BULL. AUSTRAL. MATH. SOC. 06A35
VOL. 16 (1977), 1-13.

Coproducts of De Morgan algebras

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The dual of the category of De Morgan algebras is described in terms of compact totally ordered-disconnected ordered topological spaces which possess an involutorial homeomorphism that is also a dual order-isomorphism. This description is used to study the coproduct of an arbitrary collection of De Morgan algebras and also to represent the coproduct of two De Morgan algebras in terms of the continuous order-preserving functions from the Priestley space of one algebra to the other algebra, endowed with the discrete topology. In addition, it is proved that the coproduct of a family of Kleene algebras in the category of De Morgan algebras is the same as the coproduct in the subcategory of Kleene algebras if and only if at most one of the algebras is not boolean.

1. Preliminaries

A *De Morgan algebra* $(M; \vee, \wedge, \sim, 0, 1)$ is an algebra of type $\langle 2, 2, 1, 0, 0\rangle$ such that $(M; \vee, \wedge, 0, 1)$ is a distributive lattice with largest element 1 and smallest element 0 and $\sim$ is an involutorial dual order-(lattice-)isomorphism; that is, the equations $\sim\sim x = x$, $\sim(x \vee y) = \sim x \wedge \sim y$, $\sim(x \wedge y) = \sim x \vee \sim y$, $\sim 0 = 1$, and $\sim 1 = 0$ are identically satisfied. A *Kleene algebra* is a De Morgan algebra which satisfies the identity associated with the inequality $x \wedge \sim x \leq y \vee \sim y$. In this paper, a *boolean algebra* will be regarded as a De Morgan algebra satisfying the identity $x \wedge \sim x = 0$; of course, a boolean algebra is a Kleene algebra. The chain

$$\{0, x_1, \ldots, x_{n-2}, 1 : 0 = x_0 < x_1 < \ldots < x_{n-2} < x_{n-1} = 1\}$$

Received 13 August 1976.

with n elements possesses a unique involutorial dual order-isomorphism $\sim$ such that it becomes a Kleene (De Morgan) algebra; for $i = 0, \ldots, n-1$, $\sim x_i = x_{n-i-1}$, and so if n is odd the element $x_{(n-1)/2}$ is fixed under the involution. We will denote the De Morgan algebra associated with the chain possessing n elements by $\mathbf{n}$. If a De Morgan algebra M is not boolean then there exists $x \in M$ such that $0 < x \wedge \sim x \leq x \vee \sim x < 1$ and so a De Morgan algebra is not boolean if and only if it possesses a subalgebra isomorphic to either $\mathbf{3}$ or $\mathbf{4}$. The category whose objects are members of the variety of De Morgan algebras and whose morphisms are the associated algebra-homomorphisms is denoted by M; the subcategories which correspond to the subvarieties of Kleene algebras and boolean algebras are denoted by K and B, respectively. We will use D to denote the category whose objects are distributive lattices $(L; \vee, \wedge, 0, 1)$ with 0 and 1, and whose morphisms are the associated algebra-homomorphisms. For reasons of emphasis, it will sometimes be convenient to describe the homomorphisms, congruences, and subalgebras associated with De Morgan algebras as $\sim$-homomorphisms, $\sim$-congruences, and $\sim$-subalgebras, respectively. Let X be any one of the categories M, K, B or D. Then the coproduct in the category X of a set $\{Y_i\}$ of objects in X is the same as their free product in the variety X ([1, Theorem 7, p. 34]) and is denoted by $\coprod_{\mathsf{X}} Y_i$; $Y \coprod_{\mathsf{X}} Z$ is used to denote the coproduct of two objects Y, Z of X. Coproducts of an arbitrary set of objects in D are described, in detail, in Grätzer [9, Section 12, pp. 128-137] and Balbes and Dwinger [1, Chapter 7, pp. 132-150], while coproducts in M are described in [1, Section 11.4, pp. 216-218]. Representations of the coproduct of two objects L_1, L_2 of D in terms of all the continuous monotone (order-preserving) functions from the Priestley space of L_1 into L_2, endowed with the discrete topology, have been given independently by the first author [7] and Davey [8]. One of the aims of this paper is to extend the representations of [7], [8] to M.

