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Large algebraic theories with small algebras

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The aim of the paper is to study the interrelation between several natural smallness conditions on an algebraic theory with a proper class of operations. The conditions concern the existence of sets of data determining algebras, homomorphisms, subalgebras, and congruences.

I. Introduction and results

Let us consider an algebraic theory (Ω, E) where Ω is a (possibly proper) class of finitary or infinitary operation symbols (shortly, operations) and E is a class of equations. Denote $\overline{\Omega}$ the clone of (Ω, E) ; thus $\overline{\Omega}$ is the class of all operations obtained by transfinite recursion from the basic ones (those from Ω) and the trivial ones (projections) by means of composition of the form $\omega(\omega_i;\ i \in n)$ where ω is n-ary and the ω_i's and the result have the same arity. Of course, two operations from $\overline{\Omega}$ are regarded to be equal if their equality is derivable from E . If the class of all n-ary operations in $\overline{\Omega}$ is a set for every set n then the theory is said to be *varietal* [4].

Algebraic theories involving a proper class of operations were considered for the first time by triple theorists in order to include some categories of algebraic nature; see [5] for references and historical remarks. It turned out that triples (equivalently, varietal theories) do not include the theory of complete lattices and the theory of complete boolean algebras [1], [2]. That is why Linton [4] considered also non-

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varietal theories.

However, the notion of an algebraic theory seems to be too general. Indeed, the "large" theories have been introduced to describe algebraic categories whose objects are small by which we mean that each of them is described by a *set* of data. For instance, a complete lattice on a set X is described by means of two X-ary operations sup and inf . We suggest that only those algebraic theories are reasonable for which the algebras are small.

Without any restriction on the character of data describing an algebra, the requirement in question is as follows:

LEG: the conglomerate of all (Ω, E)-algebras is equipotent with a proper class.

(LEG for legitimacy: the number of (Ω, E)-algebras does not, in contrary to a general case, exceed the cardinality of the universum we work in.) Another, perhaps more natural condition is

FIB: for every set X , algebras whose underlying set is X are determined by a set of operations in the sense: there is a *set* $\Omega(X) \subset \overline{\Omega}$ such that for any two (Ω, E)-algebras $A = (X, \{\omega^A\})$, $B = (X, \{\omega^B\})$ we have $A = B$ if and only if $\omega^A = \omega^B$ for every $\omega \in \Omega(X)$.

(FIB for "small fibred": we shall show that (Ω, E) satisfies FIB if and only if for every set X , algebras whose underlying set is X form a set; see II.6.) A simple example (III.4) shows that the obvious implication FIB $\Rightarrow$ LEG can not be reversed.

Unfortunately, we are not able to give an intrinsic characterization of theories with LEG and FIB respectively. However, in [6], a condition on an algebraic theory has been discovered which ensures FIB and which is fulfilled by various non-varietal theories of nature (for example, by complete lattices and complete boolean algebras). It is, as we shall prove, actually stronger than FIB .

An algebraic theory (Ω, E) is *locally small based* if

LSB: $\overline{\Omega}$ is generated by $\Omega' \subset \overline{\Omega}$ where $\Omega' = \bigcup_n \Omega'_n$ $(\Omega'_n$ being the class of all n-ary operations in Ω') such that

(i) each Ω'_n is a set

(ii) for every $\omega \in \Omega'_m$ and every $f : m \to n$, $\omega f^* \in \Omega'_n$;

here $\omega f^*(x_i;\ i \in n) = \omega(x_{f(j)};\ j \in m)$.

Further conditions concerning selection of homomorphisms, subalgebras, and congruences are as follows. In each of them, the existence of a *set* $\Omega(X) \subset \overline{\Omega}$ is required for every set X such that

HOM: a mapping $f : X \to Y$ is a homomorphism from $A = (X,\ \{\omega^A\})$ to $B = (Y,\ \{\omega^B\})$ if and only if it is compatible with all operations $\omega \in \Omega(X)$;

SUB: the set X carries a subalgebra of an algebra $B = (Y,\ \{\omega^B\})$ (where $X \subset Y$) if and only if it is closed in B under all operations $\omega \in \Omega(X)$;

CON: an equivalence on X is a congruence on an algebra $A = (X,\ \{\omega^A\})$ if and only if it is a congruence with respect to all operations $\omega \in \Omega(X)$.

