with $\{\phi_{jk}\}$ corresponding to the set of standard matrix units $\{e_{jk}\}$ and θr to the diagonal matrix $\operatorname{diag}(r, \ldots, r)$. We often treat this (natural) isomorphism as an identification.

The free right module R^n acquires the structure of a left $\operatorname{End}({}^nR)$-module in two ways, once of itself and once as the dual of ${}^nR$, but these structures can easily be seen to be the same. Thus there are two ways of making R^n into a left $M_n(R)$-module, which coincide provided that we choose the basis of R^n dual to the original basis of ${}^nR$.

4.1.5 The evaluation homomorphisms

We associate with a left R-module W and its dual $W^* = \mathrm{Hom}({}_RW, {}_RR)$ two *evaluation homomorphisms* σ and τ that play an important role. As before, let $S = \mathrm{End}({}_RW)$.

We define the R-R-bimodule homomorphism

$$\sigma : W \otimes_S W^* \to R$$

by

$$\sigma(w \otimes \phi) = w\phi \quad \text{for } w \in W \text{ and } \phi \in W^*.$$

We also define the S-S-module homomorphism

$$\tau : W^* \otimes_R W \to S$$

by

$$\tau(\phi \otimes w) = \phi w \quad \text{for } w \in W \text{ and } \phi \in W^*,$$

where the endomorphism ϕw of W is defined by the equation

$$w'(\phi w) = (w'\phi)w \quad \text{for } w' \in W.$$

This equation may be reformulated as the commutativity of the square

$$\begin{array}{ccc} W \otimes_S W^* \otimes_R W & \xrightarrow{id \otimes \tau} & W \otimes_S S \\ \sigma \otimes id \downarrow & & \downarrow \\ R \otimes_R W & \longrightarrow & W \end{array}$$

In general, properties such as this flow readily from the associativity we have built into our notation. As an example, the reader may like to use the S-R-bimodule structure of W^*, expressed in (4.1.1) by the equation $w(s\phi r) = ((ws)\phi)r$, to deduce the commutativity of

$$\begin{array}{ccc} W^* \otimes_R W \otimes_S W^* & \xrightarrow{id \otimes \sigma} & W^* \otimes_R R \\ \tau \otimes id \downarrow & & \downarrow \\ S \otimes_S W^* & \longrightarrow & W^* \end{array}$$

For another viewpoint here, note that the Adjointness Theorem (3.1.19) shows that there is an isomorphism

$$\mathrm{Hom}_{R-R}(W \otimes_S W^*, R) \cong \mathrm{Hom}_{S-R}(W^*, W^*);$$

then σ is the distinguished element of the left-hand term corresponding to id_{W^*}. Although similar considerations apply to τ, the formulas are messier in general. In the cases of interest to us they simplify, as in (4.1.14) below.

Suppose in particular that $W = {}^nR$ with basis $\{f_i\}$, that $\{f_j^*\}$ is the dual basis of R^n (4.1.3), and that $\{\phi_{jk}\}$ is the corresponding set of matrix units for $\mathrm{End}({}^nR) = M_n(R)$ (4.1.4).

Then the map $\sigma : {}^nR \otimes_{M_n(R)} R^n \to R$ is given by $f_i \otimes f_j^* \mapsto f_i f_j^*$, while $\tau : R^n \otimes_R {}^nR \to \mathrm{End}({}^nR)$ sends $f_j^* \otimes f_k$ to ϕ_{jk}. Clearly, both σ and τ are isomorphisms. (The fact τ is an isomorphism can also be regarded as a special case of (3.1.15).) When $\{f_i\}$ and $\{f_j^*\}$ are the standard bases, the maps σ and τ can be regarded as the multiplication of 'a row by a column' and 'a column by a row'.

4.1.6 Projective modules

We next find a characterization of finitely generated projective left R-modules W in terms of the homomorphism $\tau : W^* \otimes_R W \to S$, where $S = \mathrm{End}({}_RW)$, and we derive some consequences.

4.1.7 Theorem

The left R-module ${}_RW$ is finitely generated and projective if and only if the map $\tau : W^* \otimes_R W \to S$ is a surjection.

Proof

Suppose that W is finitely generated and projective. This means that, for some integer n, we have ${}^nR \cong W \oplus V$ as left R-modules. There is no loss of generality in replacing W by an isomorphic module (if necessary) so that there is an internal direct sum decomposition ${}^nR = W \oplus V$.

Let $\{f_i\}$ be a basis for ${}^nR$, and write $f_i = w_i \oplus v_i$ for $i = 1, \ldots, n$, with w_i in W and v_i in V. For $j = 1, \ldots, n$, let ϕ_j be the restriction to W of the member f_j^* of the dual basis in $({}^nR)^*$.

For any x in ${}^nR$, we have $x = \sum(xf_j^*)f_j$; thus for x in W we have $x = \sum(x\phi_j)w_j$ and hence

$$1 = \sum \phi_j w_j = \tau(\sum \phi_j \otimes w_j)$$

in S. Now τ is S-S-bilinear, so $\mathrm{Im}\,\tau$ is an ideal in S, and thus τ is surjective.

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Conversely, suppose that we have $\{\phi_j\}$ in W^* and $\{w_j\}$ in W with $1 = \sum_{j=1}^{n} \phi_j w_j$. Choose a basis $\{f_i\}$ for ${}^nR$ and define an R-homomorphism ρ from ${}^nR$ to W by $f_i\rho = w_i$ for each i. Define also a map η from W to ${}^nR$ by $w\eta = \sum (w\phi_j)f_j$. Then η is an R-homomorphism which splits ρ, making W a direct summand of ${}^nR$. □

The collection of elements $\{w_j\}$ together with the maps $\{\phi_j\}$ is sometimes called a *projective coordinate system* for W, since an element w of W can be written

$$w = (w\phi_1)w_1 + \cdots + (w\phi_n)w_n,$$

$(w\phi_1, \ldots, w\phi_n)$ being not necessarily unique 'coordinates' for w.

4.1.8 Some identifications

By the left-handed version of part (i) of (3.3.24), any finitely generated projective left R-module W is isomorphic to one of the form ${}^nR\epsilon$ for some idempotent ϵ of the ring $\mathrm{End}({}^nR)$.

Let $\xi : W \to {}^nR\epsilon$ be such an isomorphism. Then there is a ring isomorphism from S to the ring $S' = \mathrm{End}({}^nR\epsilon)$, given by $s \mapsto \xi^{-1}s\xi$, as in the commuting diagram below.

$$\begin{array}{ccc} W & \xrightarrow{\quad s \quad} & W \\ \uparrow \xi^{-1} & & \downarrow \xi \\ {}^nR\epsilon & \xrightarrow{\quad \xi^{-1}s\xi \quad} & {}^nR\epsilon \end{array}$$

There is also an induced isomorphism from W^* to $({}^nR\epsilon)^*$, given by $\phi \mapsto \xi^{-1}\phi$, that converts the S-R-bimodule structure on W^* to the S'-R-bimodule structure on $({}^nR\epsilon)^*$. If we identify W with ${}^nR\epsilon$ via ξ, these isomorphisms become identities.

These identifications are used in the proof of the following result.

4.1.9 Proposition

Let ${}_RW$ be a finitely generated projective left R-module, with $S = \mathrm{End}({}_RW)$. Then, for some n, the following hold.

(i) *$W \cong {}^nRe$, where e is an idempotent element of $M_n(R)$.*
(ii) *$W^* \cong eR^n$.*
(iii) *There is a ring isomorphism $S \cong eM_n(R)e$.*
(iv) *$\tau : W^* \otimes_R W \to S$ is an isomorphism of S-S-bimodules.*
(v) *$W^{**} \cong W$ as an R-S-bimodule.*

Proof

(i) By the preceding remarks, we can write $W = {}^nR\epsilon$ for some idempotent ϵ in $\mathrm{End}({}^nR)$, and we can realise ϵ as an idempotent matrix e in $M_n(R)$ if we fix some basis of ${}^nR$ that consists of n elements.

(ii) We have $W \oplus V = {}^nR$ with $V = {}^nR(1-e)$. If ϕ is in $R^n = ({}^nR)^*$, clearly $e\phi$ is in W^*. On the other hand, any element ψ in W^* extends to an element ψ' of R^n, by setting $\psi' = 0$ on elements of V, and then $\psi = e\psi'$.

(iii) Proceeding as in (ii), we note that if s' is an endomorphism of ${}^nR$, then $es'e$ is an endomorphism of W. Conversely, any element s in S extends to an element s' of $M_n(R)$, by setting $s' = 0$ on elements of V, and then $s = es'e$. Moreover, these inverse constructions respect composition of endomorphisms.

(iv) We need only check that τ is an injection. Expanding the tensor product, we have a direct decomposition (as abelian groups)

$$R^n \otimes_R {}^nR = (W^* \otimes_R W) \oplus (V^* \otimes_R W) \oplus (W^* \otimes_R V) \oplus (V^* \otimes_R V),$$

where $V^* = (1-e)R^n$.

The homomorphism $\tau : W^* \otimes_R W \to S$ is induced by restriction of the corresponding homomorphism τ' from $R^n \otimes {}^nR$ to $M_n(R)$. Since τ' is injective, so is τ.

(v) Repeating the arguments for right modules, we have

$$(eR^n)^* \cong W^{**} \cong {}^nRe. \qquad \square$$

Thus, finitely generated projectives may be characterized as those left R-modules W for which $\tau : W^* \otimes_R W \to S$ is an isomorphism. We next describe those W for which $\sigma : W \otimes_S W^* \to R$ is an isomorphism.

4.1.10 Generators

A left R-module W is called a *generator* of ${}_R\mathcal{M}\textsc{od}$ if there is a surjective homomorphism $W^n \to R$ of left R-modules for some integer n. Then given any left R-module N, there is a surjection $W^\Lambda \to N$ for some index set Λ (which depends on N). This fact explains the term 'generator'.

There is a nice symmetry between the concepts of generator and of finitely generated projective. For, since ${}_RR$ is a projective R-module, the surjective homomorphism $W^n \to R$ must be split, making R a direct summand of some W^n whenever W is a generator. On the other hand, any finitely generated projective left R-module is a direct summand of some ${}^nR$. This symmetry is developed further in the next result, to be compared with (4.1.7) and (iv) of (4.1.9) above.

4.1.11 Proposition

The finitely generated left R-module W is a generator if and only if the homomorphism

$$\sigma : W \otimes_S W^* \longrightarrow R, \qquad \sigma(w \otimes \phi) = w\phi,$$

is surjective.

Further, if σ is surjective then it is an isomorphism of R-R-bimodules.

Proof

Suppose that there is a surjection $f : W^n \to R$, and choose $x = (x_i)$ in W^n with $xf = 1$. Let $\sigma_i : W \to W^n$ be inclusion at the i th coordinate and put $\phi_i = \sigma_i f : W \to R$. Then

$$\sigma\left(\sum x_i \otimes \phi_i\right) = \sum x_i \sigma_i f = 1.$$

Conversely, given $\{x_i\}$ and $\{\phi_i\}$ with $\sigma(\sum x_i \otimes \phi_i) = 1$, define $f : W^n \to R$ by $(w_i)f = \sum w_i \phi_i$. Thus $(x_i)f = 1$, making W a generator.