Let $C(X)$ be the family of all compact-open subsets of a topological space X. Then the topological space X is called a *spectral space* if it satisfies each of the following properties:

(S1) X is a compact T_0-space;

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(S2) $C(X)$ is a ring of subsets of X and a base for the open sets;

(S3) if F is a closed set in X and C_1 is any subfamily of $C(X)$ such that $\cap F_1 \cap F \neq \emptyset$ for any finite subfamily F_1 of C_1 then $\cap C_1 \cap F \neq \emptyset$.

A function between topological spaces is called *strongly continuous* if it is continuous and the inverse image of each compact-open subset is itself compact-open. The category whose objects are spectral spaces and whose morphisms are strongly continuous functions is denoted by *Spec*.

An *ordered topological space* $(X; \leq, \tau)$ is a partially ordered set $(X; \leq)$ and a topological space $(X; \tau)$; when there is no possibility of confusion, we will denote an ordered topological space $(X; \leq, \tau)$ by means of the underlying set X. An ordered topological space X is *totally order-disconnected* if, whenever $x \nleq y$ $(x, y \in X)$, there exist a clopen increasing subset U and a clopen decreasing subset V such that $U \cap V = \emptyset$ and $x \in U$, while $y \in V$ – a subset U of a partially ordered set X is *increasing (decreasing)* if $x \leq y$ and $x \in U$ $(y \in U)$ imply $y \in U$ $(x \in U)$. The set $\mathcal{D}(X)$ of all clopen increasing subsets of an ordered topological space X is a bounded distributive lattice with respect to set-union and set-intersection. In a compact totally order-disconnected ordered topological space X, $\mathcal{D}(X) \cup \{X \backslash U : U \in \mathcal{D}(X)\}$ is a sub-base for the open sets (*cf.* [6, Section 1]). The category whose objects are compact totally order-disconnected ordered topological spaces and whose morphisms are continuous monotone functions is denoted by *Todc*. In [6, Theorem 2.3] it is shown that *Spec* and *Todc* are isomorphic categories. It is well known that *Spec* is the dual of the category **D** (*cf.* [9, Section 11, pp. 117-125], [1, Chapter 4, pp. 75-84]) and hence, as noted in [6], *Todc* is also the dual of **D**.

Let L be an object of **D** and let $\Sigma(L)$ be the set of all prime ideals of L. When $\Sigma(L)$ is endowed with the so-called hull-kernel or Stone topology, whose basic open sets are those sets of the form $\{P \in \Sigma(L) : a \notin P\}$ for some $a \in L$, it is a spectral space and the map $a \mapsto \{P \in \Sigma(L) : a \notin P\}$ is an isomorphism of L onto $C(\Sigma(L))$. The space $\Sigma(L)$ is called the *Stone space* of L. Order $\Sigma(L)$ with that

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partial order which is the converse of set-inclusion and endow $\Sigma(L)$ with the topology whose sub-basic open sets are of the form $\{P \in \Sigma(L) : a \notin P\}$ or $\{P \in \Sigma(L) : a \in P\}$, where $a \in L$. That is,

$$C(\Sigma(L)) \cup \{X \backslash V : V \in C(\Sigma(L))\}$$

is a sub-base for the new topology. Then $\Sigma(L)$ becomes an object in *Todc* (*cf.* [6], [5, Corollary 1.5]). This ordered space is called the *Priestley space* of L and is denoted by $\mathrm{Pr}(L)$; $C(\Sigma(L))$ can be identified with $\mathcal{D}(\mathrm{Pr}(L))$ and the map $a \mapsto \{P \in \mathrm{Pr}(L) : a \notin P\}$ is an isomorphism of L onto $\mathrm{Pr}(L)$.

