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On lattices in semi-stable representations: a proof of a conjecture of Breuil

Tong Liu

Abstract

For $p \geqslant 3$ an odd prime and a nonnegative integer $r \leqslant p-2$, we prove a conjecture of Breuil on lattices in semi-stable representations, that is, the anti-equivalence of categories between the category of strongly divisible lattices of weight r and the category of Galois stable $\mathbb{Z}_p$-lattices in semi-stable p-adic Galois representations with Hodge–Tate weights in $\{0, \dots, r\}$.

Contents

1. Introduction

Let k be a perfect field of characteristic $p > 2$, $W(k)$ its ring of Witt vectors, $K_0 = W(k)[1/p]$, K/K_0 a finite totally ramified extension and $e = e(K/K_0)$ the absolute ramification index. We are interested in understanding semi-stable p-adic Galois representations of $G := \mathrm{Gal}(\bar{K}/K)$. An important result in this direction is proved by Colmez and Fontaine [CF00]: semi-stable p-adic Galois representations are classified by weakly admissible filtered (φ, N)-modules. Since G is compact, any continuous representation $\rho : G \to \mathrm{GL}_n(\mathbb{Q}_p)$ admits a G-stable $\mathbb{Z}_p$-lattice. It is thus natural to ask whether there also exists a corresponding integral structure on the side of filtered (φ, N)-modules. Fontaine and Laffaille [FL82] first attacked this question by defining $W(k)$-lattices in filtered (φ, N)-modules. Unfortunately, their theory only works for the case $e = 1$, $N = 0$ and Hodge–Tate weights in $\{0, \dots, p-2\}$. In the late 1990s, Breuil introduced the theory of filtered (φ, N)-modules over S to study semi-stable Galois representations [Bre97, Bre98b, Bre99a], where S is the p-adic completion of divided power envelope of $W(k)[u]$ with respect to the ideal $(E(u))$, and $E(u)$ is the Eisenstein polynomial for a fixed uniformizer π of K. Breuil proved that the knowledge of filtered (φ, N)-modules over S is equivalent to that of filtered (φ, N)-modules (see Theorem 2.2.1 for the precise statement). Furthermore, it turns out that there are integral structures, strongly

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divisible lattices, which naturally live inside filtered (φ, N)-modules over S. These structures allow for arbitrary ramification of K/K_0. For a strongly divisible lattice $\mathcal{M}$, Breuil constructed a G-stable $\mathbb{Z}_p$-lattice $T_{\mathrm{st}}(\mathcal{M})$ in a semi-stable Galois representation and raised the following conjecture (the main conjecture in [Bre02]).

Conjecture 1.0.1 (Breuil's conjecture). *Fix a nonnegative integer $r \leqslant p-2$. The functor T_{st} establishes an anti-equivalence of categories between the category of strongly divisible lattices of weight r and the category of G-stable $\mathbb{Z}_p$-lattices in semi-stable representations of G with Hodge–Tate weights in $\{0, \dots, r\}$.*

If $r \leqslant 1$, the conjecture has been proved by Breuil in [Bre00] and [Bre02]. The case $e = 1$ was shown by Fontaine and Laffaille in [FL82] for crystalline representations. In [Bre99a], Breuil proved that there at least exists a strongly divisible lattice in the side of filtered (φ, N)-modules over S if $er < p-1$. Based on this result, Breuil [Bre99c] proved the case $e = 1$ for general semi-stable representations, and Caruso [Car05] proved Breuil's conjecture for $er < p-1$. Their ideas involve a weak version of Conjecture 1.0.1; see the end of § 2.3 for details. In [Fal99], Faltings proved that the restriction of T_{st} to the subcategory of *filtered free* strongly divisible lattices is *fully faithful*.

In this paper, we give a complete proof for the above conjecture by using results of Kisin [Kis06]. Let $K_\infty = \bigcup_{n \geqslant 1} K(\sqrt[p^n]{\pi})$, $G_\infty = \mathrm{Gal}(\bar{K}/K_\infty)$ and $\mathfrak{S} = W(k)[\![u]\!]$. We equip $\mathfrak{S}$ with the endomorphism φ which acts via Frobenius on $W(k)$, and sends u to u^p. Let $\mathrm{Mod}^{\varphi}_{/\mathfrak{S}}$ denote the category of finite free $\mathfrak{S}$-modules $\mathfrak{M}$ equipped with a φ-semi-linear map $\varphi_{\mathfrak{M}} : \mathfrak{M} \to \mathfrak{M}$ such that the cokernel of $\mathfrak{S}$-linear map $1 \otimes \varphi_{\mathfrak{M}} : \mathfrak{S} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M} \to \mathfrak{M}$ is killed by $E(u)^r$. In [Kis06], Kisin proved that any G_∞-stable $\mathbb{Z}_p$-lattice T in a semi-stable Galois representation comes from an object $(\mathfrak{M}, \varphi)$ in $\mathrm{Mod}^{\varphi}_{/\mathfrak{S}}$. Using the functor $\mathfrak{M} \rightsquigarrow S \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}$ provided by Breuil, Kisin's theory allows us to construct 'quasi-strongly divisible lattices', i.e. strongly divisible lattices without considering monodromy, to establish an anti-equivalence between the category of quasi-strongly divisible lattices and the category of G_∞-stable $\mathbb{Z}_p$-lattices in semi-stable Galois representations. Furthermore, we prove that a quasi-strongly divisible lattice is strongly divisible if and only if the corresponding G_∞-stable $\mathbb{Z}_p$-lattice is G-stable (see Theorem 3.5.4 for the more precise statement). Conjecture 1.0.1 then follows.

The paper proceeds as follows. In § 2, after briefly reviewing the theory of semi-stable p-adic Galois representations, filtered (φ, N)-modules over S and definition of (quasi-)strongly divisible lattices, we are then able to give a precise statement of our main theorem. Section 3 is devoted to reviewing Kisin's theory from [Kis06], which allows us to construct quasi-strongly divisible lattices and establishes an anti-equivalence between the category of quasi-strongly divisible lattices and the category of G_∞-stable $\mathbb{Z}_p$-lattices in semi-stable Galois representations; and the full faithfulness of T_{st} follows from this. In the next two sections, we prove that a quasi-strongly divisible lattice is strongly divisible if and only if the corresponding G_∞-stable $\mathbb{Z}_p$-lattice is G-stable. The idea is to use an extended version of Falting's theorem [Fal99, Theorem 5], The proof of such a theorem (Theorem 4.3.4) mainly depends on the construction of the Cartier dual for quasi-strongly divisible lattices from [Car05], which we discuss in § 4. In the last section, we combine our previous preparations to prove the essential surjectivity of T_{st}.