Suppose further that $\sigma(\sum w_j \otimes \psi_j) = 0$ for some w_j and ψ_j. Then, noting that each $\phi_i w_j$ is a member of S, we have

$$\begin{aligned} \textstyle\sum_j w_j \otimes \psi_j &= \textstyle\sum_j \sum_i (x_i \phi_i w_j \otimes \psi_j) \\ &= \textstyle\sum_j \sum_i (x_i \otimes \phi_i w_j \psi_j) \\ &= \textstyle\sum_i x_i \otimes \phi_i (\sum_j w_j \psi_j) \\ &= 0, \end{aligned}$$

which shows that σ is injective. □

4.1.12 Progenerators

A left R-module W is said to be a *progenerator* (of ${}_R\mathcal{M}\textsc{od}$), or a *left R-progenerator*, if it is a finitely generated projective generator. Some texts use the term *faithfully projective* module instead of progenerator. This terminology derives from an alternative definition of a generator – see Exercise 4.1.8 and [Bass 1968], Chapter II, Proposition 1.2 and Corollary 4.8.

Collecting our previous results (4.1.7), (iv) of (4.1.9) and (4.1.11), and writing $S = \mathrm{End}({}_R W)$ and $W^* = \mathrm{Hom}({}_R W, {}_R R)$ as usual, we obtain the following important result.

4.1.13 Theorem

The left R-module W is a progenerator if and only if

$$\sigma : W \otimes_S W^* \cong R$$

and

$$\tau : W^* \otimes_R W \cong S$$

are bimodule isomorphisms. □

It is clear that the preceding definitions and results all have right-handed counterparts which we invoke in both the statement and the proof of the next result, the keystone of the Morita theory.

4.1.14 Theorem

Suppose that W is a left R-progenerator. Then the following statements are true.

(i) $W^* \cong \mathrm{Hom}(W_S, S_S)$ *as an S-R-bimodule.*

(ii) $R \cong \mathrm{End}(W_S)$ *as a ring.*

(iii) W_S *is a right S-progenerator.*

(iv) W^* *is both a left S-progenerator and a right R-progenerator, and there are ring isomorphisms $R \cong \mathrm{End}({}_S W^*)$ and $S \cong \mathrm{End}(W^*_R)$.*

Proof

We make repeated use of the isomorphisms σ and τ.

(i) Given $\xi \in \mathrm{Hom}(W_S, S_S)$, define $\alpha(\xi) \in \mathrm{Hom}(R_R, W^*_R) \cong W^*$ as the composition

$$R \xrightarrow{\sigma^{-1}} W \otimes_S W^* \xrightarrow{\xi \otimes id} S \otimes_S W^* \xrightarrow{\cong} W^*.$$

To see that α is injective, observe that $\alpha(\xi)$ can be zero only if

$$\xi \otimes id : W \otimes_S W^* \longrightarrow S \otimes_S W^*$$

is zero. But then the commutative diagram

$$\begin{array}{ccc} W \otimes_S W^* \otimes_R W & \xrightarrow{\xi \otimes id \otimes id} & S \otimes_S W \otimes_R W^* \\ \cong \Big\downarrow id \otimes \tau & & \cong \Big\downarrow id \otimes \tau \\ W \otimes_S S & \xrightarrow{\xi \otimes id} & S \otimes_S S \\ \cong \Big\downarrow & & \cong \Big\downarrow \\ W & \xrightarrow{\xi} & S \end{array}$$

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shows that ξ is zero too. In formulas, if

$$\sigma^{-1}(1) = \sum x_i \otimes \phi_i \in W \otimes_S W^*,$$

then $\alpha(\xi) = \sum \xi(x_i)\phi_i$, where we recall that $w(s\phi) = (ws)\phi$.

This construction is reversible. Define

$$\beta : W^* \longrightarrow \mathrm{Hom}(W_S, S_S)$$

by

$$\beta(\phi)(w) = \phi w \in S,$$

where we recall that $w'(\phi w) = (w'\phi)w$. In other words, β corresponds to τ under the adjunction (see (3.1.19))

$$\mathrm{Hom}({}_SW^* \otimes_R W_S, {}_SS_S) \cong \mathrm{Hom}({}_SW^*_R, {}_S\mathrm{Hom}(W_S, S_S)_R).$$

Evidently

$$\alpha\beta(\phi) = \sum \beta(\phi)(x_i)\phi_i = \sum (\phi x_i)\phi_i = \phi \sum x_i\phi_i = \phi,$$

so that α and β are inverse isomorphisms.

Note that, using these isomorphisms, we have an adjunction

$$\mathrm{Hom}_{S-S}(W^* \otimes_R W, S) \cong \mathrm{Hom}_{S-R}(W^*, \mathrm{Hom}(W_S, S_S)) \cong \mathrm{Hom}_{S-R}(W^*, W^*)$$

with τ the distinguished element corresponding to id_{W^*}.

(ii) We use the homomorphism $\widehat{\tau} : W \otimes_S \mathrm{Hom}(W_S, S_S) \to \mathrm{End}(W_S)$ which plays the same role for the right module W_S as τ does for the left module ${}_RW$. There is a commutative square

$$\begin{array}{ccc} W \otimes_S W^* & \xrightarrow[\cong]{\sigma} & R \\ \cong \Big\downarrow id \otimes \sigma & & \Big\downarrow \iota \\ W \otimes_S \mathrm{Hom}(W_S, S_S) & \xrightarrow{\widehat{\tau}} & \mathrm{End}(W_S) \end{array}$$

where the natural action of R on W gives a ring homomorphism $\iota : R \to \mathrm{End}(W_S)$. Since σ is an isomorphism, the identity element id_W of $\mathrm{End}(W_S)$ is in the image of $\widehat{\tau}$. But $\widehat{\tau}$ is an $\mathrm{End}(W_S)$-bimodule homomorphism, hence it is surjective. By the right-handed version of (4.1.7), $\widehat{\tau}$ is an isomorphism. Thus ι must be a ring isomorphism.

(iii) Since $\widehat{\tau}$ is surjective, W is projective as a right S-module. To show that W_S is a generator, we have to show that the homomorphism

$$\widehat{\sigma} : W^* \otimes_{\mathrm{End}(W_S)} W \longrightarrow S$$

is an isomorphism. However, by using the isomorphism ι, we can define an isomorphism $\theta : W^* \otimes_{\mathrm{End}(W_S)} W \to W^* \otimes_R W$ making the following square commutative:

$$\begin{array}{ccc} W^* \otimes_{\mathrm{End}(W_S)} W & \xrightarrow{\ \widehat{\sigma}\ } & S \\ \cong \big\downarrow \theta & & \big\downarrow id_S \\ W^* \otimes_R W & \xrightarrow[\cong]{\ \tau\ } & S \end{array}$$

The result follows.

(iv) Since we have an R-S-bimodule isomorphism $W^{**} \cong W$, we can repeat the arguments with the roles of W and W^* interchanged. □

We note also that progenerators can be characterized in terms of 'invertibility'.

4.1.15 Theorem

Let R and S be arbitrary rings and let W be an R-S-bimodule. Suppose that there is an S-R-bimodule W' together with bimodule isomorphisms

$$\sigma' : W \otimes_S W' \cong R \quad \text{and} \quad \tau' : W' \otimes_R W \cong S.$$

Then W is a left R-progenerator with ring isomorphisms $S \cong \mathrm{End}({}_RW)$ and $R \cong \mathrm{End}(W_S)$, and there is an S-R-bimodule isomorphism $W' \cong W^*$, where $W^* = \mathrm{Hom}({}_RW, {}_RR)$.

Proof

Using the adjunction

$$\mathrm{Hom}({}_RW \otimes_S W'_R, {}_RR_R) \cong \mathrm{Hom}({}_SW'{}_R, {}_SW^*_R),$$

we define an S-R-bimodule homomorphism $\theta : W' \to W^*$ by

$$w(\theta w') = \sigma'(w \otimes w').$$

Then the commuting triangle

$$\begin{array}{ccc} W \otimes_R W' & \xrightarrow{\ id \otimes \theta\ } & W \otimes_S W^* \\ & {\scriptstyle \sigma'} \searrow \qquad \swarrow {\scriptstyle \sigma} & \\ & R & \end{array}$$

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shows that W is an R-generator. Tensoring again on the left with W' and using τ', we see that θ is an isomorphism.

In the commuting square of (4.1.5)

$$\begin{array}{ccc} W \otimes_S W^* \otimes_R W & \xrightarrow{id \otimes \tau} & W \otimes_S S \\ {\scriptstyle \sigma \otimes id} \downarrow & & \downarrow \\ R \otimes_R W & \longrightarrow & W \end{array}$$

we again cancel W on the left, by tensoring with W' and using τ', to deduce that τ also is an isomorphism. It follows that S and $\mathrm{End}({}_RW)$ are both S-S isomorphic to $W' \otimes_R W$ and hence to each other. The identification of S with $\mathrm{End}({}_RW)$ allows us to apply part (ii) of (4.1.14), yielding $R \cong \mathrm{End}(W_S)$. □

4.1.16 Commutative domains

Some interesting explicit computations of duals can be made for modules over a commutative domain $\mathcal{O}$. Here, we lay the foundations for these computations by showing that a fractional ideal of $\mathcal{O}$ is a progenerator precisely when it is invertible, which gives an alternative view of the definition of a Dedekind domain (2.3.20). Further developments of this discussion, particularly for modules over $\mathcal{O}$-orders, will be given in the exercises below and in the following chapter.

Let $\mathcal{K}$ be the field of fractions of $\mathcal{O}$. By definition, a fractional ideal of $\mathcal{O}$ is a nonzero finitely generated $\mathcal{O}$-submodule $\mathfrak{a}$ of $\mathcal{K}$.

The ring $\mathrm{End}(\mathcal{K}_{\mathcal{K}})$ can be identified as $\mathcal{K}$ itself, where the elements of $\mathcal{K}$ are viewed as operating on the right module $\mathcal{K}$ by left multiplication. This identification leads to the identification $\mathcal{O} = \mathrm{End}(\mathcal{O}_{\mathcal{O}})$, and, for any pair of fractional ideals $\mathfrak{a}$ and $\mathfrak{b}$ of $\mathcal{O}$, we have an identification

$$\mathrm{Hom}_{\mathcal{O}}(\mathfrak{a}, \mathfrak{b}) = \{x \in \mathcal{K} \mid x\mathfrak{a} \subseteq \mathfrak{b}\}.$$

This is again a fractional ideal of $\mathcal{O}$ when $\mathcal{O}$ is Noetherian, since $\mathrm{Hom}_{\mathcal{O}}(\mathfrak{a}, \mathfrak{b}) \subseteq a^{-1}\mathfrak{b}$ for any nonzero element a of $\mathfrak{a}$. We see below that $\mathrm{Hom}_{\mathcal{O}}(\mathfrak{a}, \mathfrak{b})$ is also a fractional ideal if $\mathfrak{a}$ is projective.

In particular, the dual of $\mathfrak{a}$ is

$$\mathfrak{a}^* = \{x \in \mathcal{K} \mid x\mathfrak{a} \subseteq \mathcal{O}\}.$$

We have also shown (3.2.16) that the tensor product $\mathfrak{a} \otimes_{\mathcal{O}} \mathfrak{b}$ of fractional ideals $\mathfrak{a}$ and $\mathfrak{b}$ can be identified with the product $\mathfrak{a}\mathfrak{b}$ evaluated in $\mathcal{K}$. Thus the

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criterion in (4.1.7) for $\mathfrak{a}$ to be (left or right) finitely generated and projective over $\mathcal{O}$ now reads

$$\mathfrak{a}\mathfrak{a}^* = \mathrm{End}(\mathfrak{a}_{\mathcal{O}}) = \{x \in \mathcal{K} \mid x\mathfrak{a} \subseteq \mathfrak{a}\}. \tag{1}$$

Since $\mathfrak{a}\mathfrak{a}^* \subseteq \mathcal{O} \subseteq \mathrm{End}(\mathfrak{a}_{\mathcal{O}})$ always, equation (1) gives

$$\mathfrak{a}\mathfrak{a}^* = \mathcal{O}. \tag{2}$$

On the other hand, if equation (2) holds, then $\mathcal{O} = \mathrm{End}(\mathfrak{a}_{\mathcal{O}})$ by (4.1.15), which shows that (2) is a restatement of (1).