2. Coproducts

Let P be a prime ideal in a De Morgan algebra M and let $g(P) = M \backslash {\sim}P$, where for any subset A of M, ${\sim}A = \{{\sim}a : a \in A\}$. Then $g(P)$ is also a prime ideal and the map $P \mapsto g(P)$ is an involution on the Stone space $\Sigma(M)$. Since, for any $a \in M$,

$$\{P \in \Sigma(M) : {\sim}a \notin P\} = \Sigma(M) \backslash g(\{P \in \Sigma(M) : a \notin P\}) ,$$

we see that for each $V \in C(\Sigma(M))$, $\Sigma(M) \backslash g(V) \in C(\Sigma(M))$. In addition, if we define ${\sim}V$ to be $\Sigma(M) \backslash g(V)$ for any $V \in C(\Sigma(M))$ then $C(\Sigma(M))$ becomes a De Morgan algebra and the map $a \mapsto \{P \in \Sigma(M) : a \notin P\}$ is a ${\sim}$-isomorphism of M onto $C(\Sigma(M))$; this was first proved by Białynicki-Birula and Rasiowa [3]. Let g-*Spec* be the category whose objects are spectral spaces X possessing an involution g such that $X \backslash g(V) \in C(X)$ for each $V \in C(X)$ and whose morphisms are strongly continuous functions $f : X_1 \to X_2$ ((X_1, g_1), (X_2, g_2) objects in g-*Spec*) such that $g_2 \circ f = f \circ g_1$. Then Petrescu has extended the duality between D and *Spec* to a duality between M and g-*Spec*.

LEMMA 2.1 (Petrescu [12, Corollary 2.5]). *The category* g-*Spec is isomorphic to the dual of* M.

Let X be a spectral space and f be a morphism in *Spec*. Then the relation $x \le y$ $(x, y \in X)$ if and only if x is in the closure of $\{y\}$, is a partial order on X, and $\Omega(X) = (X; \le, \tau)$, where τ is the topology whose sub-basic open sets are the members of $C(X) \cup \{X \backslash V : V \in C(X)\}$, is an object in *Todc*. Let $\Omega(f) = f$. Then

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$\Omega(f)$ is a morphism in $Todc$. On the other hand, if Y is an object in $Todc$ and $\Psi(Y) = (Y; \sigma)$ is the set Y endowed with the topology σ whose basic open sets are the members of $\mathcal{D}(Y)$ then $\Psi(Y)$ is an object in $Spec$. If h is a morphism in $Todc$ and $\Psi(h) = h$ then $\Psi(h)$ is a morphism in $Spec$ and we have

LEMMA 2.2 ([*6*, Theorem 2.3]). *Ω and Ψ are mutually inverse isomorphisms between the categories $Spec$ and $Todc$.*

Let g-$Todc$ be the category whose objects are compact totally order-disconnected ordered topological spaces X which possess an involution g, which is both a homeomorphism and a dual order-isomorphism, and whose morphisms are functions $f : X_1 \to X_2$ $\left((X_1, g_1), (X_2, g_2)\right.$ objects in g-$Todc$$\left.\right)$ such that $g_2 \circ f = f \circ g_1$.

THEOREM 2.3. *The category* g-$Todc$ *is isomorphic to the dual of the category* M.

Proof. If (X, g) is an object in g-$Spec$ then both $g(V)$ and $X \backslash g(V) = g(X \backslash V)$ are clopen in $\Omega(X)$ for any V in $C(X)$. Because the topology τ on $\Omega(X)$ has $C(X) \cup \{X \backslash V : V \in C(X)\}$ as a sub-base we see that g is an open mapping on $\Omega(X)$. But $g^2 = 1_X$, and so g is a homeomorphism. In addition, if $x, y \in X$ and $x \leq y$, then x is in all basic closed sets which contain y. Thus $g(y) \leq g(x)$, as otherwise $g(y) \nleq g(x)$ and so there is $W \in C(X)$ such that $g(y) \in W$ and $g(x) \in X \backslash W$ and so $y \in X \backslash V$, $V = X \backslash g(W) \in C(X)$, and yet $x \notin X \backslash V$. Since $g^2 = 1_X$ it follows that g is a dual order-isomorphism. The rest of the theorem follows immediately from Lemmas 2.1 and 2.2, since $C(X) = \mathcal{D}(\Omega(X))$.