2. Preliminaries and the main result

This paper discusses lots of categories and functors. For the convenience of readers, we begin by summarizing their relations and our main results as the following diagram.

$$\begin{tikzcd}
\mathrm{Mod}_{/\mathcal{O}}^{\varphi,N\nabla} \arrow[r, "D", "\sim"'] \arrow[rr, bend left, "\mathcal{D}_{\mathcal{O}}", "\sim"'] & \mathrm{MF}(\varphi,N) \arrow[r, "\mathcal{D}", "\sim"'] & \mathcal{MF}(\varphi,N) \arrow[rr, "V_{\mathrm{cris}}"] & & \mathrm{Rep}_{\mathbb{Q}_p}(G_\infty) \\
\mathrm{Mod}_{/\mathfrak{S}}^{\varphi,N} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p & \mathrm{MF}^{\mathrm{w}}(\varphi,N) \arrow[l, "\Theta"'] \arrow[u, hook] \arrow[r, "\mathcal{D}", "\sim"'] & \mathcal{MF}^{\mathrm{w}}(\varphi,N) \arrow[u, hook] \arrow[r, "V_{\mathrm{st}}", "\sim"'] & \mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{st}}(G) \arrow[r] & \mathrm{Rep}_{\mathbb{Q}_p}(G_\infty) \arrow[u, equal] \\
\mathrm{Mod}_{/\mathfrak{S}}^{\varphi,N} \arrow[u] \arrow[dd] & & \mathrm{Mod}_{/S}^{\varphi,N} \arrow[u] \arrow[d] \arrow[r, "T_{\mathrm{st}}", "\sim"'] \arrow[dr, Rightarrow] & \mathrm{Rep}_{\mathbb{Z}_p}^{\mathrm{st}}(G) \arrow[u] \arrow[d] \arrow[r] & \mathrm{Rep}_{\mathbb{Z}_p}(G_\infty) \arrow[u] \arrow[d, equal] \\
& & \widetilde{\mathrm{Mod}}_{/S}^{\varphi} \arrow[d, hook] \arrow[r, "T_{\mathrm{cris}}", "\sim"'] & \mathrm{Rep}_{\mathbb{Z}_p}^{\mathrm{st}}(G_\infty) \arrow[r, hook] & \mathrm{Rep}_{\mathbb{Z}_p}(G_\infty) \arrow[d, equal] \\
\mathrm{Mod}_{/\mathfrak{S}}^{\varphi} \arrow[rr, hook, "\mathcal{M}_{\mathfrak{S}}"] \arrow[rrrr, bend right, hook, "T_{\mathfrak{S}}"'] & & \mathrm{Mod}_{/S}^{\varphi} \arrow[rr, "T_{\mathrm{cris}}"] & & \mathrm{Rep}_{\mathbb{Z}_p}(G_\infty)
\end{tikzcd}$$

Here is a general explanation of the above diagram.

(i) Injection arrows $\hookrightarrow$ symbolize fully faithful functors and $\xrightarrow[\sim]{}$ symbolizes equivalence or anti-equivalence. The notation $\mathrm{Rep}^{\mathrm{st}}$ symbolizes the categories of semi-stable representations with Hodge–Tate weights in $\{0,\ldots,r\}$. For example, $\mathrm{Rep}^{\mathrm{st}}_{\mathbb{Z}_p}(G_\infty)$ symbolizes the category of G_∞-stable $\mathbb{Z}_p$-lattices in semi-stable representations.

(ii) The main goal of this paper is to prove that T_{st} is an anti-equivalence. To achieve this, we first prove that T_{st} is fully faithful by showing that T_{cris} restricted to $\widetilde{\mathrm{Mod}}{}^{\varphi}_{/S}$ (the category of quasi-strongly divisible lattices) is an anti-equivalence in § 3, and then we prove the essential surjectivity of T_{st} in §§ 4 and 5.

(iii) The first column is about Kisin's theory on φ-modules over $\mathfrak{S}$. The second column is about classical modules in Fontaine's theory and the third about Breuil's theory on S-modules.These three theories can be connected by auxiliary categories in the first row (see the end of § 2.2, the end of § 3.1 and § 3.2). The last two columns are about the Galois sides. Note that representations of G_∞ (e.g. G_∞-stable $\mathbb{Z}_p$-lattices inside semi-stable representations) can be more conveniently described by Kisin's theory (see §§ 3.3 and 3.4).

(iv) The second row is about the theory over $\mathbb{Q}_p$ whereas the third row is about the theory over $\mathbb{Z}_p$, which also is the key result of this paper. Many important inputs depend on the last two rows which are about Kisin's theory (via $T_{\mathfrak{S}}$) and Breuil's theory (via T_{cris}) on $\mathbb{Z}_p$-representations of G_∞ (see §§ 3.3 and 3.4).

2.1 Semi-stable Galois representations and weakly admissible modules

Fix an odd prime p. Recall that a p-adic representation is a continuous linear representation of $G := \mathrm{Gal}(\bar{K}/K)$ on a finite dimensional $\mathbb{Q}_p$-vector space V and a p-adic representation V of G is called *semi-stable* [Fon94b] if

$$\dim_{K_0}(B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V)^G = \dim_{\mathbb{Q}_p} V, \tag{2.1.1}$$

where B_{st} is the period ring constructed by Fontaine; see for example [Fon94a] or § 2.2 for the construction.

In [CF00] and [Fon94b], Fontaine and Colmez give an alternative description of semi-stable p-adic representations. Recall that a filtered (φ, N)-module is a finite dimensional K_0-vector space D endowed with:

(1) a Frobenius semi-linear injection: $\varphi : D \to D$;

(2) a linear map $N : D \to D$ such that $N\varphi = p\varphi N$;

(3) a decreasing filtration $(\mathrm{Fil}^i D_K)_{i\in\mathbb{Z}}$ on $D_K := K \otimes_{K_0} D$ by K-vector spaces such that $\mathrm{Fil}^i D_K = D_K$ for $i \ll 0$ and $\mathrm{Fil}^i D_K = 0$ for $i \gg 0$.

If D is a one dimensional (φ, N)-module, and $v \in D$ is a basis vector, then $\varphi(v) = \alpha v$ for some $\alpha \in K_0$. We write $t_N(D)$ for the p-adic valuation of α (p-adic valuation of α does not depend on choice of v) and $t_H(D)$ for the unique integer i such that $\mathrm{gr}^i D_K$ is nonzero. If D has dimension $d > 1$, then we write $t_N(D) = t_N(\wedge^d D)$ and $t_H(D) = t_H(\wedge^d D)$. Recall that a filtered (φ, N)-module is called *weakly admissible* if $t_H(D) = t_N(D)$ and for any (φ, N)-submodule $D' \subset D$, $t_H(D') \leqslant t_N(D')$, where $D'_K \subset D_K$ is equipped with the induced filtration.

The aforementioned result of Colmez and Fontaine [CF00] is that the functor

$$D_{\mathrm{st},*} : V \to (B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V)^G$$

establishes an equivalence of categories between the category of semi-stable p-adic representations of G and the category of weakly admissible filtered (φ, N)-modules.