THEOREM. LSB $\Rightarrow$ SUB $\Longleftrightarrow$ HOM $\Rightarrow$ FIB $\Rightarrow$ CON , FIB $\Rightarrow$ LEG . *None of the implications* SUB $\Rightarrow$ LSB, FIB $\Rightarrow$ HOM, CON $\Rightarrow$ FIB, LEG $\Rightarrow$ CON, CON $\Rightarrow$ LEG, *is valid.*

II. Proofs

II.1. Let $A = (X,\ \{\omega^A\})$ be an $(\Omega,\ E)$-algebra. Then all ω^A with $\omega \in \overline{\Omega}$ are determined by X-ary operations from $\overline{\Omega}$ by

$$\omega^A(\alpha) = (\omega\alpha^*)^A(1_X)$$

whenever ω is n-ary and $\alpha \in X^n$.

II.2. *A mapping* $f : X \to Y$ *is a homomorphism from an* $(\Omega,\ E)$*-algebra* $A = (X,\ \{\omega^A\})$ *to an* $(\Omega,\ E)$*-algebra* $B = (Y,\ \{\omega^B\})$ *if and only if it is compatible with all* X*-ary operations from* $\overline{\Omega}$.

Proof. Let $\omega \in \Omega$ be n-ary and let $\alpha \in X^n$. If f is compatible

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with X-ary operations then $f\omega^A(\alpha) = f(\omega\alpha^*)^A(1_X) = (\omega\alpha^*)^B(f) = \omega^B(f\alpha)$. So f is compatible with ω, too.

II.3. *A set* $X \subset Y$ *carries a subalgebra of* $B = (Y, \{\omega^B\})$ *if and only if* X *is closed in* B *under all* X*-ary operations from* $\bar{\Omega}$.

Proof. If $\omega \in \Omega$, $\alpha \in X^n$, and $j : X \to Y$ is the inclusion, then $\omega^B(j\alpha) = (\omega\alpha^*)^B(j) \in X$ if X is closed in B under X-ary operations.

II.4. *An equivalence* $\sim$ *on a set* X *is a congruence on an* (Ω, E)*-algebra* $A = (X, \{\omega^A\})$ *if and only if it is a congruence with respect to all* $X \times X$*-ary operations from* $\bar{\Omega}$.

Proof. Let $\omega \in \Omega$ be n-ary, let $\alpha, \beta \in X^n$, $\alpha(t) \sim \beta(t)$ for all $t \in n$. We have to prove $\omega^A(\alpha) \sim \omega^A(\beta)$. Define $\bar{\alpha}, \bar{\beta} : X \times X \to X$ as follows: $\bar{\alpha}(x, y) = \alpha(t)$, $\bar{\beta}(x, y) = \beta(t)$ if $(x, y) = (\alpha(t), \beta(t))$ for some t, $\bar{\alpha}(x, y) = \bar{\beta}(x, y) = x_0$ otherwise, where $x_0 \in X$ is arbitrary but fixed. Define $\gamma : n \to X \times X$ by $\gamma(t) = (\alpha(t), \beta(t))$. Then $\alpha = \bar{\alpha}\gamma$, $\beta = \bar{\beta}\gamma$, and so $\omega^A(\alpha) = (\omega\gamma^*)^A(\bar{\alpha})$, $\omega^A(\beta) = (\omega\gamma^*)^A(\bar{\beta})$, $\omega\gamma^*$ is $X \times X$-ary, and $\bar{\alpha}(t) \sim \bar{\beta}(t)$ for every t. Thus if $\sim$ is a congruence with respect to all $X \times X$-ary operations, then $(\omega\gamma^*)^A(\bar{\alpha}) \sim (\omega\gamma^*)^A(\bar{\beta})$; that is $\omega^A(\alpha) \sim \omega^A(\beta)$.

II.5. In each of the conditions FIB, SUB, HOM, CON, we may assume that $\Omega(X) \subset \Omega(Y)$ whenever card $X \leq$ card Y; in particular, that $\Omega(X)$ depends on card X only.

Indeed, for every set X, $\Omega(X)$ can be replaced by $\Omega(n)$ where $n =$ card X and so by $\bigcup_m \Omega(m)$ where m runs through all cardinals less than or equal to n.

II.6. *A theory* (Ω, E) *satisfies* FIB *if and only if for every set* X, (Ω, E)*-algebras whose underlying set is* X *form a set.*

Proof. Necessity: Obvious.

Sufficiency: Let $\bar{\Omega}_X$ be the class of all X-ary operations in $\bar{\Omega}$. For $\omega, \sigma \in \Omega_X$ and for every algebra A the underlying set of which is X, put $\omega \sim_A \sigma$ if and only if $\omega^A = \sigma^A$. Then $\overline{\Omega}_X/\sim_A$ is a set. As the algebras A in question form a set, also $\overline{\Omega}_X/\sim$ is a set where $\sim$ is the intersection of all $\sim_A$. So there exists a set $\Omega(X) \subset \overline{\Omega}$ such that for every $\omega \in \overline{\Omega}_X$ there is $\sigma \in \Omega(X)$ such that $\omega \sim \sigma$; that is, $\omega^A = \sigma^A$ for all A in question. Now FIB follows by II.1.