As both $\mathcal{O}$ and $\mathrm{End}(\mathfrak{a}_{\mathcal{O}})$ are domains having field of fractions $\mathcal{K}$, $\mathfrak{a}\mathfrak{a}^*$ can be considered to be the tensor product over either $\mathcal{O}$ or $\mathrm{End}(\mathfrak{a}_{\mathcal{O}})$. It follows that equation (2) is also equivalent to the criterion for a fractional ideal $\mathfrak{a}$ be a generator (4.1.11).

Now suppose that equation (2) holds. Since $\mathfrak{a} \subseteq \mathfrak{a}^{**}$, we have

$$\mathcal{O} = \mathfrak{a}^*\mathfrak{a} \subseteq \mathfrak{a}^*\mathfrak{a}^{**} \subseteq \mathcal{O},$$

and hence that

$$\mathfrak{a}^*\mathfrak{a} = \mathfrak{a}^*\mathfrak{a}^{**}$$

and

$$\mathfrak{a} = \mathfrak{a}\mathfrak{a}^*\mathfrak{a} = \mathfrak{a}\mathfrak{a}^*\mathfrak{a}^{**} = \mathfrak{a}^{**}.$$

Thus $\mathfrak{a}^*$ is itself a fractional ideal and a progenerator, by (4.1.15) again.

Recall that a fractional ideal $\mathfrak{a}$ of a domain $\mathcal{O}$ is said to be invertible if there is some fractional ideal $\mathfrak{a}^{-1}$ with $\mathfrak{a}\mathfrak{a}^{-1} = \mathcal{O}$ (whence (4.1.15) yet again applies). Our discussion is summarized in the following result.

4.1.17 Proposition

Let $\mathcal{O}$ be a commutative domain, and let $\mathfrak{a}$ be a fractional ideal of $\mathcal{O}$. Then the following assertions are equivalent.

(i) $\mathfrak{a}$ *is invertible as an ideal.*

(ii) $\mathfrak{a}$ *is a finitely generated projective $\mathcal{O}$-module.*

(iii) $\mathfrak{a}$ *is a generator (as an $\mathcal{O}$-module).*

(iv) $\mathfrak{a}\mathfrak{a}^* = \mathcal{O}$.

Furthermore, if these statements hold, then $\mathfrak{a}^{**} = \mathfrak{a}$, and, for any fractional ideal $\mathfrak{b}$ of $\mathcal{O}$, $\mathrm{Hom}_{\mathcal{O}}(\mathfrak{a}, \mathfrak{b}) = \mathfrak{a}^*\mathfrak{b}$ *is also a fractional ideal.* □

Our definition of a Dedekind domain (2.3.20) is that a Dedekind domain is a commutative domain $\mathcal{O}$ for which every nonzero fractional ideal is invertible. Thus the following corollary is immediate.

4.1.18 Corollary

Let $\mathcal{O}$ be a commutative domain. Then the following assertions are equivalent.

(i) *$\mathcal{O}$ is a Dedekind domain.*
(ii) *Every fractional ideal of $\mathcal{O}$ is a (left and right) projective $\mathcal{O}$-module.*
(iii) *Every fractional ideal of $\mathcal{O}$ is a (left and right) $\mathcal{O}$-generator.* □

Exercises

4.1.1 Let R be a ring. Show that a direct sum of R-generators is also an R-generator.

Let the ring $R = S_1 \times S_2$ be the internal direct product of two nontrivial rings.

Show that S_1 is not a generator when considered as an R-module. Deduce that a direct summand of a generator need not be a generator.

4.1.2 For this exercise, the discussions in (4.1.3), (4.1.4) and (4.1.5) are helpful.

Let R be a ring, and let the 'row-space' ${}^nR$ be regarded as a right $M_n(R)$-module in the usual way.

Verify that there are natural isomorphisms

(i) $R \cong \mathrm{End}(({}^nR)_{M_n(R)})$ as rings,
(ii) $(({}^nR)_{M_n(R)})^* \cong R^n$ as $M_n(R)$-R-bimodules,

and deduce that ${}^nR$ is an $M_n(R)$-progenerator.

4.1.3 **Morita context**

The situation described in (4.1.5) can easily be generalized, as follows. Let R and S be rings, and let V be an S-R-bimodule and W an R-S-bimodule. Then a *generalized Morita context* consists of these objects together with an R-R-bimodule homomorphism $\sigma : W \otimes_S V \to R$ and an S-S-module homomorphism $\tau : V \otimes_R W \to S$ which make the following squares commute.

$$\begin{array}{ccc} W \otimes_S V \otimes_R W & \xrightarrow{id \otimes \tau} & W \otimes_S S \\ \sigma \otimes id \Big\downarrow & & \Big\downarrow \\ R \otimes_R W & \longrightarrow & W \end{array}$$

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$$\begin{array}{ccc} V \otimes_R W \otimes_S V & \xrightarrow{id \otimes \sigma} & V \otimes_R R \\ {\scriptstyle \tau \otimes id} \downarrow & & \downarrow \\ S \otimes_S V & \longrightarrow & V \end{array}$$

(a) Show that a generalized Morita context defines a ring

$$\begin{pmatrix} R & W \\ V & S \end{pmatrix}.$$

In the case $W = {}^nR$, $V = W^*$ and $S = \mathrm{End}(W)$ of (4.1.4), this ring is just $M_{n+1}(R)$.

(b) Let $\mathcal{C}$ be a preadditive (left) category with two nonzero objects x, y, and let $R = \mathrm{Mor}(x, x)$, $S = \mathrm{Mor}(y, y)$, $W = \mathrm{Mor}(x, y)$ and $V = \mathrm{Mor}(y, x)$. Show that we have a generalized Morita context.

Show conversely that a generalized Morita context gives such a preadditive category.

Remark. When the homomorphisms σ, τ are both isomorphisms, we have a *Morita context*, and then R and S are Morita equivalent, which phenomenon we discuss in the next section.

4.1.4 **Morita context for firm nonunital rings**

Extending the previous exercise, a Morita context may be defined for nonunital rings and their modules. We continue the data of Exercise 3.2.5, that is, we consider firm nonunital rings R and S with firm R-modules V_R and ${}_RW$, an R-R-bimodule isomorphism $\sigma : W \otimes_S V \to R$ and an S-S-isomorphism $\tau : V \otimes_R W \to S$. Show that we have a Morita context defining a firm nonunital ring

$$\begin{pmatrix} R & W \\ V & S \end{pmatrix}$$

which is isomorphic to

$$\begin{pmatrix} R \\ V \end{pmatrix} \otimes_R \begin{pmatrix} R & W \end{pmatrix}.$$

We may therefore describe W as a generalized left R-progenerator.

4.1.5 Let $\mathcal{K}$ be a field. Show that the ideal (X, Y) in the polynomial ring $\mathcal{K}[X, Y]$ is neither projective nor a generator.

Extend this result to the ideal $(X_1, \ldots, X_k)$ of the polynomial ring $\mathcal{K}[X_1, \ldots, X_n]$ for $n \geq k \geq 2$.

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(These results are considered from a more elementary viewpoint in [BK: IRM] Exercises 5.1.6 and 6.1.2.)

4.1.6 Let $\mathcal{O}$ be a Dedekind domain with field of fractions $\mathcal{K}$ and let $\mathcal{O}'$ be a proper subring of $\mathcal{O}$ which also has field of fractions $\mathcal{K}$. Show that $\mathcal{O}$ is neither projective nor a generator as an $\mathcal{O}'$-module.

(This is a variation on [BK: IRM] Exercise 5.1.10.)

4.1.7 Let $\mathcal{O}$ be a commutative domain and let $\mathfrak{a}$ be an *integral* ideal of $\mathcal{O}$, that is, $\mathfrak{a} \subseteq \mathcal{O}$. This long example investigates the nature of some modules over the tiled order $R = \begin{pmatrix} \mathcal{O} & \mathfrak{a} \\ \mathcal{O} & \mathcal{O} \end{pmatrix}$, comprising those matrices in $M_2(\mathcal{O})$ with 1,2 entry belonging to $\mathfrak{a}$. (See Exercise 2.4.5 for the general definition of an order.) Orders of this shape are worthy of a detailed analysis because it is possible to make explicit computations which exhibit some fundamental differences between the properties of commutative and noncommutative rings. This class of order also includes prototypes of hereditary and maximal orders (7.3.30), which are themselves of great interest.

We obtain explicit criteria for a certain type of right ideal to be projective or a generator, and we show how to find a right projective generator I with $\operatorname{End}(I_R) \neq R$. The results refine and extend those in [BK: IRM] Exercise 5.1.12, and will be revisited in Exercises 4.2.12, 7.2.5 and 7.3.2.

Throughout, $\mathcal{K}$ is the field of fractions of $\mathcal{O}$ and $\{e_{11}, e_{12}, e_{21}, e_{22}\}$ is the standard set of matrix units for $M_2(\mathcal{O})$, and unadorned $\otimes$, Hom, End and dual will be understood to be computed with respect to $\mathcal{O}$. Successively stronger conditions are imposed on the ring $\mathcal{O}$ and ideal $\mathfrak{a}$ as we progress.

(i) Let L be a right R-module. Deduce from the inclusion of rings $\mathcal{O}e_{11} \oplus \mathcal{O}e_{22} \subseteq R$ that $L = Le_{11} \oplus Le_{22}$ as an $\mathcal{O}e_{11} \oplus \mathcal{O}e_{22}$-module, where Le_{11} and Le_{22} are both $\mathcal{O}$-modules and $\mathcal{K}Le_{11} \cong \mathcal{K}Le_{22}$ as a $\mathcal{K}$-space.
Conversely, given $\mathcal{O}$-modules M, N with $\mathcal{K}M = \mathcal{K}N$, show that the 'row vector' $L = (M \ N) \subseteq (\mathcal{K}M \ \mathcal{K}N)$ is a right R-module under the obvious matrix multiplication if and only if $M\mathfrak{a} \subseteq N \subseteq M$, in which case $Le_{11} = M$ and $Le_{22} = N$.

(ii) Suppose from now on that $L = (M \ N)$ is in fact a right R-module, and that $\mathcal{K}M = \mathcal{K}^r$, the column space of dimension r. Show that

$$\operatorname{End}(L_R) = \operatorname{End}(M) \cap \operatorname{End}(N) \subseteq M_r(\mathcal{K}) = \operatorname{End}_{\mathcal{K}}(\mathcal{K}^r).$$

(iii) Using the identification of $\mathrm{Hom}_{\mathcal{K}}(\mathcal{K}^r, \mathcal{K})$ with ${}^r\mathcal{K}$, verify that $(L_R)^* = M^* \cap N^* \subseteq {}^r\mathcal{K}$.

(iv) Put $S = \mathrm{End}(L_R)$. Deduce that

$$L^* \otimes_S L \cong (M^* \cap N^*)M + (M^* \cap N^*)N \subseteq R$$

and

$$L \otimes_R L^* \cong M(M^* \cap N^*) + N(M^* \cap N^*) \subseteq S \subseteq M_r(\mathcal{K}).$$

Show that L is a projective right R-module if and only if

$$(M^* \cap N^*)M + (M^* \cap N^*)N = R,$$

and that L is a right R-generator if and only if

$$M(M^* \cap N^*) + N(M^* \cap N^*) = \mathrm{End}(M) \cap \mathrm{End}(N).$$

(v) Assume now that $L = (\mathfrak{b}\ \mathfrak{c})$ where $\mathfrak{b}$ and $\mathfrak{c}$ are fractional ideals of $\mathcal{O}$ (so that $r = 1$). Interpret the above results to obtain:

(a) L is an R-module if and only if $\mathfrak{c} \subseteq \mathfrak{b}$ and $\mathfrak{a}\mathfrak{b} \subseteq \mathfrak{c}$.