If (X, g) is an object in g-$Spec$ then the topology τ which converts X to $\Omega(X)$ is precisely the coarsest topology on X for which g is continuous or even a homeomorphism. Thus, we see the advantage of using g-$Todc$ as a representation of the dual of M. Indeed, we now give a simple proof of a theorem due to Berman and Dwinger (see [*1*, Theorem 2, p. 216]).

THEOREM 2.4. *The product of any set $\{(X_i, g_i)\}$ of objects in*

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g-*Todc is* $(\prod X_i, \prod g_i)$, *where* $\prod X_i$ *is the cartesian product of the spaces* X_i *endowed with the direct product order, and* $\prod g_i$ *is the product of the functions* g_i . *Hence, for any set* $\{N_i\}$ *of De Morgan algebras,* $\coprod_{\mathrm{M}} N_i$ *and* $\coprod_{\mathrm{D}} N_i$ *are isomorphic lattices.*

Proof. The theorem follows immediately from duality and the fact that $\prod g_i : \prod X_i \to \prod X_i$ is continuous if and only if each g_i is continuous.

COROLLARY 2.5. *Let* $\{M_i : i \in I\}$ *be a set of De Morgan algebras and* J *be a non-empty subset of* I *such that for each* $j \in J$, N_j *is a* $\sim$-*subalgebra of* M_j . *Then, provided that each* M_i *is regarded as a* $\sim$-*subalgebra of* $\coprod_{i \in I}{}_{\mathrm{M}} M_i$, *the* $\sim$-*subalgebra of* $\coprod_{i \in I}{}_{\mathrm{M}} M_i$ *generated by* $\bigcup_{j \in J} N_j$ *is isomorphic to* $\coprod_{j \in J}{}_{\mathrm{M}} N_j$.

Proof. Because of Grätzer [9, Theorem 5, p. 131; *cf*. Corollary 6, p. 132] the sublattice of $\coprod_{i \in I}{}_{\mathrm{D}} M_i$ generated by $\bigcup_{j \in J} N_j$ is isomorphic to $\coprod_{j \in J}{}_{\mathrm{D}} N_j$. In view of Theorem 2.4, this sublattice is a De Morgan algebra and, of course, the restrictions of the $\sim$-operations of $\coprod_{i \in I}{}_{\mathrm{M}} M_i$ and the sublattice generated by $\bigcup_{j \in J} N_j$ coincide on each N_j . Hence, the sub lattice generated by $\bigcup_{j \in J} N_j$ is a $\sim$-subalgebra of $\coprod_{i \in I}{}_{\mathrm{M}} M_i$, and the result follows.

COROLLARY 2.6. *Let* $\{B_i\}$ *be a set of boolean algebras. Then,* $\coprod_{\mathrm{B}} B_i$, $\coprod_{\mathrm{K}} B_i$, $\coprod_{\mathrm{M}} B_i$, *and* $\coprod_{\mathrm{D}} B_i$ *are isomorphic lattices.*

Proof. Due to Theorem 2.4 and [1, Theorem 2, p. 133], $\coprod_{\mathrm{M}} B_i$ is a boolean algebra.

Let X be an ordered topological space and L be a bounded distributive lattice endowed with the discrete topology. Then $C_m(X, L)$ denotes the set of all continuous monotone functions mapping X into L ,

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and $\overline{C}_m(X, L)$ is the set of all members of $C_m(X, L)$ whose range is finite. Under pointwise operations $\overline{C}_m(X, L)$ is a bounded distributive lattice and in [7, Theorem 2.1] it is shown that $\overline{C}_m(X, L)$ is isomorphic to $\mathcal{D}(X) \bigsqcup_{\mathrm{D}} L$. Hence, if A is another bounded distributive lattice and $X = \mathrm{Pr}(A)$, $\overline{C}_m(X, L) = C_m(X, L)$ and $C_m(X, L)$ is isomorphic to $A \bigsqcup_{\mathrm{D}} L$ (*cf.* [7, Corollary 2.3] and Davey [8]). For $y \in L$, let $c(y)$ be the constant function on X such that $c(y)(x) = y$ for all $x \in X$; the map $y \mapsto c(y)$ is a lattice-embedding of L into $\overline{C}_m(X, L)$. For $V \in \mathcal{D}(X)$, let χ_V be the characteristic function of V ; that is, $\chi_V(x) = 1$ if $x \in V$ and $\chi_V(x) = 0$ if $x \in X \backslash V$; the map $V \mapsto \chi_V$ is a lattice-embedding of $\mathcal{D}(X)$ into $\overline{C}_m(X, L)$.