II.7. LSB $\Rightarrow$ SUB is proved in [6]. FIB $\Rightarrow$ LEG is obvious.

II.8. SUB $\Rightarrow$ FIB follows by the observation that for $A = (X, \{\omega^A\})$, $B = (X, \{\omega^B\})$, we have $A = B$ if and only if the diagonal in $X \times X$ carries a subalgebra of $A \times B$.

II.9. FIB $\Rightarrow$ CON .

Proof. We use II.6 and proceed quite analogously as in the proof of II.6 to obtain a set $\Omega(X)$ of $X \times X$-ary operations such that for each $X \times X$-ary operation ω there is $\sigma \in \Omega(X)$ such that $\omega^A = \sigma^A$ for all algebras with underlying set X . Then we apply II.4.

II.10. SUB $\Rightarrow$ HOM .

Proof. Suppose SUB . Then also FIB and CON by II.8 and II.9. Suppose II.5 and put

$$\Omega(X) = \Omega_{\mathrm{CON}}(X) \cup \Omega_{\mathrm{SUB}}(X) \cup \Omega_{\mathrm{FIB}}(X) ,$$

where $\Omega_{\mathrm{CON}}(X)$ stands for $\Omega(X)$ of CON and so on. Let f, X, Y, A, B be as in HOM . Let f be compatible with all $\omega \in \Omega(X)$. Then the equivalence "$x \sim y$ if and only on $f(x) = f(y)$ " is a congruence with respect to all $\omega \in \Omega(X)$. As $\Omega(X) \supset \Omega_{\mathrm{CON}}(X)$, $\sim$ is a congruence on A and f is a homomorphism from A to some algebra $A' = (f(X), \{\omega^{A'}\})$ where $\omega^{A'}$ is a restriction of ω^B for every $\omega \in \Omega(X)$. It follows also that $f(X)$ is closed in B under all $\omega \in \Omega(X)$. As $\Omega(X) \supset \Omega_{\mathrm{SUB}}(X) \supset \Omega_{\mathrm{SUB}}(f(X))$, $f(X)$ carries a subalgebra B' of B . We have $\omega^{A'} = \omega^{B'}$ for all $\omega \in \Omega(X)$, and so $A' = B'$, because $\Omega(X) \supset \Omega_{\mathrm{FIB}}(X) \supset \Omega_{\mathrm{FIB}}(f(X))$. Thus f is a homomorphism from A to B .

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II.11. HOM $\Rightarrow$ SUB .

Proof. Write Ω as $\Omega = \bigcup_\alpha \Omega_\alpha$ where α runs through ordinals, each Ω_α is a set, and $\Omega_\alpha \subset \Omega_\beta$ whenever $\alpha \leq \beta$. Let E_α consist of all equations between operations derived from Ω_α which hold in every (Ω, E)-algebra. Suppose (Ω, E) does not fulfil SUB . Then there exists a set X such that for every α there is an (Ω, E)-algebra $A_\alpha = \left(Y_\alpha, \{\omega^{A_\alpha}\}\right)$ such that X is closed in A_α under all Ω_α-operations - and thus carries an $(\Omega_\alpha, E_\alpha)$-algebra B_α - but X does not carry a subalgebra of A_α . As for every α , all $(\Omega_\alpha, E_\alpha)$-algebras whose underlying set is X form a set, we can, using induction, redefine the family $\{A_\alpha\}$ in such a way that B_β extends B_α whenever $\alpha \leq \beta$ (in the sense that $\omega^{B_\alpha} = \omega^{B_\beta}$ for all $\omega \in \Omega_\alpha$). Then there is an (Ω, E)-algebra B which extends all B_α . Now, suppose HOM . Then we may assume that $\Omega(X)$ of HOM equals some Ω_α . But the inclusion $X \to Y_\alpha$ is not a homomorphism from B to A_α although it is compatible with all Ω_α-operations, a contradiction.