(b) Suppose that L is a right R-module. Then

$$\mathrm{End}(L_R) = \mathrm{End}(\mathfrak{b}) \cap \mathrm{End}(\mathfrak{c})$$

and

$$(L_R)^* = \begin{pmatrix} \mathfrak{b}^* \cap \mathrm{Hom}(\mathfrak{c}, \mathfrak{a}) \\ \mathfrak{b}^* \end{pmatrix}.$$

(vi) From now on, assume that the ideal $\mathfrak{a}$ is invertible.
Show that if $x \in \mathrm{Hom}_{\mathcal{O}}(\mathfrak{c}, \mathfrak{a})$, then $x\mathfrak{a}\mathfrak{b} \subseteq x\mathfrak{c} \subseteq \mathfrak{a}$ and so $x\mathfrak{b} \subseteq \mathrm{End}(\mathfrak{a}) = \mathcal{O}$, that is, $x \in \mathfrak{b}^*$. Obtain the simplified formula

$$(L_R)^* = \begin{pmatrix} \mathrm{Hom}(\mathfrak{c}, \mathfrak{a}) \\ \mathfrak{b}^* \end{pmatrix}.$$

(vii) Derive the formulas

$$L_R \otimes_R (L_R)^* \cong \mathfrak{b} \cdot \mathrm{Hom}(\mathfrak{c}, \mathfrak{a}) + \mathfrak{c}\mathfrak{b}^* \subseteq \mathcal{K}$$

and

$$(L_R)^* \otimes_S L \cong \begin{pmatrix} \mathfrak{b} \cdot \mathrm{Hom}(\mathfrak{c}, \mathfrak{a}) & \mathfrak{c} \cdot \mathrm{Hom}(\mathfrak{c}, \mathfrak{a}) \\ \mathfrak{b}^*\mathfrak{b} & \mathfrak{b}^*\mathfrak{c} \end{pmatrix} \subseteq L_2(\mathcal{K}),$$

where $S = \mathrm{End}(\mathfrak{b}) \cap \mathrm{End}(\mathfrak{c})$.

(viii) Show that $\mathrm{End}({}_SL) = \begin{pmatrix} \mathrm{End}(\mathfrak{b}) & \mathrm{Hom}(\mathfrak{b},\mathfrak{c}) \\ \mathrm{Hom}(\mathfrak{c},\mathfrak{b}) & \mathrm{End}(\mathfrak{c}) \end{pmatrix} \subseteq M_2(\mathcal{K})$, and deduce that L cannot be a right R-generator (save in the trivial case that $\mathfrak{a} = \mathcal{O}$ and $\mathfrak{b} = \mathfrak{c}$ is invertible).

(ix) Prove that L is right R-projective if and only if

$$\mathfrak{b} \cdot \mathrm{Hom}(\mathfrak{c},\mathfrak{a}) + \mathfrak{c}\mathfrak{b}^* = \mathrm{End}(\mathfrak{b}) \cap \mathrm{End}(\mathfrak{c}).$$

Using the relations $\mathfrak{c} \subseteq \mathfrak{b}$ and $\mathrm{Hom}(\mathfrak{c},\mathfrak{a}) \subseteq \mathfrak{b}^*$, deduce that L is R-projective if and only if $\mathfrak{b}$ is invertible and

$$\mathfrak{b} \cdot \mathrm{Hom}(\mathfrak{c},\mathfrak{a}) + \mathfrak{c}\mathfrak{b}^* = \mathcal{O}.$$

Prove further that L is R-projective if and only if $L = \mathfrak{b}(\mathcal{O}\ \mathfrak{d})$ where $\mathfrak{b}$ is invertible, $\mathfrak{a} \subseteq \mathfrak{d} \subseteq \mathcal{O}$ and $\mathrm{Hom}(\mathfrak{d},\mathfrak{a}) + \mathfrak{d} = \mathcal{O}$.
(Of course, $(\mathcal{O}\ \mathfrak{a})$ is projective regardless of whether or not $\mathfrak{a}$ is an invertible ideal.)

(x) For simplicity, assume from now on that $\mathcal{O}$ is a Dedekind domain. Let $\mathfrak{a} = \mathfrak{p}_1^{a(1)} \cdots \mathfrak{p}_k^{a(k)}$ and $\mathfrak{d} = \mathfrak{p}_1^{d(1)} \cdots \mathfrak{p}_k^{d(k)}$, where $a(i) \geq d(i) \geq 0$, be the prime factorizations of $\mathfrak{a}$ and $\mathfrak{d}$. Show that $(\mathcal{O}\ \mathfrak{d})$ is R-projective if and only if for each i, we have either $a(i) = d(i)$ or $d(i) = 0$.
Remark. If $\mathfrak{a}$ is squarefree, that is, all $a(i) = 1$, then all modules $(\mathcal{O}\ \mathfrak{d})$ must be projective. This is a key step in showing that R is then a hereditary order, that is, every submodule of a projective (right) R-module is again projective (7.3.30). We consider this topic again in Exercises 7.2.5 and 7.3.2.

(xi) Let I be a *fractional right ideal* of R in $M_2(\mathcal{K})$, by which we mean a finitely generated right R-submodule of $M_2(\mathcal{K})$ with $\mathcal{K}I = M_2(\mathcal{K})$. (This extends the definition given in [BK: IRM] Exercise 5.1.11, and anticipates (4.2.22).) By part (i), we can write I in the form $I = \begin{pmatrix} \mathfrak{b} & \mathfrak{b}\mathfrak{d} \\ \mathfrak{f} & \mathfrak{f}\mathfrak{h} \end{pmatrix}$ where $\mathfrak{b}$, $\mathfrak{f}$ are fractional ideals of $\mathcal{O}$ and $\mathfrak{d}$, $\mathfrak{h}$ integral ideals with $\mathfrak{a} \subseteq \mathfrak{d} \subseteq \mathcal{O}$ and $\mathfrak{a} \subseteq \mathfrak{h} \subseteq \mathcal{O}$.
Verify that I is a left R-module if and only if the relations $\mathfrak{a}\mathfrak{f} \subseteq \mathfrak{b} \subseteq \mathfrak{f}$ and $\mathfrak{a}\mathfrak{f}\mathfrak{h} \subseteq \mathfrak{b}\mathfrak{d} \subseteq \mathfrak{f}\mathfrak{h}$ hold.

(xii) Write $T = \mathrm{End}(I_R)$ and $U = \mathrm{End}({}_TI)$, and let $L = (\mathfrak{b}\ \mathfrak{b}\mathfrak{d})$ and $P = (\mathfrak{f}\ \mathfrak{f}\mathfrak{h})$ denote the 'rows' of I; both L and P are right R-modules. Using (ii), verify that

$$\mathrm{End}({}_TI) = \mathrm{End}({}_{\mathcal{O}}L) \cap \mathrm{End}({}_{\mathcal{O}}P) = \begin{pmatrix} \mathcal{O} & \mathfrak{d} \cap \mathfrak{h} \\ \mathfrak{d}^{-1} \cap \mathfrak{h}^{-1} & \mathcal{O} \end{pmatrix}.$$

(Notice that we do not need to know T explicitly!)

(xiii) Before proceeding, we remind ourselves of a definition. Two integral ideals $\mathfrak{d}$ and $\mathfrak{h}$ of the Dedekind domain $\mathcal{O}$ are said to be *coprime* if $\mathfrak{d}+\mathfrak{h}=\mathcal{O}$, which is the same as asserting that they have no common prime ideal as a factor.
Now deduce that $\mathrm{End}({}_TI)=R$ if and only if $\mathfrak{d}$ and $\mathfrak{h}$ are coprime ideals with $\mathfrak{a}=\mathfrak{d}\mathfrak{h}$. If this holds, show also that L and P are both right R-projective and that I is a right R-progenerator.

(xiv) Let C and D denote the 'column' components of I. Obtain the formula

$$\begin{aligned}\mathrm{End}(I_U)&=\mathrm{End}(C_{\mathcal{O}})\cap\mathrm{End}(D_{\mathcal{O}})\\&=\begin{pmatrix}\mathcal{O} & \mathfrak{b}\mathfrak{f}^{-1}\cap\mathfrak{b}\mathfrak{f}^{-1}\mathfrak{d}\mathfrak{h}^{-1}\\ \mathfrak{b}^{-1}\mathfrak{f}\cap\mathfrak{b}^{-1}\mathfrak{f}\mathfrak{d}^{-1}\mathfrak{h} & \mathcal{O}\end{pmatrix}.\end{aligned}$$

Put $\mathfrak{m}=\mathfrak{b}\mathfrak{f}^{-1}$ and $\mathfrak{n}=\mathfrak{b}\mathfrak{f}^{-1}\mathfrak{d}\mathfrak{h}^{-1}$, which are integral ideals because of the relations $\mathfrak{b}\subseteq\mathfrak{f}$ and $\mathfrak{b}\mathfrak{d}\subseteq\mathfrak{f}\mathfrak{h}$.
Show that if $\mathrm{End}(I_U)=R$, then $\mathfrak{m}$ and $\mathfrak{n}$ are coprime ideals with $\mathfrak{m}\mathfrak{n}=\mathfrak{a}$, and that

$$I=\begin{pmatrix}\mathfrak{b} & \mathfrak{b}\mathfrak{d}\\ \mathfrak{f} & \mathfrak{f}\mathfrak{h}\end{pmatrix}=\begin{pmatrix}\mathfrak{f}\mathfrak{m} & \mathfrak{f}\mathfrak{m}\mathfrak{d}\\ \mathfrak{f} & \mathfrak{f}\mathfrak{h}\end{pmatrix}=\begin{pmatrix}\mathfrak{f}\mathfrak{m} & \mathfrak{n}\mathfrak{f}\mathfrak{h}\\ \mathfrak{f} & \mathfrak{f}\mathfrak{h}\end{pmatrix},$$

where $\mathfrak{m}\mathfrak{d}=\mathfrak{n}\mathfrak{h}$.

(xv) Suppose that $\mathrm{End}(I_R)=R=\mathrm{End}({}_RI)$. Show that $\mathfrak{m}=\mathfrak{h}$, $\mathfrak{d}=\mathfrak{n}$ and $\mathfrak{n}\mathfrak{h}=\mathfrak{a}$, and deduce that $I=\mathfrak{f}\begin{pmatrix}\mathfrak{h} & \mathfrak{a}\\ \mathcal{O} & \mathfrak{h}\end{pmatrix}$, where $\mathfrak{f}$ is a fractional ideal of $\mathcal{O}$ and $\mathfrak{h}$ is a factor of $\mathfrak{a}$ such that $\mathfrak{h}$ and $\mathfrak{a}\mathfrak{h}^{-1}$ are coprime.
Prove the converse: if I has this form, then

$$\mathrm{End}(I_R)=R=\mathrm{End}({}_RI),$$

and I is an R-R-progenerator.

(xvi) We can now manufacture ideals which are right R-progenerators but have $\mathrm{End}(I_R)\neq R$. Suppose that $\mathfrak{a}=\mathfrak{p}^2$ where $\mathfrak{p}$ is prime. Take $\mathfrak{b}=\mathfrak{p}$, $\mathfrak{d}=\mathfrak{p}^2$ and $\mathfrak{f}=\mathfrak{h}=\mathcal{O}$, so that $\mathfrak{m}=\mathfrak{p}$, $\mathfrak{n}=\mathfrak{p}^3$ and $I=\begin{pmatrix}\mathfrak{p} & \mathfrak{p}^3\\ \mathcal{O} & \mathcal{O}\end{pmatrix}$. Then $\mathrm{End}(I_R)=\begin{pmatrix}\mathcal{O} & \mathfrak{p}^3\\ \mathfrak{p}^{-1} & \mathcal{O}\end{pmatrix}$.