Now suppose (X, g) is an ordered topological space with an involutorial homeomorphism g which is also a dual order-isomorphism. If $V \in \mathcal{D}(X)$ and we define $\sim V = X \backslash g(V)$ then we see that $\mathcal{D}(X)$ becomes a De Morgan algebra. Thus, if L is also a De Morgan algebra, Theorem 2.4 implies that $\overline{C}_m(X, L)$ is a De Morgan algebra which is isomorphic to $\mathcal{D}(X) \bigsqcup_{\mathrm{M}} L$. The purpose of the next result is to give a formula for the involution on $\overline{C}_m(X, L)$.

THEOREM 2.7. *Let* (X, g) *be an ordered topological space* X *with an involutorial homeomorphism* g *which is also a dual order-isomorphism. Let* L *be a De Morgan algebra. Then* $\overline{C}_m(X, L)$ *is a De Morgan algebra, where, for each* $f \in \overline{C}_m(X, L)$, $\sim f$ *is given by* $(\sim f)(x) = \sim\big(f(g(x))\big)$ *for each* $x \in X$ *, and* $\overline{C}_m(X, L)$ *is isomorphic to* $\mathcal{D}(X) \bigsqcup_{\mathrm{M}} L$.

Proof. Let $f \in \overline{C}_m(X, L)$. Then $\sim f$, as defined above, is equal to the product $\sim \circ f \circ g$. Hence $\sim f$ has finite range, is monotone as it is the composition of a monotone function together with two antitone functions, and is continuous since f and g are continuous and the $\sim$-operation on L is continuous because L has the discrete topology. Since $\sim\sim f = \sim \circ (\sim \circ f \circ g) \circ g = \sim^2 \circ f \circ g^2 = 1_L \circ f \circ 1_X$ and because

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it is easy to see that $\sim 0 = 1$ and $f_1 \le f_2$ $(f_1, f_2 \in \overline{C}_m(X, L))$ implies $\sim f_2 \le \sim f_1$, $\overline{C}_m(X, L)$ is a De Morgan algebra with the $\sim$-operation, as defined above. In order to prove that $\overline{C}_m(X, L)$ is isomorphic to $\mathcal{D}(X) \coprod_{\mathrm{M}} L$, it is sufficient, in the light of our remarks preceding this theorem, to check that the maps $y \mapsto c(y)$ and $V \mapsto \chi_V$ are $\sim$-homomorphisms of L and $\mathcal{D}(X)$, respectively, into $\overline{C}_m(X, L)$. But it is not hard to verify that $c(\sim y) = \sim c(y)$ for any $y \in L$ and $\sim \chi_V = \chi_{X \setminus g(V)} = \chi_{\sim V}$ for any $V \in \mathcal{D}(X)$, and so the theorem follows.

The representation of $A \coprod_{\mathrm{D}} L$ (A, L objects in D) by means of $C_m(\mathrm{Pr}(A), L)$ can be simplified when A is a chain with n elements (see, for example, [7, Example 2.13]). We now note the corresponding result for M, when $A = \mathrm{n}$ and L is a De Morgan algebra.