III. Counterexamples

III.1. SUB $\not\Rightarrow$ LSB . The simplest counterexample is provided by a non-LSB theory which degenerates in the sense that it has only trivial models. For instance, the theory in [3, p. 558] works. But our ambition is to present a counterexample which admits no degeneration: if $\omega, \sigma \in \overline{\Omega}$ and $\omega^A = \sigma^A$ for every algebra A then $\omega = \sigma$. Put

$$\Omega = \{\alpha_i ; i \in \mathrm{Ord}\} \cup \{*, 0\} , \quad \alpha_i \text{ unary}, \quad * \text{ binary}, \quad 0 \text{ nullary},$$

$$E : \alpha_i(x) * \alpha_j(x) = \alpha_j(x) \text{ for } i < j , \quad x * x = 0 * x = 0 .$$

(a) Let us prove SUB . Let X be a set and k an ordinal whose cofinal is bigger than card X^X . Put $\Omega(X) = \{\alpha_i ; i < k\} \cup \{*, 0\}$. Let $B = (Y, \{\omega^B\})$. Let $X \subset Y$ be closed under all $\Omega(X)$-operations in B .

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Then there exists a cofinal set K in k such that $\alpha_i^B/X = \alpha_j^B/X$ for $i, j \in K$ where $/X$ means the restriction to X. Let $i, j \in K$ and $x \in X$. Then $\alpha_j^B(x) = \alpha_i^B(x) * \alpha_j^B(x) = \alpha_i^B(x) * \alpha_i^B(x) = 0^B$. Thus, if $s > k$, then $\alpha_s^B(x) = \alpha_j^B(x) * \alpha_s^B(x) = 0^B * \alpha_s^B(x) = 0^B$ and so X is closed in B under all operations.

(b) The theory does not degenerate. Indeed, let $\omega, \sigma \in \overline{\Omega}$ and let $\omega^A = \sigma^A$ for every algebra A. There exists an ordinal k such that $\omega, \sigma \in \overline{\Omega}_1$, where $\overline{\Omega}_1$ is the clone of (Ω_1, E_1) where $\Omega_1 = \{\alpha_i;\ i < k\} \cup \{*, 0\}$, and E_1 consists of equations between $\overline{\Omega}_1$-operations which can be derived from E. Let A be the (Ω_1, E_1)-free algebra over n where n is the arity of ω, σ. The point is that A can be made an (Ω, E)-algebra by putting $\alpha_i(x) = 0$ for $i \geq k$ and for every x. Thus $\omega^A = \sigma^A$ and so the equation $\omega = \sigma$ can be derived within (Ω_1, E_1).

(c) The theory is not LSB. Indeed, consider algebras $A_\beta = (\beta, \{\alpha^{A_\beta}\}) \cup \{*^A, 0^A\}$ where β runs through cardinals and for every β and $x, y \in \beta$,

$$\alpha_i^{A_\beta}(x) = x + i + 1 \quad \text{for } i < \beta, \quad \alpha_i^{A_\beta}(x) = 0 \quad \text{otherwise,}$$

$$x *^A y = \max(x, y) \quad \text{if } 0 \neq x \neq y,$$

$$x *^A x = 0 *^A x = 0 = 0^A.$$

Then $\alpha_i^{A_\beta} \neq \alpha_j^{A_\beta}$ for $i < j < \beta$ and so $\alpha_i \neq \alpha_j$ for $i \neq j$. It follows that the theory is non-varietal and we can use the following:

LEMMA. LSB + *bounded* $\Rightarrow$ *varietal.*

Proof. Bounded means that arities of all $\omega \in \Omega$ are less than k for some cardinal k. Let $\Omega' = \bigcup_n \Omega'_n$ be as in LSB. For every $\delta \in \Omega'_k$ and every mapping $t : k \to k$ choose $\sigma \in \overline{\Omega}$ with $\sigma t^* = \delta$ (if any).

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Collect these σ to obtain a *set* Ω''_k. Let n be a set, card $n > k$, and let $\omega \in \Omega'_n$. Then $\omega = \beta f^*$ for some $f : k \to n$ and $\beta \in \overline{\Omega}$, β k-ary. Write f as $f = mt$ with $t : k \to k$ and with $m : k \to n$ injective; choose $r : n \to k$ with $rm = 1_k$. Then $\beta t^* = \beta t^* m^* r^* = \omega r^* \in \Omega'_k$. By the definition of Ω''_k, $\beta t^* = \sigma t^*$ for some $\sigma \in \Omega''_k$. Thus $\omega = \beta t^* m^* = \sigma t^* m^*$. This proves that the set $\bigcup_{i<k} \Omega'_i \cup \Omega''_k$ generates $\overline{\Omega}$, too. Hence the theory is varietal.