4.1.8 The term 'faithfully projective' is used sometimes in place of 'progenerator' since a generator can be defined alternatively as follows.

A left R-module G is a generator if and only if $\mathrm{Hom}_R(G,-)$ *is a faithful functor.*

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Show that our definition (4.1.10) does imply that $\mathrm{Hom}(G, -)$ is a faithful functor. Here is an outline of the proof of the converse.

(i) For any maximal left ideal $\mathfrak{m}$ of R, there is a homomorphism $\alpha : G \to R$ with $G\alpha \not\subseteq \mathfrak{m}$. (Note that there are two R-module homomorphisms from R to $R/\mathfrak{m}$, the canonical and the zero homomorphisms.)

(ii) Deduce that the left ideal $\sum\{G\alpha \mid \alpha \in \mathrm{Hom}(G, R)\}$ of R must be R. (Note that any proper ideal of a ring is contained in a maximal ideal [BK: IRM](1.2.22).)

(iii) $1_R = g_1\alpha_1 + \cdots + g_n\alpha_n$ for some elements g_i of G and homomorphisms $\alpha_i : G \to R$.

(iv) There is a surjection $G^n \to R$.

4.2 MORITA EQUIVALENCE

Given rings R and S, an R-S-bimodule W defines a functor $- \otimes_R W : \mathcal{M}od_R \to \mathcal{M}od_S$. If W is an R-progenerator and $S \cong \mathrm{End}(_RW)$, then this functor is an equivalence of categories, and R and S are said to be Morita equivalent through W.

The prime examples of Morita equivalences are those between a ring R and the matrix rings $M_n(R)$, $n \geq 1$, which arise through the free left R-modules ${}^nR$. However, it is possible for R to be equivalent to a ring which is not a full matrix ring over R; for this to happen, R must have a nonfree projective generator.

When R and S are Morita equivalent, the categories $\mathcal{M}_R$ and $\mathcal{M}_S$ are also equivalent, as are $\mathcal{P}_R$ and $\mathcal{P}_S$, and the rings R and S share many ring-theoretic properties.

It can be shown that any equivalence between $\mathcal{M}od_R$ and $\mathcal{M}od_S$ must arise through an R-progenerator, so that Morita equivalence can be defined alternatively in terms of the equivalence of categories, but we do not use this approach in this text.

There may be several essentially distinct ways in which a pair of rings are Morita equivalent, since different modules may induce equivalences from $\mathcal{M}od_R$ to $\mathcal{M}od_S$ which are not isomorphic as functors. The amount of variation can be described by the Picard group $\mathrm{Pic}(R)$ of R, which measures the distinct self-equivalences of $\mathcal{M}od_R$. This group can be regarded as a generalization of the class group of a Dedekind domain to rings in general.

In our discussion, we view a left R-module as an operator on the category

$\mathcal{M}_{ODR}$ of right modules. However, it turns out that the theory is left-right symmetric.

4.2.1 The definition and first results

Let R and S be rings and let W be a left R-progenerator. If $S \cong \mathrm{End}({}_RW)$, then R and S are, by definition, *Morita equivalent* through W. Of course, there may be many left R-progenerators that give rise to Morita equivalences between R and S.

By (4.1.14), the dual $W_R^* = \mathrm{Hom}({}_RW, {}_RR)$ is a right R-progenerator with $S \cong \mathrm{End}(W_R^*)$, so the definition is left-right symmetric in that we could equally work with right R-progenerators acting on left modules.

Again appealing to (4.1.14), we see that ${}_SW^*$ is a left S-progenerator with $R \cong \mathrm{End}({}_SW^*)$. Therefore Morita equivalence is a symmetric relation beween rings. It is obviously reflexive and the next lemma shows that it is transitive, so that Morita equivalence is genuinely an equivalence relation on the category $\mathcal{R}_{ING}$ of rings.

4.2.2 Lemma

Suppose that the ring R is Morita equivalent to the ring S through the left R-progenerator W and that S is Morita equivalent to the ring T through the left S-progenerator X.

Then R is Morita equivalent to T through the left R-progenerator $W \otimes_S X$, and there is an isomorphism $(W \otimes_S X)^* \cong X^* \otimes_S W^*$ of T-R-bimodules.

Proof

We have isomorphisms

$$\sigma_W : W \otimes_S W^* \cong R, \qquad \tau_W : W^* \otimes_R W \cong S,$$

$$\sigma_X : X \otimes_T X^* \cong S \quad \text{and} \quad \tau_X : X^* \otimes_S X \cong T.$$

Since the tensor product is associative, we can use σ_X and σ_W in turn to provide an isomorphism

$$(W \otimes_S X) \otimes_T (X^* \otimes_S W^*) \cong R.$$

Similarly, $(X^* \otimes_S W^*) \otimes_R (W \otimes_S X) \cong T$, so the result follows by (4.1.15). □

The most important consequence of the existence of a Morita equivalence between rings R and S is that it guarantees that any (reasonable) category of modules over R is equivalent, as a category, to its counterpart over S.

The basic result is the following.

4.2.3 Theorem

Suppose that the rings R and S are Morita equivalent through the left R-progenerator W. Then the following statements hold.

(i) *The functors*

$$-\otimes_R W : \mathcal{M}od_R \longrightarrow \mathcal{M}od_S \text{ and } -\otimes_S W^* : \mathcal{M}od_S \longrightarrow \mathcal{M}od_R$$

are mutually inverse equivalences between the categories of right modules.

(ii) *The restrictions of these functors are mutually inverse equivalences between $\mathcal{M}_R$ and $\mathcal{M}_S$, the categories of finitely generated modules, and between $\mathcal{P}_R$ and $\mathcal{P}_S$, the categories of finitely generated projective modules.*

(iii) *The functors*

$$W^* \otimes_R - : {}_R\mathcal{M}od \longrightarrow {}_S\mathcal{M}od \text{ and } W \otimes_S - : {}_S\mathcal{M}od \longrightarrow {}_R\mathcal{M}od$$

are also mutually inverse equivalences which induce equivalences between ${}_R\mathcal{M}$ and ${}_S\mathcal{M}$ and between ${}_R\mathcal{P}$ and ${}_S\mathcal{P}$.

Proof

The composite of the functors $-\otimes_R W : \mathcal{M}od_R \to \mathcal{M}od_S$ and $-\otimes_S W^* : \mathcal{M}od_S \to \mathcal{M}od_R$ is the functor $-\otimes_R W \otimes_S W^* : \mathcal{M}od_R \to \mathcal{M}od_R$. There is a natural transformation from this composite functor to the identity functor on $\mathcal{M}od_R$ which is exhibited by the commuting diagram

$$\begin{array}{ccccc} M \otimes_R W \otimes_S W^* & \xrightarrow{id \otimes \sigma} & M \otimes_R R & \xrightarrow{\cong} & M \\ \mu \otimes id \otimes id \downarrow & & \downarrow \mu \otimes id & & \downarrow \mu \\ N \otimes_R W \otimes_S W^* & \xrightarrow{id \otimes \sigma} & N \otimes_R R & \xrightarrow{\cong} & N \end{array}$$

for any R-homomorphism $\mu : M \to N$. Since σ is an R-R-bimodule isomorphism (4.1.13), this transformation is a natural isomorphism.

Similarly for the other composition.

The remaining assertions follow by (3.1.17) since W and W^* are finitely generated and projective on both sides, and by symmetry. □

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4.2.4 Further developments

1. The converse of the preceding result is true: if the categories $\mathcal{M}od_R$ and $\mathcal{M}od_S$ are equivalent, the equivalence must be given by $-\otimes_R W$ for some progenerator W. A weak version of this result is indicated in Exercise 4.2.13 below; for full details see [Rowen 1988] §4.1, for example.
2. There is an extension of the concept of progenerator to an arbitrary abelian category. It can then be shown that an abelian category is equivalent to $\mathcal{M}od_R$ for some ring R precisely when it has a progenerator (and contains arbitrary coproducts of copies of the progenerator); see [Pareigis 1970] §4.11 Theorem 1.

   Another far-reaching categorical generalization of Morita theory is given in §2 of [Kuhn 1994].
3. [Quillen] has generalized the preceding result to firm nonunital rings, as defined in Exercise 3.2.5. His results require an appropriate generalization of the category $\mathcal{M}od_R$ to the case that R may be nonunital. Let us here use the same notation $\mathcal{M}od_R$ for this category. Then there is a bijective correspondence between equivalences from $\mathcal{M}od_R$ to $\mathcal{M}od_S$ with R and S firm nonunital rings and firm generalized R-progenerators W (as in Exercise 4.1.4).

   His approach uses a notion of relative Morita equivalence which we outline in Exercise 6.3.6.

   The refinement of Theorem 4.2.3 presented there leads to a definition of $\mathcal{M}od_R$ for nonunital rings in Exercise 6.3.7.
4. Recently, the Morita theory has been applied to mathematical physics. For a discussion, which involves setting up the corresponding machinery for categories of modules over C^*-algebras, see [Schwarz 1998].

4.2.5 Properties preserved by Morita equivalence

Our aim now is to show that many important properties of rings are preserved by Morita equivalence. We start by examining the effect of a Morita equivalence on the submodules of a module.

4.2.6 Theorem

Suppose that R and S are Morita equivalent through the left R-progenerator W. Let M be a right R-module and N a right S-module. Then there are order-preserving bijections

(i) *$M' \mapsto M' \otimes_R W$ between R-submodules of M and S-submodules of $M \otimes_R W$.*

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(ii) *$N' \mapsto N' \otimes_S W^*$ between S-submodules of N and R-submodules of $N \otimes_S W^*$.*

(iii) *$\mathfrak{a} \mapsto \tau(W^* \otimes_R \mathfrak{a}W)$ between the twosided ideals of R and those of S.* □

If we make the informal identification $W^*W = S$, then part (iii) can be stated more expressively as '$\mathfrak{a} \mapsto W^*\mathfrak{a}W$ gives a bijection between the twosided ideals of R and those of S'.

4.2.7 Corollary

(i) *M is Noetherian or Artinian if and only if $M \otimes_R W$ is Noetherian or Artinian respectively.*

(ii) *N is Noetherian or Artinian if and only if $N \otimes_S W^*$ is Noetherian or Artinian respectively.*

(iii)

$$\mathrm{rad}(M \otimes_R W) = \mathrm{Im}((\mathrm{rad}\, M) \otimes_R W \to M \otimes_R W)$$

and

$$\mathrm{rad}(N \otimes_S W^*) = \mathrm{Im}((\mathrm{rad}\, N) \otimes_S W^* \to N \otimes_S W^*),$$

where the homomorphisms are injections induced by the obvious inclusions.

(iv) $\mathrm{rad}(R) \mapsto \tau(W^* \otimes_R (\mathrm{rad}(R)W)) = \mathrm{rad}(S)$.

Proof

Parts (i) and (ii) are immediate from the characterizations of these properties in terms of the ascending and descending chain conditions ([BK: IRM], (3.1.6) and (4.1.1) respectively). Part (iii) follows directly from the definition of the radical $\mathrm{rad}(L)$ of a module L as the intersection of the maximal submodules of L ([BK: IRM] (4.3.1)), and for (iv), we use the fact that the Jacobson radical $\mathrm{rad}(R)$ of a ring R is a twosided ideal of R which is equally the intersection of the maximal right ideals of R and of the maximal left ideals of R – see [BK: IRM], (4.3.7) and (4.3.11) respectively. □

We come to our main result.

4.2.8 Theorem

Suppose that rings R and S are Morita equivalent. Then the following statements are true.