In the sequel, we will instead use the contravariant functor $D_{\mathrm{st}}(V) := D_{\mathrm{st},*}(V^\vee)$, where $V^\vee$ is the dual representation of V. The advantage of this is that the Hodge–Tate weights of V are exactly the $i \in \mathbb{Z}$ such that $\mathrm{gr}^i D_{\mathrm{st}}(V)_K \neq 0$. A quasi-inverse to D_{st} is then given by

$$V_{\mathrm{st}}(D) := \mathrm{Hom}_{\varphi,N}(D, B_{\mathrm{st}}) \cap \mathrm{Hom}_{\mathrm{Fil}^\cdot}(D_K, K \otimes_{K_0} B_{\mathrm{st}}). \tag{2.1.2}$$

Convention 2.1.1. Here we use slightly different notation from [Bre02] and [CF00]: D_{st} here is D^*_{st} in [Bre02] and [CF00]; V_{st} here is V^*_{st} in [Bre02] and [CF00]. Also we will use T_{st} to denote T^*_{st} in [Bre02] and [Bre99a] later. The reason for using such notation is that we will always use *contravariant* functors instead of covariant functors in this paper. Removing '$*$' from the superscript looks more neat and convenient.

A filtered (φ, N)-module is called *positive* if $\mathrm{Fil}^0 D = D$. In this paper, we only consider positive filtered (φ, N)-modules. We denote the category of positive filtered (φ, N)-modules by $\mathrm{MF}(\varphi, N)$ and the category of positive weakly admissible filtered (φ, N)-modules by $\mathrm{MF}^{\mathrm{w}}(\varphi, N)$.

2.2 Breuil's theory on filtered (φ, N)-modules over S

Throughout the paper we will fix a uniformizer $\pi \in \mathcal{O}_K$, and $E(u) \in W(k)[u]$ the Eisenstein polynomial of π. We denote by S the p-adic completion of the divided power envelope of $W(k)[u]$ with respect to $\mathrm{Ker}(s)$, where $s : W(k)[u] \to \mathcal{O}_K$ is the canonical surjection by sending u to π. For any positive integer i, let $\mathrm{Fil}^i S \subset S$ be the p-adic closure of the ideal generated by the divided powers $\gamma_j(u) = E(u)^j/j!$ for all $j \geqslant i$. There is a unique continuous map $\varphi : S \to S$ which extends the Frobenius on $W(k)$ and satisfies $\varphi(u) = u^p$. We define a continuous $W(k)$-linear derivation $N : S \to S$ such that $N(u) = -u$. It is easy to check that $N\varphi = p\varphi N$ and $\varphi(\mathrm{Fil}^i S) \subset p^i S$ for $0 \leqslant i \leqslant p-1$, and we write $\varphi_i = p^{-i}\varphi|_{\mathrm{Fil}^i S}$ and $c_1 = \varphi_1(E(u))$. Note that c_1 is a unit in S. Finally, we put $S_{K_0} := S \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ and $\mathrm{Fil}^i S_{K_0} := \mathrm{Fil}^i S \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$.

Let $\mathcal{MF}(\varphi, N)$ be a category whose objects are finite free S_{K_0}-modules $\mathcal{D}$ with:

(i) a $\varphi_{S_{K_0}}$-semi-linear morphism $\varphi_{\mathcal{D}} : \mathcal{D} \to \mathcal{D}$ such that the determinant of $\varphi_{\mathcal{D}}$ is invertible in S_{K_0} (the invertibility of the determinant does not depend on the choice of basis);

(ii) a decreasing filtration over $\mathcal{D}$ of S_{K_0}-modules, i.e. $\mathrm{Fil}^i(\mathcal{D})$, $i \in \mathbb{Z}$, such that $\mathrm{Fil}^0(\mathcal{D}) = \mathcal{D}$ and that $\mathrm{Fil}^i S_{K_0} \mathrm{Fil}^j(\mathcal{D}) \subset \mathrm{Fil}^{i+j}(\mathcal{D})$;

(iii) a K_0-linear map (monodromy) $N : \mathcal{D} \to \mathcal{D}$ such that

(1) for all $f \in S_{K_0}$ and $m \in \mathcal{D}$, $N(fm) = N(f)m + fN(m)$,

(2) $N\varphi = p\varphi N$,
(3) $N(\mathrm{Fil}^i\mathcal{D}) \subset \mathrm{Fil}^{i-1}(\mathcal{D})$.

We call objects in $\mathcal{MF}(\varphi, N)$ *filtered* (φ, N)*-modules over* S. Let $D \in \mathrm{MF}(\varphi, N)$ be a filtered (φ, N)-module. We can associate an object $\mathcal{D} \in \mathcal{MF}(\varphi, N)$ by the following:

$$\mathcal{D} := S \otimes_{W(k)} D \tag{2.2.1}$$

and

- $\varphi := \varphi_S \otimes \varphi_D : \mathcal{D} \to \mathcal{D}$,
- $N := N \otimes \mathrm{Id} + \mathrm{Id} \otimes N : \mathcal{D} \to \mathcal{D}$,
- $\mathrm{Fil}^0(\mathcal{D}) := \mathcal{D}$ and by induction:
$$\mathrm{Fil}^{i+1}\mathcal{D} := \{x \in \mathcal{D} \mid N(x) \in \mathrm{Fil}^i\mathcal{D} \text{ and } f_\pi(x) \in \mathrm{Fil}^{i+1}D_K\},$$
where $f_\pi : \mathcal{D} \twoheadrightarrow D_K$ is defined by $\lambda \otimes x \mapsto s(\lambda)x$.

For a $\mathcal{D} \in \mathcal{MF}(\varphi, N)$, Breuil associated a $\mathbb{Q}_p[G]$-module $V_{\mathrm{st}}(\mathcal{D})$. Several period rings have to be defined before we can describe this functor. Let $R = \varprojlim \mathcal{O}_{\bar{K}}/p$, where the transition maps are given by Frobenius. By the universal property of Witt vectors $W(R)$ of R, there is a unique surjective map $\theta : W(R) \to \widehat{\mathcal{O}_{\bar{K}}}$ to the p-adic completion $\widehat{\mathcal{O}_{\bar{K}}}$, which lifts the projection $R \to \mathcal{O}_{\bar{K}}/p = \widehat{\mathcal{O}_{\bar{K}}}/p$ onto the first factor in the inverse limit. We denote by A_{cris} the p-adic completion of the divided power envelope of $W(R)$ with respect to the $\mathrm{Ker}(\theta)$, and write $B^+_{\mathrm{cris}} := A_{\mathrm{cris}}[1/p]$.

For each $n \geqslant 0$, fix $\pi_n \in \bar{K}$ a p^nth root of π such that $\pi^p_{n+1} = \pi_n$. Write $\underline{\pi} = (\pi_n)_{n \geqslant 0} \in R$, and let $[\underline{\pi}] \in W(R)$ be the Teichmüller representation. We embed the $W(k)$-algebra $W(k)[u]$ into $W(R)$ by $u \mapsto [\underline{\pi}]$. Since $\theta([\underline{\pi}]) = \pi$ this embedding extends to an embedding $S \hookrightarrow A_{\mathrm{cris}}$, and $\theta|_S$ is the map $s : S \to \mathcal{O}_K$ sending u to π. The embedding is compatible with Frobenius endomorphisms. As usual, we denote by B^+_{st} the ring obtained by formally adjoining the element '$\log[\underline{\pi}]$' to B^+_{cris}, and by B^+_{dR} the $\mathrm{Ker}(\theta)$-adic completion of $W(R)[1/p]$. Choose a generator t of $\mathbb{Z}_p(1) \subset A_{\mathrm{cris}}$. Such t can be constructed by $t := \log([\epsilon])$ for $\epsilon = (\epsilon_i)_{i \geqslant 0} \in R$, where ϵ_i is a primitive p^ith root of unity such that $\epsilon^p_{i+1} = \epsilon_i$. We denote $B^+_{\mathrm{st}}[1/t]$ by B_{st}.