COROLLARY 2.8. *Let L be a De Morgan algebra. Then $\mathrm{n} \coprod_{\mathrm{M}} L$ is isomorphic to the subalgebra of the $(n-1)$-fold direct power of L consisting of all $(n-1)$-tuples $(y_0, y_1, \ldots, y_{n-2})$ such that $y_0 \ge y_1 \ge \cdots \ge y_{n-2}$, and $\sim(y_0, y_1, \ldots, y_{n-2}) = (z_0, z_1, \ldots, z_{n-2})$, where $z_i = \sim y_{n-2-i}$ for each $i = 0, 1, \ldots, n-2$.*

3. Kleene algebras

If P is a prime ideal of a De Morgan algebra M, let $R(P)$ be the largest lattice-congruence having P as a congruence-class. Of course, the partition of M associated with $R(P)$ is $\{P, M \setminus P\}$. The partition associated with the lattice-congruence $R(P) \cap R(g(P))$ is $\{M \setminus (P \cup g(P)), P \setminus g(P), g(P) \setminus P, P \cap g(P)\}$, though, of course, we are not implying that these four congruence-classes are necessarily distinct. If $x \in M \setminus (P \cup g(P))$ then $\sim x \in P \cap g(P)$; if $x \in P \cap g(P)$ then $\sim x \in M \setminus (P \cup g(P))$; if $x \in P \setminus g(P)$ then $\sim x$ is also in $P \setminus g(P)$; if $x \in g(P) \setminus P$ then $\sim x$ is also in $g(P) \setminus P$. Hence, $R(P) \cap R(g(P))$ is a $\sim$-congruence and $M/(R(P) \cap R(g(P)))$ is isomorphic to 2 if and only if $P = g(P)$, $M/(R(P) \cap R(g(P)))$ is isomorphic to 3 if and only if P and $g(P)$ are comparable but not equal, and $M/(R(P) \cap R(g(P)))$ is isomorphic to $\mathrm{T} = \{0, a, b, 1 : 0 = a \wedge b < a, b < a \vee b = 1, \sim 0 = 1, \sim a = a, \sim b = b\}$

if and only if all four congruence-classes are distinct. In order to describe the dual of K we need a consequence of the following result; we will need the full generality of the result later in this section.

THEOREM 3.1. *Let* Φ *be a* $\sim$*-congruence on a De Morgan algebra* M *and let* $\theta : M \to M/\Phi$ *be the canonical epimorphism. Then*

$$\Phi = \bigcap\{R(\theta^{-1}(P)) \cap R(g(\theta^{-1}(P))) : P \in \mathrm{Pr}(M/\Phi)\} .$$

Proof. Let $x, y \in M$ be such that $x \equiv y(\Phi)$. Then $\theta(x) = \theta(y)$. Of course, $\theta^{-1}(P) \in \mathrm{Pr}(M)$ and, as θ is a $\sim$-homomorphism, $g(\theta^{-1}(P)) = M\backslash{\sim}\theta^{-1}(P) = \theta^{-1}(M/\Phi\backslash{\sim}P) = \theta^{-1}(g(P))$ for each $P \in \mathrm{Pr}(M/\Phi)$. As $x \in \theta^{-1}(P)$ if and only if $y \in \theta^{-1}(P)$ and $x \in \theta^{-1}(g(P))$ if and only if $y \in \theta^{-1}(g(P))$ for each $P \in \mathrm{Pr}(M/\Phi)$, we have

$$x \equiv y \cap \{R(\theta^{-1}(P)) \cap R(g(\theta^{-1}(P))) : P \in \mathrm{Pr}(M/\Phi)\} .$$

Conversely, suppose $a, b \in M$ and yet $a \not\equiv b(\Phi)$. Then $\theta(a) \neq \theta(b)$ and so there exists $P \in \mathrm{Pr}(M/\Phi)$ such that $\theta(a) \in P$ and $\theta(b) \notin P$, or vice-versa. Then $a \not\equiv b(R(\theta^{-1}(P)))$ and $a \not\equiv b(R(\theta^{-1}(P)) \cap R(g(\theta^{-1}(P))))$, and the result follows.