(i) *R is right Noetherian if and only if S is right Noetherian.*

(ii) *R is right Artinian if and only if S is right Artinian.*

(iii) *R is Artinian semisimple if and only if S is Artinian semisimple.*

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(iv) *R is simple (as a ring) if and only if S is simple.*

Furthermore, the left-handed versions of (i) *and* (ii) *also hold.*

Proof

(i) – (iii) are most conveniently deduced from the categorical characterizations of these properties given in (2.3.9): for example, R is right Noetherian if and only if the category $\mathcal{N}_{OETH}{}_R$ of Noetherian modules is the same as the category $\mathcal{M}_R$ of finitely generated modules. Alternatively, one may argue directly from the definitions.

For (iv), recall that a simple ring is one without any proper twosided ideals except 0. □

Next, we show that the centre $Z(R)$ of a ring is preserved under Morita equivalence.

4.2.9 Proposition

Suppose that rings R and S are Morita equivalent. Then there is a ring isomorphism $Z(R) \cong Z(S)$.

Proof

Let W be a left R-progenerator giving the equivalence. We use the ring isomorphisms given in (4.1.15) to identify S with $\mathrm{End}({}_RW)$ and R with $\mathrm{End}(W_S)$. If $z \in Z(R)$, there is an R-endomorphism αz of W defined by $w(\alpha z) = zw$ for w in W, and $\alpha : Z(R) \to Z(S)$ is clearly a ring homomorphism. Similarly, an element $z' \in Z(S)$ gives an S-endomorphism $\beta z'$ of W by $(\beta z')w = wz'$, and α and β are mutually inverse. □

As an application, we recall that, for a commutative domain $\mathcal{O}$, a ring R is an $\mathcal{O}$-order if $\mathcal{O}$ embeds in $Z(R)$ and if, as both a left and right $\mathcal{O}$-module, R is finitely generated over $\mathcal{O}$ and torsion-free.

4.2.10 Corollary

Let $\mathcal{O}$ be a commutative domain and suppose that R is an $\mathcal{O}$-order. If a ring S is Morita equivalent to R, then S is also an $\mathcal{O}$-order.

Proof

Since $\mathcal{O}$ is contained in the centre of R, it can also be embedded in the centre of S. Since R is finitely generated as an $\mathcal{O}$-module, and ${}_RW$ and W^*_R are finitely generated as R-modules, they are finitely generated as $\mathcal{O}$-modules, and hence $S \cong W^* \otimes_R W$ is also finitely generated as an $\mathcal{O}$-module. Essentially the same argument shows that S is torsion-free over $\mathcal{O}$, for R is torsion-free and hence the projective R-modules W and W^* are also torsion-free. □

Recall from Exercise 2.4.5 that, for an $\mathcal{O}$-order R, $\mathcal{T}_{\mathcal{ORO},R}$ is the full subcategory of $\mathcal{M}_R$ given by the finitely generated R-modules which are $\mathcal{O}$-torsion, and $\mathcal{TF}_{\mathcal{O},R}$ is that given by the finitely generated R-modules which are $\mathcal{O}$-torsion-free. We record:

4.2.11 Proposition

Suppose that R and S are Morita equivalent $\mathcal{O}$-orders. Then there are induced equivalences of categories

$$\mathcal{T}_{\mathcal{ORO},R} \simeq \mathcal{T}_{\mathcal{ORO},S}$$

and

$$\mathcal{TF}_{\mathcal{O},R} \simeq \mathcal{TF}_{\mathcal{O},S}.$$

□

4.2.12 An illustration: matrix rings

For any ring R and integer n, the full matrix ring $M_n(R)$ is Morita equivalent to R, since we have $M_n(R) \cong \mathrm{End}({}_R({}^nR))$ by (4.1.4). Thus we have shown that the matrix ring inherits most of the significant properties of R.

If R has a direct product decomposition $R = R_1 \times \cdots \times R_k$ as a ring, then there is a corresponding decomposition of the matrix ring. To see this, we review some facts about idempotents that are discussed in detail in [BK: IRM], (2.6.2).

An idempotent f of a ring R is *central* if f is in the centre $Z(R)$ of R.

Idempotents $f_1, \dots, f_k$ are *orthogonal* if

$$f_i f_j = f_j f_i = 0 \text{ whenever } i \neq j.$$

A set $\{f_1, \dots, f_k\}$ of orthogonal, central idempotents is called a *full set of orthogonal central idempotents* for R if further

$$f_1 + \cdots + f_k = 1_R.$$

There is a bijective correspondence between direct product decompositions of R (as a ring) and full sets of orthogonal idempotents in R, in which the idempotents f_i are the identity elements of the components $R_i = Rf_i$ of R ([BK: IRM], Proposition 2.6.3).

Since a full set of orthogonal central idempotents of R is evidently a full set of orthogonal central idempotents of $M_n(R)$, the following result is straightforward.

4.2.13 Proposition

Suppose that $R = R_1 \times \cdots \times R_k$, a direct product of rings. Then

(i) $M_n(R) = M_n(R_1) \times \cdots \times M_n(R_k)$;

(ii) *R is Morita equivalent to $M_{n_1}(R_1) \times \cdots \times M_{n_k}(R_k)$ for any set of integers $n_1, \ldots, n_k > 0$.* □

Taking $R_1, \ldots, R_k$ all to be division rings, we obtain an alternative proof of the fact that a direct sum of matrix rings over division rings is Artinian semisimple [BK: IRM] (4.2.6). The theory of Artinian semisimple rings can actually be derived from the Morita theory, as in [Bass 1968] Chapter III.

We also note a characterization of Morita equivalence in terms of matrix rings, which is suggested by (4.1.9).

4.2.14 Proposition

The following statements are equivalent.

(i) *The rings R and S are Morita equivalent.*

(ii) *There is a ring isomorphism $S \cong eM_n(R)e$, where n is an integer and e is an idempotent element of $M_n(R)$ such that*

$$M_n(R) = M_n(R)eM_n(R).$$

Proof

In view of (4.1.9) it suffices to consider a bimodule W of the form ${}^nRe$. Then W^* is eR^n and the image of τ is $eR^n \cdot {}^nRe$ in S. However, $R^n \cdot {}^nR = M_n(R)$ (see (3.1.15)), so that the image of τ is isomorphic to $eM_n(R)e$. Now τ is an isomorphism precisely when the image contains 1, that is, $S \cong eM_n(R)e$. On the other hand, σ is an isomorphism so long as ${}^nReR^n$ contains 1; this is just the requirement that the twosided ideal of $M_n(R)$ generated by e should contain the identity matrix and thus equal $M_n(R)$. □

4.2.15 An illustration: orders over Dedekind domains

In the discussion so far, our explicit examples of Morita equivalences between rings all derive from the equivalence of a ring R with its full matrix rings $M_n(R)$, $n \geq 1$. Some examples of Morita equivalence which are not of this type can be found by taking the base ring R to be a Dedekind domain which has non-principal ideals.

Let $\mathcal{O}$ be a Dedekind domain with field of fractions $\mathcal{K}$. Our definition of a Dedekind domain (2.3.20) requires that any fractional ideal $\mathfrak{a}$ of $\mathcal{O}$ is invertible. By (4.1.17), the inverse $\mathfrak{a}^{-1}$ of $\mathfrak{a}$ can be identified as its dual $\mathfrak{a}^*$:

$$\mathfrak{a}^{-1} = \{x \in \mathcal{K} \mid \mathfrak{a}x \subseteq \mathcal{O}\},$$

and $\mathfrak{a}$ is a finitely generated (left and right) projective $\mathcal{O}$-module, in fact a progenerator.

Let P be a finitely generated projective (left) $\mathcal{O}$-module. Then (2.3.20 - C)

$$P = \mathfrak{a}_1 \oplus \cdots \oplus \mathfrak{a}_r$$

for some fractional ideals $\mathfrak{a}_1, \ldots, \mathfrak{a}_r$. Further, by Steinitz' theorem (2.3.20 - D), P is isomorphic to a module in the standard form ${}^{r-1}\mathcal{O} \oplus \mathfrak{a}$, where $\mathfrak{a} = \mathfrak{a}_1 \cdots \mathfrak{a}_r$ and the rank r and ideal class $\{\mathfrak{a}\}$ in the class group $\mathrm{Cl}(\mathcal{O})$ are invariants of the isomorphism class of P. (See [BK: IRM] Chapter 5, for a fuller discussion.)

Given a fractional ideal $\mathfrak{a}$, we therefore have $\mathfrak{a} \oplus \mathfrak{a} \cong \mathcal{O} \oplus \mathfrak{a}^2$. Thus the fact that $\mathcal{O}$ is a generator shows again that $\mathfrak{a}$ is also a generator, and hence that any finitely generated projective module is a generator. Now $\mathcal{K}P = {}^r\mathcal{K}$, the row-space, and the dual $({}^r\mathcal{K})^*$ can be identified as the column-space $\mathcal{K}^r$; then

$$P^* = \{y \in \mathcal{K}^r \mid Py \subseteq \mathcal{O}\} = \mathcal{O}^{r-1} \oplus \mathfrak{a}^{-1}.$$

Put $S = \mathrm{End}({}_{\mathcal{O}}P)$. By (4.2.10), S is an $\mathcal{O}$-order in $M_r(\mathcal{K})$, and, computing naively in $\mathcal{K}^r \cdot {}^r\mathcal{K} = M_r(\mathcal{K})$, we find that S can be represented as a tiled order

$$S = P^* \otimes_{\mathcal{O}} P = \begin{pmatrix} \mathcal{O} & \cdots & \mathcal{O} & \mathfrak{a} \\ \vdots & \ddots & \vdots & \vdots \\ \mathcal{O} & \cdots & \mathcal{O} & \mathfrak{a} \\ \mathfrak{a}^{-1} & \cdots & \mathfrak{a}^{-1} & \mathcal{O} \end{pmatrix}.$$

(More details of the calculation are given in [BK: IRM] Exercise 6.1.3.) This description of S makes it plausible that S is not the matrix ring $M_r(\mathcal{O})$. However, it may happen that S is isomorphic to $M_r(\mathcal{O})$ as a ring. For example, take $r = 2$ and $\mathfrak{a} = a\mathcal{O}$ principal, and let $t = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$; then $s \mapsto t^{-1}st$ gives such an isomorphism. In Exercise 4.2.5 below, we indicate the argument which shows that, for $r = 2$, there can be no such isomorphism in general.

The full answer for arbitrary r can be found in [Curtis & Reiner 1987], §55.39: $\mathrm{End}({}^{r-1}\mathcal{O} \oplus \mathfrak{a}) \cong M_r(\mathcal{O})$ if and only if $\{\mathfrak{a}\}$ is an rth power in the class group $\mathrm{Cl}(\mathcal{O})$.

4.2.16 The Picard Group

We next investigate, for an arbitrary ring R, those R-R-bimodules that give rise to self-equivalences on the category $\mathcal{M}od_R$. Such bimodules define a group, the Picard group $\mathrm{Pic}(R)$, which provides a generalization to rings in general of the ideal class group of a Dedekind domain.

The Picard group also measures the different Morita equivalences between a pair of rings, since if U and V are both R-S-progenerators for some rings R and S, then $U \otimes_S V^*$ will be an R-R-progenerator and so give rise to an element of $\mathrm{Pic}(R)$. An interpretation of this fact in terms of groupoids is indicated in Exercise 4.2.10.

We start with a result which extends (3.3.19).

4.2.17 Lemma

Suppose that W and X are R-S-bimodules.

Then the functors $-\otimes_R W$ and $-\otimes_R X$ are naturally isomorphic if and only if there is an R-S-bimodule isomorphism from W to X.