Let $\widehat{A_{\mathrm{st}}}$ be the p-adic completion of the divided power polynomial algebra $A_{\mathrm{cris}}\langle X\rangle$. We endow $\widehat{A_{\mathrm{st}}}$ with a continuous G-action, a Frobenius φ, a monodromy operator N and positive filtration Fil^i as the following.

For any $g \in G$, let $\underline{\epsilon}(g) = g([\underline{\pi}])/[\underline{\pi}] \in A_{\mathrm{cris}}$. We extend the natural G-action and Frobenius on A_{cris} to $\widehat{A_{\mathrm{st}}}$ by putting $g(X) = \underline{\epsilon}(g)X + \underline{\epsilon}(g) - 1$ and $\varphi(X) = (1 + X)^p - 1$. We define a monodromy operator N on $\widehat{A_{\mathrm{st}}}$ to be a unique A_{cris}-linear derivation such that $N(X) = 1 + X$. For any $i \geqslant 0$, we define

$$\mathrm{Fil}^i\widehat{A_{\mathrm{st}}} = \left\{\sum_{j=0}^{\infty} a_j\gamma_j(X),\ a_j \in A_{\mathrm{cris}},\ \lim_{j\to\infty} a_j = 0,\ a_j \in \mathrm{Fil}^{i-j}A_{\mathrm{cris}},\ 0 \leqslant j \leqslant i\right\}.$$

Finally, [Bre97, § 4.2], we have an isomorphism $S \xrightarrow{\sim} (\widehat{A_{\mathrm{st}}})^G$ compatible with all structures given by $u \mapsto [\underline{\pi}](1 + X)^{-1}$. Therefore, $\widehat{A_{\mathrm{st}}}$ is an S-algebra.

For any $\mathcal{D} \in \mathcal{MF}(\varphi, N)$, one can associate a $\mathbb{Q}_p[G]$-module

$$V_{\mathrm{st}}(\mathcal{D}) := \mathrm{Hom}_{S,\mathrm{Fil}^\cdot,\varphi,N}(\mathcal{D}, \widehat{A_{\mathrm{st}}}[1/p]).$$

The following theorem is one of main results in [Bre97].

Theorem 2.2.1 (Breuil). *The functor $\mathcal{D} : D \to S \otimes_{W(k)} D$ defined in (and below) (2.2.1) induces an equivalence between the category $\mathrm{MF}(\varphi, N)$ and $\mathcal{MF}(\varphi, N)$ and there is a natural isomorphism $V_{\mathrm{st}}(D) \simeq V_{\mathrm{st}}(\mathcal{D})$ as $\mathbb{Q}_p[G]$-modules.*

From now on, we always identify $V_{\mathrm{st}}(D)$ with $V_{\mathrm{st}}(\mathcal{D})$ as the same Galois representations, and denote $\mathcal{MF}^{\mathrm{w}}(\varphi, N)$ the essential image of $\mathcal{D}$ restricted to $\mathrm{MF}^{\mathrm{w}}(\varphi, N)$.

2.3 The Main Theorem

Theorem 2.2.1 shows that the knowledge of filtered (φ, N)-modules over S is equivalent to that of filtered (φ, N)-modules. It turns out that integral structures can be more conveniently defined inside filtered (φ, N)-modules over S. However, when working on integral p-adic Hodge theory via S-modules, the following technical restriction always has to be assumed.

Assumption 2.3.1. Fix a positive integer $r \leqslant p-2$. The filtration on the weakly admissible filtered (φ, N)-module D is such that $\mathrm{Fil}^0 D_K = D_K$ and $\mathrm{Fil}^{r+1} D_K = 0$. Equivalently, the Hodge–Tate weights of the semi-stable p-adic Galois representation under consideration are always contained in $\{0, \dots, r\}$.

Remark 2.3.2. (1) Conjecture 1.0.1 has been proved for $r = 0$ in [Bre02, § 3.1]. So we only consider the case $r > 0$ from now on ($r = 0$ will cause a little trouble only in the end).

(2) Up to the twist of the (φ, N)-module of a power of the cyclotomic character, all modules whose filtration length does not exceed r satisfy the above assumption.

Following [Bre02, § 2.2], we define the integral structures inside $\mathcal{D}$ to correspond to the Galois stable $\mathbb{Z}_p$-lattices.

Definition 2.3.3. Let D be a weakly admissible filtered (φ, N)-module satisfying Assumption 2.3.1 and $\mathcal{D} := \mathcal{D}(D) \in \mathcal{MF}^{\mathrm{w}}(\varphi, N)$. A *quasi-strongly divisible lattice of weight* r in $\mathcal{D}$ is an S-submodule $\mathcal{M}$ of $\mathcal{D}$ such that:

(1) $\mathcal{M}$ is S-finite free and $\mathcal{M}[\frac{1}{p}] \xrightarrow{\sim} \mathcal{D}$;
(2) $\mathcal{M}$ is stable under φ, i.e. $\varphi(\mathcal{M}) \subset \mathcal{M}$;
(3) $\varphi(\mathrm{Fil}^r \mathcal{M}) \subset p^r \mathcal{M}$ where $\mathrm{Fil}^r \mathcal{M} := \mathcal{M} \cap \mathrm{Fil}^r \mathcal{D}$.

A *strongly divisible lattice of weight* r in $\mathcal{D}$ is a quasi-strongly divisible lattice $\mathcal{M}$ in $\mathcal{D}$ such that $N(\mathcal{M}) \subset \mathcal{M}$.

It will be more convenient and explicit to describe the category of (quasi-)strongly divisible lattices by projective limits of torsion objects. Let $'\mathrm{Mod}_{/S}^{\varphi, N}$ denote the category whose objects are 4-tuples $(\mathcal{M}, \mathrm{Fil}^r \mathcal{M}, \varphi_r, N)$, consisting of:

(1) an S-module $\mathcal{M}$;
(2) an S-submodule $\mathrm{Fil}^r \mathcal{M} \subset \mathcal{M}$ containing $\mathrm{Fil}^r S \cdot \mathcal{M}$;
(3) a φ-semi-linear map $\varphi_r : \mathrm{Fil}^r \mathcal{M} \to \mathcal{M}$ such that for all $s \in \mathrm{Fil}^r S$ and $x \in \mathcal{M}$ we have $\varphi_r(sx) = (c_1)^{-r} \varphi_r(s) \varphi_r(E(u)^r x)$;
(4) a $W(k)$-linear morphism $N : \mathcal{M} \to \mathcal{M}$ such that
   (a) for all $s \in S$ and $x \in \mathcal{M}$, $N(sx) = N(s)x + sN(x)$,
   (b) $E(u) N(\mathrm{Fil}^r \mathcal{M}) \subset \mathrm{Fil}^r \mathcal{M}$,
   (c) the following diagram commutes.

$$\begin{tikzcd} \mathrm{Fil}^r \mathcal{M} \arrow[r, "\varphi_r"] \arrow[d, "E(u)N"'] & \mathcal{M} \arrow[d, "c_1 N"] \\ \mathrm{Fil}^r \mathcal{M} \arrow[r, "\varphi_r"] & \mathcal{M} \end{tikzcd} \tag{2.3.1}$$

Morphisms are given by S-linear maps preserving the Fil and commuting with φ_r and N. A sequence is defined to be *short exact* if it is short exact as a sequence of S-module, and induces a short exact sequence on the Fil.