Let Φ be ω, the smallest congruence on M, whereby $x \equiv y(\omega)$ $(x, y \in M)$ if and only if $x = y$. Then $M/\Phi = M$ and so Theorem 3.1 implies that

$$\bigcap\{R(P) \cap R(g(P)) : P \in \mathrm{Pr}(M)\} = \omega .$$

Hence we obtain the well known result that any De Morgan algebra is isomorphic to a subdirect product of copies of T and its subalgebras 3 and 2. Of course, T, 3, and 2 are the only subdirectly irreducible (simple) De Morgan algebras. All this was first established by Białynicki-Birula [2, Theorem 2.1] and Kalman [*11*, Lemma 2], using other techniques. Since 2 and 3 are Kleene algebras while T is not a Kleene algebra, we see that a De Morgan algebra is a Kleene algebra if and only if it is isomorphic to a subdirect product of copies of 2 and 3. In other words, we have

THEOREM 3.2. *The dual of* K *is isomorphic to the subcategory of* g-Todc *whose objects* (X, g) *are such that* x *and* $g(x)$ *are comparable for each* $x \in X$.

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LEMMA 3.3. *Each of the coproducts* $3\coprod_M 3$, $3\coprod_M 4$, *and* $4\coprod_M 4$ *is not a Kleene algebra.*

Proof. The lemma can be proved by obtaining the required coproducts via Corollary 2.8; $3\coprod_M 3$ has a six-element planar graph, $3\coprod_M 4$ has a ten element planar graph, while $4\coprod_M 4$ has a twenty element non-planar graph. However, it is a simple matter to establish the lemma by making use of Theorems 2.4 and 3.2; $(\mathrm{Pr}(3), g)$ is the discrete space $\{x_0, x_1 : x_0 > x_1, g(x_0) = x_1, g(x_1) = x_0\}$, $(\mathrm{Pr}(4), g)$ is the discrete space $\{y_0, y_1, y_2 : y_0 > y_1 > y_2, g(y_0) = y_2, g(y_1) = y_1, g(y_2) = y_0\}$. In $(\mathrm{Pr}(3), g) \times (\mathrm{Pr}(3), g)$, $g((x_0, x_1)) = (g(x_0), g(x_1)) = (x_1, x_0)$, which is incomparable with (x_0, x_1). Similarly $g((y_0, y_2)) = (y_2, y_0)$ in $(\mathrm{Pr}(4), g) \times (\mathrm{Pr}(4), g)$ and (y_2, y_0) is incomparable with (y_0, y_2). Similarly $g((x_0, y_2)) = (x_1, y_0)$ in $(\mathrm{Pr}(3), g) \times (\mathrm{Pr}(4), g)$ and (x_0, y_2) is incomparable with (x_1, y_0). The proof is now complete.

LEMMA 3.4. *Let* B *be a boolean algebra and* K *be a Kleene algebra. Then* $B\coprod_M K$ *is a Kleene algebra.*

Proof. Since all the prime ideals are maximal in a boolean algebra B, $(\mathrm{Pr}\ B, g)$ is an ordered space in which the partial order is equality and the involution g is the identity function. Thus, if $h.k \in C_m(\mathrm{Pr}(B), K)$ and $x \in \mathrm{Pr}(B)$,

$$(h \wedge \sim h)(x) = h(x) \wedge (\sim h)(x) = h(x) \wedge \sim h(g(x)) = h(x) \wedge \sim(h(x)) \leq k(x) \vee \sim k(x) = (k \vee \sim k)(x) .$$

Thus the lemma follows from Theorem 2.7.

THEOREM 3.5. *Let* $\{K_i\}$ *be a set of Kleene algebras. Then* $\coprod_M K_i$ *is a Kleene algebra if and only if at most one of the* K_i *is not a boolean algebra.*

Proof. Due to Corollary 2.6, Lemma 3.4, and the "associativity" of coproducts, $\coprod_M K_i$ is a Kleene algebra when all but possibly one of the K_i is not boolean.

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On the other hand, suppose the coproduct in M of the K_i is a Kleene algebra and yet there exist two, namely K_{i_1} and K_{i_2}, which are not boolean. Then K_{i_1} and K_{i_2} each have either 3 or 4 as a subalgebra and so one of $3 \sqcup_M 3$, $3 \sqcup_M 4$ or $4 \sqcup_M 4$ is isomorphic to a subalgebra of $\bigsqcup_M K_i$, due to Corollary 2.5 and its proof. Then Lemma 3.4 supplies the required contradiction.