Proof

Suppose that $-\otimes_R W$ and $-\otimes_R X$ are naturally isomorphic. By the definition (1.3.1), this means that for each right R-module M there is an isomorphism

$$\eta_M : M \otimes_R W \to M \otimes_R X$$

of right S-modules, and that for each homomorphism

$$\alpha : M' \to M,$$

the diagram

$$\begin{array}{ccc} M' \otimes_R W & \xrightarrow{\eta_{M'}} & M' \otimes_R X \\ {\scriptstyle \alpha\otimes id}\downarrow & & \downarrow{\scriptstyle \alpha\otimes id} \\ M \otimes_R W & \xrightarrow{\eta_M} & M \otimes_R X \end{array}$$

commutes.

In particular, there must be an isomorphism of right S-modules η_R from $R \otimes_R W$ to $R \otimes_R X$, which we can regard as an S-isomorphism from W to X.

For r in R, let $\lambda(r) : R \to R$ be the right R-homomorphism given by left multiplication. Then the equality

$$(\lambda(r) \otimes 1)\eta_R = \eta_R(\lambda(r) \otimes 1)$$

shows that η_R is also a homomorphism of left R-modules.

The converse is clear. □

4.2.18 Definition of Pic(R)

Suppose that W defines a Morita equivalence from R to itself. Then W must be an R-R-bimodule, and by (4.1.15) it is *invertible*, that is, there is an *inverse R-R-bimodule* W^* such that

$$W \otimes_R W^* \cong R \cong W^* \otimes_R W$$

as R-R-bimodules. Conversely, if W has such an inverse, then W is a left R- and right R-progenerator and defines a Morita self-equivalence on R. The preceding lemma shows that two such bimodules W, V give the same equivalence on $\mathcal{M}\textsc{od}_R$ if and only if they are isomorphic as R-R-bimodules.

We therefore define the *Picard group* Pic(R) of R to be the set of R-R-bimodule isomorphism classes of invertible R-R-bimodules. The class of W will be denoted $\{W\}$. The multiplication in Pic(R) is given by

$$\{W\} \cdot \{X\} = \{W \otimes_R X\},$$

the identity is $\{R\}$, and $\{W\}^{-1} = \{W^*\}$. Closure follows from (4.2.2) and associativity from (3.1.5).

There is no reason why Pic(R) need be abelian, but examples where it is not abelian are too complicated to include in this account; see statement 55.58 of [Curtis & Reiner 1987].

The next two results give the fundamental properties of the Picard group.

4.2.19 Proposition

Suppose that the rings R and S are Morita equivalent. Then there is an isomorphism of groups Pic(R) ≅ Pic(S).

Proof

Let $_RU$ be a progenerator giving the equivalence. Then the isomorphism is

$$\mu : \{W\} \longmapsto \{U^* \otimes_R W \otimes_R U\} \quad \text{for } \{W\} \in \text{Pic}(R).$$

It is easy to check that μ is well-defined. To see that μ is a (multiplicative)

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homomorphism, recall that the square

$$\begin{array}{ccc} \mathcal{M}\text{OD}_R & \xrightarrow{-\otimes_R U} & \mathcal{M}\text{OD}_S \\ {\scriptstyle -\otimes_R W}\downarrow & & \downarrow{\scriptstyle -\otimes_S U^* \otimes_R W \otimes_R U} \\ \mathcal{M}\text{OD}_R & \xrightarrow{-\otimes_R U} & \mathcal{M}\text{OD}_S \end{array}$$

is a commutative diagram of functors, at least to within an isomorphism afforded by σ. Now take V to be an R-R-bimodule, and evaluate the effect of these functors on $U^* \otimes_R V$ in both ways. We then see that

$$\mu(\{V\})\mu(\{W\}) = \mu(\{V\}\{W\}).$$

The fact that μ is an isomorphism follows from the observation that it has an inverse, the homomorphism induced by U^*. □

4.2.20 Proposition

Suppose that $R = R_1 \times \cdots \times R_k$, *a direct product of rings.*
Then $\text{Pic}(R) \cong \text{Pic}(R_1) \times \cdots \times \text{Pic}(R_k)$, *the direct product of groups.*

Proof

Let $e_1, \ldots, e_k$ be the full set of orthogonal central idempotents of R that gives the decomposition ([BK: IRM] (2.6.2)). If W is an R-R-bimodule, then

$$W = We_1 \oplus \cdots \oplus We_k$$

where each We_i is an R_i-R_i-bimodule, and if X is another R-R-bimodule, we have

$$W \otimes_R X \cong (We_1 \otimes_{R_1} Xe_1) \oplus \cdots \oplus (We_k \otimes_{R_k} Xe_k).$$

In the other direction, if $W_1, \ldots, W_k$ is a set of bimodules over the respective components of R, then $W = W_1 \oplus \cdots \oplus W_k$ is an R-R-bimodule with $We_i \cong W_i$ for each i.

It is now easy to check that the invertibility of W corresponds to that of each We_i, and that the map

$$\{W\} \mapsto (\{We_1\}, \ldots, \{We_k\})$$

gives an isomorphism. □

A first calculation.

4.2.21 Theorem

Let R *be an Artinian semisimple ring. Then* $\text{Pic}(R) = 1$.

Proof

By the Wedderburn-Artin Theorem ([BK: IRM] (4.2.3)),

$$R = M_{n_1}(\mathcal{D}_1) \times \cdots \times M_{n_k}(\mathcal{D}_k),$$

where each $\mathcal{D}_i$ is a division ring. The preceding results show that it is enough to prove that $\text{Pic}(\mathcal{D}) = 1$ for $\mathcal{D}$ a division ring. But this is obvious, since any $\mathcal{D}$-module is free, of unique dimension. □

4.2.22 Orders

The most direct analogy between the Picard group of a ring and the class group of a Dedekind domain occurs when R is an $\mathcal{O}$-order for some (commutative) integral domain $\mathcal{O}$. Let $\mathcal{K}$ be the field of fractions of $\mathcal{O}$. Since R is a finitely generated torsion-free $\mathcal{O}$-module, R spans a finite-dimensional $\mathcal{K}$-space $\mathcal{K}R$ which is necessarily an Artinian ring. (The existence of $\mathcal{K}R$ is an application of the localization techniques that we meet in Chapter 6 – see (6.2.1). A direct approach is given in [BK: IRM] (1.2.23).)

We extend the definition of a fractional ideal from commutative domains to orders as follows. A (twosided) *fractional ideal* $\mathfrak{a}$ of an $\mathcal{O}$-order R is an R-R-submodule of $\mathcal{K}R$ which is finitely generated (on both sides), and which spans $\mathcal{K}R$. If $\mathfrak{a}$ is instead only a left R-module, then $\mathfrak{a}$ is a *left fractional ideal*, and if it is a right R-module, it is a *right fractional ideal*. (These definitions were anticipated in Exercise 4.1.7.)

We say that a twosided fractional ideal $\mathfrak{a}$ is *invertible* if there is some fractional ideal $\mathfrak{a}'$ with $\mathfrak{a}\mathfrak{a}' = R = \mathfrak{a}'\mathfrak{a}$.

We assume also that $\mathcal{K}R$ is semisimple. This is frequently the case in situations arising from integral representation theory and algebraic number theory. For example, the integral group ring $\mathbb{Z}G$ of a finite group G spans the algebra $\mathbb{Q}G$, which is semisimple by Maschke's Theorem ([BK: IRM] Exercise 4.3.9).

Suppose that W is an invertible R-R-bimodule. As we remarked in the proof of (4.2.10), W is $\mathcal{O}$-torsion-free, and so spans a $\mathcal{K}R$-$\mathcal{K}R$-bimodule $\mathcal{K}W$, which is also invertible, with inverse $\mathcal{K}W^*$. Thus there is a homomorphism from $\text{Pic}(R)$ to $\text{Pic}(\mathcal{K}R)$. Since $\text{Pic}(\mathcal{K}R) = 1$ by the preceding theorem, we have $\mathcal{K}W \cong \mathcal{K}R$, as a bimodule.

Then the image of W under such an isomorphism must be a fractional ideal $\mathfrak{a}$. It is also obvious that $\mathfrak{a}$ must be invertible, with inverse isomorphic to W^*. Thus the Picard group can be described as the group of isomorphism classes of invertible fractional ideals of R, with product given by $\{\mathfrak{a}\} \cdot \{\mathfrak{b}\} = \{\mathfrak{a}\mathfrak{b}\}$.

If $\{\mathfrak{a}\} = 1$, then $R \cong \mathfrak{a}$ as a bimodule. Let $\theta : R \to \mathfrak{a}$ be such an isomorphism and put $\theta(1) = a_0$ in $\mathfrak{a}$. Then a_0 has the properties that $ra_0 = a_0r$ for all r in R, $\mathfrak{a} = Ra_0$, and $ra_0 = 0$ only if $r = 0$, allowing it to be termed a *regular central generator* of $\mathfrak{a}$.

Conversely, if the fractional ideal $\mathfrak{a}$ has a regular central generator a_0, then $\mathfrak{a}$ is isomorphic to R.

When $\mathcal{O}$ is a Dedekind domain, the ideal class group $\mathrm{Cl}(\mathcal{O})$ is, by definition, the quotient group

$$\mathrm{Cl}(\mathcal{O}) = \mathrm{Frac}(\mathcal{O}) / \mathrm{Pr}(\mathcal{O})$$

where $\mathrm{Frac}(\mathcal{O})$ is the group of all fractional ideals of $\mathcal{O}$ and $\mathrm{Pr}(\mathcal{O})$ is the subgroup comprising all principal ideals. (Note that $\mathrm{Frac}(\mathcal{O})$ is a group since every fractional ideal of $\mathcal{O}$ is invertible.) It is not hard to see directly ([BK: IRM] (5.1.14)) that the class $\{\mathfrak{a}\}$ of $\mathfrak{a}$ is trivial in the ideal class group precisely when $\mathfrak{a}$ is isomorphic to $\mathcal{O}$. Thus the fractional ideals with a regular central generator play the role taken by the principal ideals for Dedekind domains.

This discussion makes it plain that if $\mathcal{O}$ is a Dedekind domain, then $\mathrm{Pic}(\mathcal{O}) = \mathrm{Cl}(\mathcal{O})$, the ideal class group. If S is a tiled $\mathcal{O}$-order of the type constructed in (4.2.15), then $\mathrm{Pic}(S) \cong \mathrm{Cl}(\mathcal{O})$ by (4.2.19).

The calculation of Picard groups for orders is an extensive subject; we give one illustrative computation as Exercise 4.2.12. A survey can be found in [Curtis & Reiner 1987], §55.

Exercises

4.2.1 Let the ring $R = S_1 \times S_2$ be the direct product of two nontrivial rings. Show that R is not Morita equivalent to $M_n(S_1)$ for any integer $n \geq 1$.

4.2.2 Let R be any ring and W any left R-module, and put $S = \mathrm{End}(_RW)$ and $T = \begin{pmatrix} R & W \\ 0 & S \end{pmatrix}$, the triangular matrix ring. Show that

$$\mathrm{End}(_R(R \oplus W)) = \begin{pmatrix} R & W \\ W^* & S \end{pmatrix}.$$

Is this the same as $\mathrm{End}(_T(R \oplus W))$? (Consider special cases.)

4.2.3 Suppose that we have a Morita context

$$\begin{pmatrix} R & W \\ V & S \end{pmatrix}, \quad \sigma : W \otimes_S V \to R, \quad \tau : V \otimes_R W \to S,$$

as in Exercise 4.1.3.

(a) Observing that $\mathrm{Hom}(M \otimes_R W \otimes_S V_R, M)$ is naturally isomorphic to $\mathrm{Hom}(M \otimes_R W_S, \mathrm{Hom}(V_R, M))$ (3.1.18), deduce that there is a natural transformation from the functor $M \mapsto M \otimes_R W$ to the functor $M \mapsto \mathrm{Hom}(V_R, M)$ from $\mathcal{M}od_R$ to $\mathcal{M}od_S$, which is a natural isomorphism in the case of Morita equivalence (that is, when σ is an isomorphism).