We denote by $'\mathrm{Mod}^{\varphi}_{/S}$ the category which forgets the operation N in the definition of $'\mathrm{Mod}^{\varphi,N}_{/S}$. Objects in $'\mathrm{Mod}^{\varphi}_{/S}$ are called *filtered φ-modules over S*. Let $\mathrm{Mod}\ \mathrm{FI}^{\varphi,N}_{/S}$ (respectively $\mathrm{Mod}\ \mathrm{FI}^{\varphi}_{/S}$) be the full subcategory of $'\mathrm{Mod}^{\varphi,N}_{/S}$ (respectively $'\mathrm{Mod}^{\varphi}_{/S}$) consisting of objects such that:

(1) as an S-module $\mathcal{M}$ is isomorphic to $\bigoplus_{i\in I} S/p^{n_i}S$, where I is a finite set and n_i is a positive number;

(2) $\varphi_r(\mathcal{M})$ generates $\mathcal{M}$ over S.

Finally we denote by $\mathrm{Mod}^{\varphi,N}_{/S}$ (respectively $\mathrm{Mod}^{\varphi}_{/S}$) the full subcategory of $'\mathrm{Mod}^{\varphi,N}_{/S}$ (respectively $'\mathrm{Mod}^{\varphi}_{/S}$) such that $\mathcal{M}$ is a finite free S-module and, for all n,

$$(\mathcal{M}_n, \mathrm{Fil}^r\mathcal{M}_n, \varphi_r, N) \in \mathrm{Mod}\ \mathrm{FI}^{\varphi,N}_{/S} \quad (\text{respectively } (\mathcal{M}_n, \mathrm{Fil}^r\mathcal{M}_n, \varphi_r) \in \mathrm{Mod}\ \mathrm{FI}^{\varphi}_{/S}),$$

where $\mathcal{M}_n = \mathcal{M}/p^n\mathcal{M}$, $\mathrm{Fil}^r\mathcal{M}_n = \mathrm{Fil}^r\mathcal{M}/p^n\mathrm{Fil}^r\mathcal{M}$, and φ_r, N are induced by modulo p^n.

Note that $\widehat{A_{\mathrm{st}}} \in {'\mathrm{Mod}^{\varphi,N}_{/S}}$. For any $\mathcal{M} \in \mathrm{Mod}^{\varphi,N}_{/S}$, define

$$T_{\mathrm{st}}(\mathcal{M}) := \mathrm{Hom}_{'\mathrm{Mod}^{\varphi,N}_{/S}}(\mathcal{M}, \widehat{A_{\mathrm{st}}}).$$

Proposition 2.3.4 (Breuil).

(1) *If $\mathcal{M}$ is a quasi-strongly divisible lattice in $\mathcal{D}$ with $\mathcal{D} \in \mathcal{MF}^{\mathrm{w}}(\varphi, N)$, then $(\mathcal{M}, \mathrm{Fil}^r\mathcal{M}, \varphi_r)$ is in $\mathrm{Mod}^{\varphi}_{/S}$ where $\varphi_r := \varphi/p^r$.*

(2) *The category of strongly divisible lattices of weight r is just $\mathrm{Mod}^{\varphi,N}_{/S}$. In particular, for any $\mathcal{M} \in \mathrm{Mod}^{\varphi,N}_{/S}$, there exists a $D \in \mathrm{MF}^{\mathrm{w}}(\varphi, N)$ such that $\mathcal{D}(D) \simeq \mathcal{M} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ as filtered (φ, N)-modules over S. Furthermore, $T_{\mathrm{st}}(\mathcal{M})$ is a G-stable $\mathbb{Z}_p$-lattice in $V_{\mathrm{st}}(D)$.*

Proof. Part (1) is the consequence of Proposition 2.1.3 in [Bre99a]. Note that, though Proposition 2.1.3 only deals with strongly divisible modules, the proof does not need monodromy at all. So the same assertion is valid for quasi-strongly divisible lattices. Part (2) is Theorem 2.2.3 and Proposition 2.2.5 in [Bre02]. ☐

From now on, we use $\mathrm{Mod}^{\varphi,N}_{/S}$ to denote the category of strongly divisible lattices of weight r and regard $\widetilde{\mathrm{Mod}}^{\varphi}_{/S}$ as a full subcategory of $\mathrm{Mod}^{\varphi}_{/S}$, where $\widetilde{\mathrm{Mod}}^{\varphi}_{/S}$ denotes the category of quasi-strongly divisible lattices. Now we can state our Main Theorem.

Theorem 2.3.5 (Main Theorem). *If $0 \leqslant r \leqslant p-2$, the functor $\mathcal{M} \to T_{\mathrm{st}}(\mathcal{M})$ establishes an anti-equivalence of categories between the category of strongly divisible lattices of weight r and the category of G-stable $\mathbb{Z}_p$-lattices in semi-stable p-adic Galois representations with Hodge–Tate weights in $\{0, \dots, r\}$.*

Remark 2.3.6. In fact, there exists a weak version of Conjecture 1.0.1. Fix a $\mathcal{D}$ inside $\mathcal{MF}^{\mathrm{w}}(\varphi, N)$. Consider the restriction of the functor T_{st}, namely,

$$T_{\mathrm{st}}|_{\mathcal{D}} : \{\text{strongly divisible lattices in } \mathcal{D}\} \to \{G\text{-stable } \mathbb{Z}_p\text{-lattices in } V_{\mathrm{st}}(\mathcal{D})\}.$$

The weak version claims that all functors $T_{\mathrm{st}}|_{\mathcal{D}}$ are equivalences. It is obvious that Conjecture 1.0.1 implies the weak one. On the other hand, from the weak version, one can deduce the essentially surjectivity of T_{st}. Therefore, if the full faithfulness of T_{st} has been known, then the weak version and the strong version are equivalent. [Car05] and [Bre98a] used this ideal to prove some special cases of Conjecture 1.0.1.