Because a Kleene algebra is isomorphic to a subdirect product of copies of 3 and its subalgebra 2, there is always a homomorphism of a given Kleene algebra into 3 which distinguishes distinct elements of the algebra. Hence, due to Grätzer [9, Exercise 9, p. 138], the category K has arbitrary coproducts. In Theorem 3.8 below, we exhibit a property of such coproducts; we present this after developing two results, which are of independent interest.

Cignoli [4, Corollaries 3.3, 4.4] showed that the categories M and K are injectively complete. From general theory [13, Proposition 2.1 and Theorem 2.3], a variety of algebras is injectively complete if and only if it is residually small and has both the congruence extension property and the amalgamation property. Thus M and K must have these three properties. Since the only subdirectly irreducible members of M are the four element algebra T together with its subalgebras 3 and 2, M and K are residually small (*cf.* Taylor [13, Theorem 1.2 (ii)]); the aim of the next two results is to provide alternative proofs of the congruence extension and amalgamation properties for M and K.

THEOREM 3.6. M, *and so* K, *has the congruence extension property.*

Proof. Let M be a subalgebra of De Morgan algebra M_1 and let Φ be a congruence on M. Due to Theorem 3.1,

$$\Phi = \bigcap\{R(\theta^{-1}(P)) \cap R(g(\theta^{-1}(P))) \ : \ P \in \Pr(M/\Phi)\} ,$$

where $\theta : M \to M/\Phi$ is the canonical epimorphism. Now for each prime ideal $P \in \Pr(M/\Phi)$, $\theta^{-1}(P)$ is a prime ideal of the sublattice M of the distributive lattice M_1 and so, by [1, Proof of Theorem 5, p. 74] (see also [9, Exercise 3, p. 100]), there is at least one prime ideal P_1 of

M_1 such that $P_1 \cap M = \theta^{-1}(P)$ and $P_1 \cap \left(M \backslash \theta^{-1}(P)\right) = \emptyset$. Since M is a $\sim$-subalgebra of M, we must also have $\sim\theta^{-1}(P) = M \cap \sim P_1$ and $\sim P_1 \cap \left(M \backslash \sim\theta^{-1}(P)\right) = \emptyset$, so that $g\left(\theta^{-1}(P)\right) = M \cap g\left(P_1\right)$ and $g\left(P_1\right) \cap \left(M \backslash g\left(\theta^{-1}(P)\right)\right) = \emptyset$. For each $P \in \Pr(M/\Phi)$, choose a prime ideal $P_1 \in \Pr\left(M_1\right)$ with the above properties. Define

$$\Phi_1 = \bigcap\left\{R\left(P_1\right) \cap R\left(g\left(P_1\right)\right) : P_1 \text{ is related to } P \in \Pr(M/\Phi), \text{ as above}\right\} .$$

Then Φ_1 is a congruence on M_1 and a routine check shows that $\Phi_1 \cap (M \times M) = \Phi$, as required.

Since M and K are each varieties with the congruence extension property and such that each subalgebra of a subdirectly irreducible algebra is itself subdirectly irreducible, we can show that each of M and K has the amalgamation property if we prove that each amalgam

$$(A;\ B,\ \gamma,\ C,\ \delta;\ \gamma : A \to B \text{ and } \delta : B \to C \text{ are monomorphisms}) ,$$

where A, B, and C are subdirectly irreducible algebras in M or K, can be amalgamated. This is due to a deep result of Grätzer and Lakser [10, Theorem 3]. The verification that such an amalgam can here be amalgamated is a simple matter and so we have

THEOREM 3.7. *Both* M *and* K *have the amalgamation property.*

Because of Theorems 3.6 and 3.7 and another result of Grätzer and Lakser [10, Theorem 4] we have

THEOREM 3.8. *Let* A *and* B *be Kleene algebras, and* A_1 *and* B_1 *be subalgebras of* A *and* B, *respectively. Then the subalgebra of* $A \sqcup_{\mathsf{K}} B$ *generated by* A_1 *and* B_1 *is isomorphic to* $A_1 \sqcup_{\mathsf{K}} B_1$.

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