Likewise, there is a natural transformation between $N \mapsto N \otimes_S V$ and $N \mapsto \mathrm{Hom}(W_S, N)$ from $\mathcal{M}od_S$ to $\mathcal{M}od_R$ which is a natural isomorphism in the case of Morita equivalence.

(b) Given a (right) R-module L and a twosided ideal $\mathfrak{a}$ of R, we sometimes say that L is $\mathfrak{a}$-*torsion* to indicate that $L\mathfrak{a} = 0$.

Let

$$ {}_M\sigma \in \mathrm{Hom}(M \otimes_R W \otimes_S V_R, M) $$

be the distinguished element determined by σ and let ${}_M\sigma'$ be its counterpart in $\mathrm{Hom}(M \otimes_R W_S, \mathrm{Hom}(V_R, M))$. Show that both $\mathrm{Ker}\, {}_M\sigma$ and $\mathrm{Cok}\, {}_M\sigma$ are $(\mathrm{Im}\, \sigma)$-torsion R-modules, while $\mathrm{Ker}\, {}_M\sigma'$ and $\mathrm{Cok}\, {}_M\sigma'$ are $(\mathrm{Im}\, \tau)$-torsion S-modules.

(There is a corresponding result with the roles of σ and τ interchanged.)

4.2.4 Let $\mathcal{K}$ be a field. Describe the rings which are Morita equivalent to the $n \times n$ triangular matrix ring $\begin{pmatrix} \mathcal{K} & \mathcal{K} & \cdots & \mathcal{K} \\ 0 & \mathcal{K} & \cdots & \mathcal{K} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathcal{K} \end{pmatrix}$.

4.2.5 Let $\mathcal{O}$ be a Dedekind domain and suppose that the class group $\mathrm{Cl}(\mathcal{O})$ contains an element $\{\mathfrak{a}\}$ which is not the square of any other element of $\mathrm{Cl}(\mathcal{O})$. For example, the class group of the ring of integers $\mathbb{Z}[\sqrt{-5}]$ of $\mathbb{Q}(\sqrt{-5})$ contains such an element since it has order 2 ([BK: IRM] (5.3.17)).

Let $P = \mathcal{O} \oplus \mathfrak{a}$. We now outline in stages the proof of the fact promised in (4.2.15): the ring $\mathrm{End}(P_{\mathcal{O}}) = \begin{pmatrix} \mathcal{O} & \mathfrak{a} \\ \mathfrak{a}^{-1} & \mathcal{O} \end{pmatrix}$ is *not* isomorphic to the matrix ring $M_2(\mathcal{O})$.

(i) Since $M_2(\mathcal{O})$ is Morita equivalent to $\mathcal{O}$ through $\mathcal{O}^2$, the $M_2(\mathcal{O})$-submodules of $\mathcal{O}^2$ are of the form $\mathfrak{b} \oplus \mathfrak{b}$, with $\mathfrak{b}$ a (fractional) ideal of $\mathcal{O}$.

(ii) If there is a ring isomorphism from $\mathrm{End}(P)$ to $M_2(\mathcal{O})$, then P is a projective $M_2(\mathcal{O})$-submodule of $\mathcal{O}^2$ by change of scalars, and so is isomorphic to $\mathfrak{b} \oplus \mathfrak{b}$ for some $\mathfrak{b}$.

(iii) This isomorphism must be given by left multiplication by an element x in $\mathcal{K}$, and so $\{\mathfrak{a}\} = \{\mathfrak{b}\}^2$, a contradiction.

4.2.6 Contrary to expectations that might be raised by the result in the previous exercise, it can happen that non-isomorphic $\mathcal{O}$-modules give rise to isomorphic orders. We now take $\mathcal{O}$ to be a Dedekind domain that has an ideal $\mathfrak{a}$ such that $\{\mathfrak{a}\}^2$ is nontrivial ([BK: IRM] (5.3.20) gives an example).

Continuing with the notation of the previous exercise, write $Q = \mathcal{O} \oplus \mathfrak{a}^{-1}$.

Show that $P \not\cong Q$ but that $\mathrm{End}(P) = \tau^{-1}\,\mathrm{End}(Q)\tau$ for the 2×2 matrix $\tau = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

4.2.7 Let $\mathcal{O}$ be a Dedekind domain and let $\mathfrak{a}$ be any integral proper ideal of $\mathcal{O}$. Put $R = \begin{pmatrix} \mathcal{O} & \mathfrak{a} \\ \mathcal{O} & \mathcal{O} \end{pmatrix}$. Show that the projective right R-module $(\mathcal{O}\ \mathfrak{a})$ cannot be R-isomorphic to $(\mathfrak{b}\ \mathfrak{b})$ for any ideal $\mathfrak{b}$ of $\mathcal{O}$, and deduce that R is not Morita equivalent to $\mathcal{O}$.

4.2.8 Let R and S be commutative rings and suppose that there is a ring homomorphism $f : R \to S$, so that S becomes an R-module by restriction of scalars. Show that extension of scalars $W \mapsto W \otimes_R S$ induces a homomorphism from $\mathrm{Pic}(R)$ to $\mathrm{Pic}(S)$, and deduce that Pic is a functor from the category of commutative rings to the category of abelian groups. (You will need (3.1.5) and (3.2.15).)

4.2.9 Let R be an $\mathcal{O}$-order spanning a semisimple $\mathcal{K}$-algebra, where $\mathcal{O}$ is a commutative domain with field of fractions $\mathcal{K}$.

Show that a twosided fractional ideal $\mathfrak{a}$ of R is invertible if and only if $\mathfrak{a}$ is both a left and a right R-progenerator.

Extend the definition of invertibility as follows. A right fractional ideal $\mathfrak{a}$ of R is *invertible* if there is a right fractional R-ideal $\mathfrak{a}'$ such that $\mathfrak{a}\mathfrak{a}' = R$ and $\mathfrak{a}'\mathfrak{a} = S$ for some $\mathcal{O}$-order S.

Show that $\mathfrak{a}$ is a right R-progenerator if and only if it is invertible, in which case S is Morita equivalent to R through $\mathfrak{a}$.

4.2.10 Let $\mathcal{O}$ be a commutative domain with field of fractions $\mathcal{K}$, and let Σ be a finite-dimensional semisimple $\mathcal{K}$-algebra.

The category $\mathcal{B}_{\mathcal{RANDT}}(\Sigma)$ has as objects the $\mathcal{O}$-orders which span Σ, a morphism from R to S being an S-R-bimodule isomorphism class $\{W\}$ of a left S-, right R-progenerator W.

Verify that $\mathcal{B}_{RANDT}(\Sigma)$ is actually a groupoid, the vertex group at R being $\mathrm{Pic}(R)$ – see Exercise 1.1.9.

(The name is in honour of H. Brandt, who first introduced this groupoid as a generalization of the ideal class group of a Dedekind domain to noncommutative arithmetic [Brandt 1926].)

4.2.11 Suppose that W is a right R-progenerator and let $S = \mathrm{End}(W_R)$, so that S is Morita equivalent to R. If there is a ring homomorphism $\theta : R \to S$, then, according to our definitions, W defines a self-equivalence on R and hence an element of $\mathrm{Pic}(R)$.

Confirm that W becomes an R-R-bimodule through restriction of scalars via θ, and hence that $\{W\}$ is trivial in $\mathrm{Pic}(R)$.

Deduce that, for R a noncommutative right principal ideal domain, $\mathrm{Pic}(R)$ is trivial.

4.2.12 We now give a calculation of a Picard group as promised in (4.2.22).

Let $\mathcal{O}$ be a Dedekind domain, let $\mathfrak{a}$ be an integral ideal of $\mathcal{O}$, and consider the tiled order $R = \begin{pmatrix} \mathcal{O} & \mathfrak{a} \\ \mathcal{O} & \mathcal{O} \end{pmatrix}$. By part (xv) of Exercise 4.1.7, an R-R-progenerator, that is, a twosided invertible fractional R-ideal, has the form $I = \mathfrak{f}\begin{pmatrix} \mathfrak{h} & \mathfrak{a} \\ \mathcal{O} & \mathfrak{h} \end{pmatrix}$, where $\mathfrak{f}$ is a fractional ideal of $\mathcal{O}$ and $\mathfrak{h}$ is a factor of $\mathfrak{a}$ such that $\mathfrak{h}$ and $\mathfrak{a}\mathfrak{h}^{-1}$ are coprime.

Verify that $\begin{pmatrix} \mathfrak{h} & \mathfrak{a} \\ \mathcal{O} & \mathfrak{h} \end{pmatrix}^2 = \mathfrak{h}R$.

Let $\mathfrak{a} = \mathfrak{q}_1 \cdots \mathfrak{q}_k$, where $\mathfrak{q}_1, \ldots, \mathfrak{q}_k$ are the distinct prime power factors of $\mathfrak{a}$, and set $M(i) = \begin{pmatrix} \mathfrak{q}_i & \mathfrak{a} \\ \mathcal{O} & \mathfrak{a}\mathfrak{q}_i^{-1} \end{pmatrix}$ for $i = 1, \ldots, k$. Denote the group of invertible twosided fractional ideals of R by $\mathrm{In}(R)$.

Show that $\mathrm{In}(R)$ is a free abelian group with generating set the ideals $M(1), \ldots, M(k)$ together with the ideals $\mathfrak{p}R$ where $\mathfrak{p}$ runs through the prime ideals of $\mathcal{O}$ that do not divide $\mathfrak{a}$.

By definition, the Dedekind domain $\mathcal{O}$ has $\mathrm{In}(R) = \mathrm{Frac}(\mathcal{O})$, the set of fractional ideals. Deduce that the natural homomorphism $\alpha : \mathrm{Frac}(\mathcal{O}) \to \mathrm{In}(R)$, defined by sending the generators $\mathfrak{p}$ of $\mathrm{Frac}(\mathcal{O})$ to $\alpha(\mathfrak{p}) = \mathfrak{p}R$, induces an injective homomorphism $\beta : \mathrm{Cl}(\mathcal{O}) \to \mathrm{Pic}(R)$, and that the cokernel $\mathrm{Cok}(\beta)$ is the direct product of k cyclic groups of order 2, with generating set the images of the classes $\{M(1)\}, \ldots, \{M(k)\}$.

4.2.13 Using (4.2.17) together with (4.1.15), prove the following result that was promised in (4.2.4), part 1.

Given rings R and S, suppose that U is a left R-module, V is a left S-module, and that $- \otimes_R U : \mathcal{M}od_R \to \mathcal{M}od_S$ and $- \otimes_S V : \mathcal{M}od_S \to \mathcal{M}od_R$ are mutually inverse equivalences of categories.

Show that U is an R-S-progenerator and $V \cong U^*$ as an S-R-bimodule.

4.2.14 **Morita equivalence not true to type**

Suppose that a ring R does not have Invariant Basis Number, that is, there are R-module isomorphisms $R^m \cong R^n$ for differing ranks m, n. Then R can be given a *type* (w, d) which measures the extent to which free modules can have bases with differing numbers of elements; (w, d) is a pair of non-negative integers such that no two of $R, R^2, \ldots, R^{w-1}$ are isomorphic, and, when $w \leq m \leq n$, $R^m \cong R^n$ if and only if $n = m + kd$ for some integer $k \geq 1$. Details are given in [BK: IRM] Exercise 2.3.2, or [Berrick & Keating 1997].

Show that $M_n(R)$ has type $(1, 1)$ if n is a sufficiently large multiple of d, and hence that the type is not preserved under Morita equivalence.

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