3. Construction of quasi-strongly divisible lattices

Let T be a G-stable $\mathbb{Z}_p$-lattice in a semi-stable Galois representation V with Hodge–Tate weights in $\{0, \ldots, r\}$. In this section, we will use the theory from [Kis06] to prove that there exists a quasi-strongly divisible lattice $\mathcal{M} \in \mathrm{Mod}^{\varphi}_{/S}$ to correspond to $T|_{G_\infty}$. As we will see later, $\mathcal{M}$ provides the ambient module for the strongly divisible lattice corresponding to T.

3.1 (φ, N_∇)-modules

We equip $K_0[\![u]\!]$ with the endomorphism $\varphi : K_0[\![u]\!] \to K_0[\![u]\!]$ which acts via the Frobenius on K_0, and sends u to u^p. Suppose that $I \subset [0, 1)$ is a subinterval. We set $\mathcal{O}_I$ the subring of $K_0[\![u]\!]$ whose elements converge for all $x \in \bar{K}$ such that $|x| \in I$. Put $\mathcal{O} = \mathcal{O}_{[0,1)}$. By [Bre97, Lemma 2.1], S can be identified as the subring of $K_0[\![u]\!]$ whose elements have the form

$$\sum_{n=0}^{\infty} w_i \frac{u^i}{q(i)!}, \quad w_i \in W(k), \ \lim_{i\to\infty} w_i = 0, \tag{3.1.1}$$

where $q(i)$ is the quotient in the Euclidean division of i by e. Therefore, for any real number μ satisfying $p^{-1/((p-1)e)} < \mu \leqslant 1$, we have natural inclusions $\mathfrak{S}[1/p] \hookrightarrow \mathcal{O}_{[0,\mu)} \hookrightarrow S_{K_0}$ compatible with Frobenius. Set $c_0 = E(0)/p \in K_0$ and

$$\lambda = \prod_{n=0}^{\infty} \varphi^n(E(u)/pc_0) \in \mathcal{O}.$$

We define a derivation $N_\nabla := -u\lambda\, d/du : \mathcal{O} \to \mathcal{O}$ and denote by the same symbol the induced derivation $\mathcal{O}_I \to \mathcal{O}_I$, for each $I \subset [0, 1)$.

By a φ-module over $\mathcal{O}$ we mean a finite free $\mathcal{O}$-module M, equipped with a φ-semi-linear, injective map $\varphi : M \to M$. A *(φ, N_∇)-module over $\mathcal{O}$* is a φ-module M over $\mathcal{O}$, together with a differential operator N^M_∇ over N_∇. That is, for any $f \in \mathcal{O}$ and $m \in M$, we have

$$N^M_\nabla(fm) = N_\nabla(f)m + fN^M_\nabla(m).$$

Here φ and N^M_∇ are required to satisfy the relation $N^M_\nabla\varphi = (1/c_0)E(u)\varphi N^M_\nabla$. We will usually write N_∇ for N^M_∇ if this will cause no confusion. The category of (φ, N_∇)-modules over $\mathcal{O}$ has a natural structure of a Tannakian category. We denote by $\mathrm{Mod}^{\varphi,N_\nabla}_{/\mathcal{O}}$ the category of (φ, N_∇)-modules M of height r, in the sense that the cokernel of $1 \otimes \varphi : \varphi^* M \to M$ is killed by $E(u)^r$ for our fixed positive integer r, where $\varphi^* M := \mathcal{O} \otimes_{\varphi,\mathcal{O}} M$.

In [Kis06, § 1.2], Kisin constructed a functor $D : \mathrm{Mod}^{\varphi,N_\nabla}_{/\mathcal{O}} \to \mathrm{MF}(\varphi, N)$. Let M be an object in $\mathrm{Mod}^{\varphi,N_\nabla}_{/\mathcal{O}}$. Define the underlying K_0-vector space of $D(M)$ as M/uM, and the operator φ and N are induced by φ, N_∇ on M. The construction of filtration on $D(M)$ is somewhat not straightforward. First we define a decreasing filtration on $\varphi^* M$ by

$$\mathrm{Fil}^i \varphi^* M := \{x \in \varphi^* M \mid 1 \otimes \varphi(x) \in E(u)^i M\}.$$

Fix any real number μ such that $p^{-1/e} < \mu < p^{-1/pe}$. Lemma 1.2.6 in [Kis06] showed that there exists a unique $\mathcal{O}_{[0,\mu)}$-linear, φ-equivariant isomorphism

$$\xi : D(M) \otimes_{K_0} \mathcal{O}_{[0,\mu)} \xrightarrow{\sim} \varphi^* M \otimes_{\mathcal{O}} \mathcal{O}_{[0,\mu)}. \tag{3.1.2}$$

The required filtration on $D(M)_K$ is defined to be the image filtration under the composite

$$D(M) \otimes_{K_0} \mathcal{O}_{[0,\mu)} \to D(M) \otimes_{K_0} \mathcal{O}/E(u)\mathcal{O} \xrightarrow{\sim} D(M) \otimes_{K_0} K = D(M)_K.$$

Theorem 1.2.8 in [Kis06] shows that the functor D induces an exact equivalence between the category $\mathrm{Mod}^{\varphi,N_\nabla}_{/\mathcal{O}}$ and $\mathrm{MF}(\varphi, N)$.

3.2 A functor from $\mathrm{Mod}_{/\mathcal{O}}^{\varphi, N_\nabla}$ to $\mathcal{MF}(\varphi, N)$

Combining the functor D in § 3.1 with the functor $\mathcal{D}$ in § 2.2 together, we obtain a functor $\mathcal{D} \circ D$ from $\mathrm{Mod}_{/\mathcal{O}}^{\varphi, N_\nabla}$ to $\mathcal{MF}(\varphi, N)$. It will be convenient to give another description of $\mathcal{D} \circ D$ for later use.

Let M be an object in $\mathrm{Mod}_{/\mathcal{O}}^{\varphi, N_\nabla}$. Define $\mathcal{D}_{\mathcal{O}}(M) := S_{K_0} \otimes_{\varphi, \mathcal{O}} M$, a $\varphi_{S_{K_0}}$-semi-linear endomorphism $\varphi_{\mathcal{D}_{\mathcal{O}}(M)} := \varphi_{S_{K_0}} \otimes \varphi_M$ (as usual, we will drop the subscript of $\varphi_{\mathcal{D}_{\mathcal{O}}(M)}$ if no confusion will arise) and decreasing filtration on $\mathcal{D}_{\mathcal{O}}(M)$ by

$$\mathrm{Fil}^i(\mathcal{D}_{\mathcal{O}}(M)) := \{m \in \mathcal{D}_{\mathcal{O}}(M) \mid (1 \otimes \varphi)(m) \in \mathrm{Fil}^i S_{K_0} \otimes_{\mathcal{O}} M\}. \tag{3.2.1}$$

Note that $\varphi(\lambda)$ is a unit in S_{K_0}, and we can define N on $\mathcal{D}_{\mathcal{O}}(M)$ by

$$N := N \otimes 1 + \frac{p}{\varphi(\lambda)} 1 \otimes N_\nabla.$$

We can naturally extend N_∇ from $\mathcal{O}$ to S_{K_0}. Note that for any $f \in S_{K_0}$ we have

$$N(\varphi(f)) = \frac{p}{\varphi(\lambda)} \varphi(N_\nabla(f)).$$

Thus it is easy to check that N is a well-defined derivation of $\mathcal{D}_{\mathcal{O}}(M)$ over the derivation N of S_{K_0} defined by $N(u) = -u$.