III.2. FIB $\not\Rightarrow$ SUB. Let us consider the theory

$\Omega = \{*, 0\} \cup \{\alpha_{ik};\ i, k$ ordinals, k a regular cardinal $i < k\}$,

$*$ binary, 0 nullary, each α_{ik} unary,

$E : \alpha_{ik}(x) * \alpha_{jk}(x) = \alpha_{ik}(x)$ whenever $i < j$, $x * x = x * 0 = 0$.

(a) Let us prove FIB. Let X be a set and k_0 a regular cardinal, $k_0 > \operatorname{card} X^X$. Put $\Omega(X) = \{*, 0\} \cup \{\alpha_{ik};\ i < k < k_0\}$. Let A be an algebra with underlying set X and let k be a regular cardinal, $k \geq k_0$. Then there is a cofinal set $K \subset k$ such that $i, j \in K$ implies $\alpha^A_{ik} = \alpha^A_{jk}$. For any $s < k$ choose $i, j \in K$ with $s < i < j$. Then

$$\alpha^A_{sk}(x) = \alpha^A_{sk}(x) * \left(\alpha^A_{ik}(x) * \alpha^A_{jk}(x)\right)$$
$$= \alpha^A_{sk}(x) * \left(\alpha^A_{ik}(x) * \alpha^A_{ik}(x)\right) = \alpha^A_{ik}(x) * 0^A = 0^A .$$

(b) Let us disprove SUB. Let $X = \beta$, β a cardinal. Suppose that there exists a set $\Omega(X)$ as in SUB. We may assume that $\Omega(X) = \{*, 0\} \cup \{\alpha_{ik};\ i < k < k_0\}$ for some $k_0 > \beta$. Let $A = \left(k_0, \{*^A, 0^A\} \cup \left\{\alpha^A_{ik}\right\}\right)$, where $x *^A y = \min(x, y)$ for $x \neq y$, $x *^A x = 0^A = 0$, $\alpha^A_{ik}(x) = x + i$ for $k = k_0$, $\alpha^A_{ik}(x) = 0$ otherwise. Then $X = \beta$ is closed in A under all $\Omega(X)$-operations but not under α_{ik_0}, a contradiction.

III.3. CON $\not\Rightarrow$ LEG (and so CON $\not\Rightarrow$ FIB) is clear: let Ω consist of

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a proper class of nullary symbols and $E = \emptyset$.

III.4. LEG $\not\to$ CON (and so LEG $\not\to$ FIB). Put

$$\Omega = \{\alpha_i;\ i \in \mathrm{Ord}\}\ ,\ \text{each}\ \alpha_i\ \text{unary},$$

$$E : \alpha_i\alpha_j(x) = \alpha_m(x)\ \text{where}\ m = \max(i, j)\ .$$

If A is an algebra, then there is a cofinal class K of ordinals such that $i, j \in K$ implies $\alpha_i^A = \alpha_j^A$. Then also for $i < k < j$ with $i, j \in K$ we have $\alpha_k^A(x) = \alpha_i^A\alpha_k^A(x) = \alpha_j^A\alpha_k^A(x) = \alpha_j^A(x)$. Thus there is an ordinal $s = s(A)$ such that $\alpha_i^A = \alpha_s^A$ for all $i > s$. It follows easily that the theory satisfies LEG .

To disprove CON , consider $X = 3$ $(= \{0, 1, 2\})$ and suppose that there exists a set $\Omega(X)$ as in CON. We may assume $\Omega(X) = \{\alpha_i;\ i < k\}$ for some k . Put $A = \left(X, \left\{\alpha_i^A\right\}\right)$ where $\alpha_i^A(1) = 2$ for $i \geq k$ and $\alpha_i^A(x) = x$ otherwise. Then the equivalence $\sim$ where $0 \sim 1$ but $0 \not\sim 2$ is a congruence with respect to all $\Omega(X)$-operations but not with respect to α_k , a contradiction.

References

[1] H. Gaifman, "Infinite Boolean polynomials I", *Fund. Math.* 54 (1964), 229-250.

[2] A.W. Hales, "On the non-existence of free complete Boolean algebras", *Fund. Math.* 54 (1964), 45-66.

[3] John R. Isbell, "General functorial semantics, I", *Amer. J. Math.* 94 (1972), 535-596.

[4] F.E.J. Linton, "Some apsects of equational categories", *Proceedings of the Conference on Categorical Algebra*, La Jolla 1965, 84-94 (Springer-Verlag, Berlin, Heidelberg, New York, 1966).

[5] Ernest G. Manes, *Algebraic theories* (Graduate Texts in Mathematics, 26. Springer-Verlag, New York, Heidelberg, Berlin, 1976).

[6] Jan Reiterman, "Algebraic theories and varieties of functor algebras", submitted.

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