Proposition 3.2.1. *The derivation N is well defined on $\mathcal{D}_{\mathcal{O}}(M)$ and $(\mathcal{D}_{\mathcal{O}}(M), \varphi, \mathrm{Fil}^i, N)$ is an object in $\mathcal{MF}(\varphi, N)$.*

Proof. Let $\mathcal{D} = \mathcal{D}_{\mathcal{O}}(M)$. We check that Frobenius, filtration and monodromy defined on $\mathcal{D}$ satisfy the required properties listed in § 2.2.

Since $E(u)^r$ kills the cokernel of $1 \otimes \varphi : \mathcal{O} \otimes_{\varphi, \mathcal{O}} M \to M$, we see that the determinant of φ_M is a divisor of $E(u)^{rd}$, where d is the $\mathcal{O}$-rank of M. Thus the determinant of $\varphi_{\mathcal{D}}$ is a divisor of $\varphi(E(u))^{rd} = p^{rd} c_1^{rd}$, and therefore is invertible in S_{K_0}. Using (3.2.1), one easily checks that $\mathrm{Fil}^i S_{K_0} \cdot \mathrm{Fil}^j \mathcal{D} \subset \mathrm{Fil}^{i+j} \mathcal{D}$. Now it suffices to check that the monodromy N satisfies the required properties.

To see that $N\varphi = p\varphi N$, for any $s \in S_{K_0}$ and $m \in M$, we have

$$\begin{aligned} N\varphi(s \otimes m) &= N(\varphi_{S_{K_0}}(s) \otimes \varphi_M(m)) \\ &= N(\varphi_{S_{K_0}}(s)) \otimes \varphi_M(m) + \frac{p}{\varphi(\lambda)} \varphi_{S_{K_0}}(s) \otimes N_\nabla(\varphi_M(m)) \\ &= p\varphi_{S_{K_0}}(N(s)) \otimes \varphi_M(m) + \frac{p}{\varphi(\lambda)} \frac{\varphi(E(u))}{\varphi(c_0)} \varphi_{S_{K_0}}(s) \otimes \varphi_M(N_\nabla(m)) \\ &= p\varphi_{\mathcal{D}}\bigg(N(s) \otimes m + \frac{p}{\varphi(\lambda)} s \otimes N_\nabla(m)\bigg) \\ &= p\varphi(N(s \otimes m)). \end{aligned}$$

To check that $N(\mathrm{Fil}^i \mathcal{D}) \subset \mathrm{Fil}^{i-1} \mathcal{D}$, note that

$$N_\nabla(E(u)^i) = -uiE(u)^{i-1} E'(u)\lambda = E(u)^i \bigg(-uiE'(u) \frac{\varphi(\lambda)}{pc_0}\bigg).$$

Thus $N_\nabla(\mathrm{Fil}^i S_{K_0} \otimes_{\mathcal{O}} M) \subset \mathrm{Fil}^i S_{K_0} \otimes_{\mathcal{O}} M$. Now let $x = \sum_i s_i \otimes m_i \in \mathrm{Fil}^i \mathcal{D}$. We claim that

$$E(u)(1 \otimes \varphi_M)(N(x)) = \frac{c_0 p}{\varphi(\lambda)} N_\nabla((1 \otimes \varphi_M)(x)). \tag{3.2.2}$$

In fact, since

$$E(u)N = \frac{c_0 p}{\varphi(\lambda)} N_\nabla \quad \text{and} \quad N_\nabla \varphi = \frac{E(u)}{c_0} \varphi N_\nabla,$$

we have

$$
\begin{aligned}
E(u)(1\otimes\varphi_M)(N(x)) &= E(u)\bigg(\sum_i N(s_i)\otimes\varphi_M(m_i)+\frac{p}{\varphi(\lambda)}s_i\otimes\varphi_M(N_\nabla(m_i))\bigg)\\
&= \frac{c_0 p}{\varphi(\lambda)}\bigg(\sum_i N_\nabla(s_i)\otimes\varphi_M(m_i)+s_i\otimes N_\nabla(\varphi_M(m_i))\bigg)\\
&= \frac{c_0 p}{\varphi(\lambda)}N_\nabla\bigg(\sum_i s_i\otimes\varphi_M(m_i)\bigg).
\end{aligned}
$$

This proves the claim (3.2.2). Finally, to prove that $N(x)\in \mathrm{Fil}^{i-1}\mathcal{D}$, it suffices to show that $(1\otimes\varphi_M)(N(x))\in \mathrm{Fil}^{i-1}S_{K_0}\otimes_{\mathcal{O}} M$. But (3.2.2) has shown us that

$$E(u)(1\otimes\varphi_M)(N(x))\in \mathrm{Fil}^i S_{K_0}\otimes_{\mathcal{O}} M.$$

Then we reduce our proof to the following lemma. ◻

Lemma 3.2.2. *Let $x\in S$ (respectively A_{cris}). If $E(u)^j x\in \mathrm{Fil}^{j+i}S$ (respectively $E([\underline{\pi}])^j x\in \mathrm{Fil}^{j+i}A_{\mathrm{cris}}$) then $x\in\mathrm{Fil}^i S$ (respectively $x\in\mathrm{Fil}^iA_{\mathrm{cris}}$).*

Proof. We have a natural embedding

$$S\overset{u\to[\underline{\pi}]}{\hookrightarrow} A_{\mathrm{cris}}\hookrightarrow B^+_{\mathrm{dR}}$$

with respect to filtration. By definition, $\mathrm{Fil}^n B^+_{\mathrm{dR}}=E([\underline{\pi}])^n B^+_{\mathrm{dR}}$ for all $n\geqslant 0$. Thus, if $E([\underline{\pi}])^j x\in \mathrm{Fil}^{i+j}B^+_{\mathrm{dR}}$ then $x\in\mathrm{Fil}^i B^+_{\mathrm{dR}}$, as required. ◻

Corollary 3.2.3. *The following equivalences of categories commute.*

$$
\begin{tikzcd}
\mathrm{MF}(\varphi,N) \arrow[r, "\mathcal{D}"] & \mathcal{MF}(\varphi,N)\\
\mathrm{Mod}^{\varphi,N_\nabla}_{/\mathcal{O}} \arrow[u, "D"] \arrow[ur, "\mathcal{D}_{\mathcal{O}}"] &
\end{tikzcd}
$$

Proof. Let $M\in\mathrm{Mod}^{\varphi,N_\nabla}_{/\mathcal{O}}$ and $\mathcal{D}=\mathcal{D}_{\mathcal{O}}(M)$. Proposition 3.2.1 has shown that $\mathcal{D}_{\mathcal{O}}(M)\in\mathcal{MF}(\varphi,N)$. By Theorem 2.2.1, there exists a unique $D\in\mathrm{MF}(\varphi,N)$ such that $\mathcal{D}_{\mathcal{O}}(M)=\mathcal{D}(D)$. It suffices to check that $D\simeq D(M)$. There exists an isomorphism $i_S: S_{K_0}\otimes_{\varphi,\mathcal{O}}M\simeq D\otimes_{K_0}S_{K_0}$ in $\mathcal{MF}(\varphi,N)$. Modulo u both sides, we get a K_0-linear isomorphism $i: D(M)\simeq D$. It is obvious that i is compatible with φ and N structures on both sides. To see that i is compatible with filtration, recall that the filtration on $D(M)$ depends on the construction of the unique $\mathcal{O}_{[0,\mu)}$-linear, φ-equivariant morphism ξ in (3.1.2):

$$\xi: D(M)\otimes_{K_0}\mathcal{O}_{[0,\mu)}\overset{\sim}{\to}\varphi^*M\otimes_{\mathcal{O}}\mathcal{O}_{[0,\mu)},$$

where μ is any fixed real number such that $p^{-1/e}<\mu<p^{-1/pe}$. Choose μ such that $p^{-1/((p-1)e)}<\mu<p^{-1/pe}$. By (3.1.1), $\mathcal{O}_{[0,\mu)}$ is a subring of S_{K_0}. Then we have an isomorphism

$$\varphi^*M\otimes_{\mathcal{O}}\mathcal{O}_{[0,\mu)}\otimes S_{K_0}\simeq M\otimes_{\mathcal{O},\varphi}S_{K_0}=\mathcal{D}_{\mathcal{O}}(M).$$

So $\xi\otimes_{\mathcal{O}_{[0,\mu)}}S_{K_0}$ and i_S induce an S_{K_0}-linear, filtration compatible isomorphism

$$(D(M)\otimes_{K_0}\mathcal{O}_{[0,\mu]})\otimes S_{K_0}\simeq D\otimes_{K_0}S_{K_0}.$$

Both sides define filtration on $D(M)$ and D by modulo $E(u)$ respectively. Therefore, filtration on $D(M)$ and D coincide. ◻

3.3 Finite φ-modules of finite height and finite $\mathbb{Z}_p$-representations of G_∞

Recall that $\mathfrak{S} = W(k)[\![u]\!]$ with the endomorphism $\varphi : \mathfrak{S} \to \mathfrak{S}$ which acts on $W(k)$ via Frobenius and sends u to u^p. In this subsection, we first recall the theory in [Fon90] on finite φ-modules over $\mathfrak{S}$ of finite height and associated finite $\mathbb{Z}_p$-representations of G_∞. Then we study the relations between finite φ-modules over $\mathfrak{S}$ of finite height and filtered φ-modules over S, and their associated finite representations of G_∞. These results have been essentially done in [Bre98c] and [Kis, § 1.1].

Denote by $'\mathrm{Mod}^{\varphi}_{/\mathfrak{S}}$ the category of $\mathfrak{S}$-modules $\mathfrak{M}$ equipped with a φ-semi-linear map $\varphi_{\mathfrak{M}} : \mathfrak{M} \to \mathfrak{M}$ such that the cokernel of the $\mathfrak{S}$-linear map: $1 \otimes \varphi_{\mathfrak{M}} : \mathfrak{S} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M} \to \mathfrak{M}$ is killed by $E(u)^r$. (We always drop the subscript $\mathfrak{M}$ of $\varphi_{\mathfrak{M}}$ if no confusion will arise.) We give $'\mathrm{Mod}^{\varphi}_{/\mathfrak{S}}$ the structure of exact category induced by that on the abelian category of $\mathfrak{S}$-modules. We denote by $\mathrm{Mod}\ \mathrm{FI}^{\varphi}_{/\mathfrak{S}}$ the full category of $'\mathrm{Mod}^{\varphi}_{/\mathfrak{S}}$ consisting of those $\mathfrak{M}$ such that as an $\mathfrak{S}$-module $\mathfrak{M}$ is isomorphic to $\bigoplus_{i\in I} \mathfrak{S}/p^{n_i}\mathfrak{S}$, where I is a finite set and n_i is a positive integer. Finally we denote by $\mathrm{Mod}^{\varphi}_{/\mathfrak{S}}$ the full subcategory of $'\mathrm{Mod}^{\varphi}_{/\mathfrak{S}}$ consisting of those $\mathfrak{M}$ which are $\mathfrak{S}$-finite free.

Recall that $[\underline{\pi}] \in W(R)$ was constructed in § 2.2. We embed $\mathfrak{S} \hookrightarrow W(R)$ by $u \mapsto [\underline{\pi}]$. This embedding is compatible with Frobenius endomorphisms. Denote by $\mathcal{O}_{\mathcal{E}}$ the p-adic completion of $\mathfrak{S}[1/u]$. Then $\mathcal{O}_{\mathcal{E}}$ is a discrete valuation ring with the residue field the Laurent series ring $k(\!(u)\!)$. We write $\mathcal{E}$ for the field of fractions of $\mathcal{O}_{\mathcal{E}}$. If $\mathrm{Fr}R$ denotes the field of fractions of R, then the inclusion $\mathfrak{S} \hookrightarrow W(R)$ extends to $\mathcal{O}_{\mathcal{E}} \hookrightarrow W(\mathrm{Fr}R)$. Let $\mathcal{E}^{\mathrm{ur}} \subset W(\mathrm{Fr}R)[1/p]$ denote the maximal unramified extension of $\mathcal{E}$ contained in $W(\mathrm{Fr}R)[1/p]$, and $\mathcal{O}^{\mathrm{ur}}$ its ring of integers. Since $\mathrm{Fr}R$ is easily seen to be algebraically closed, the residue field $\mathcal{O}^{\mathrm{ur}}/p\mathcal{O}^{\mathrm{ur}}$ is the separable closure of $k(\!(u)\!)$. We denote by $\widehat{\mathcal{E}^{\mathrm{ur}}}$ the p-adic completion of $\mathcal{E}^{\mathrm{ur}}$, and by $\widehat{\mathcal{O}^{\mathrm{ur}}}$ its ring of integers. The completion $\widehat{\mathcal{E}^{\mathrm{ur}}}$ is also equal to the closure of $\mathcal{E}^{\mathrm{ur}}$ in $W(\mathrm{Fr}R)[1/p]$. We write $\mathfrak{S}^{\mathrm{ur}} = \widehat{\mathcal{O}^{\mathrm{ur}}} \cap W(R) \subset W(\mathrm{Fr}R)$. We regard all these rings as subrings of $W(\mathrm{Fr}R)[1/p]$.

Recall that $K_\infty = \bigcup_{n\geqslant 0} K(\pi_n)$ and $G_\infty = \mathrm{Gal}(\bar{K}/K_\infty)$. We have that G_∞ continuously acts on $\mathfrak{S}^{\mathrm{ur}}$ and $\widehat{\mathcal{O}^{\mathrm{ur}}}$ and fixes the subring $\mathfrak{S} \subset W(R)$. Denote by $\mathrm{Rep}_{\mathrm{tor}}(G_\infty)$ the category of finite length $\mathbb{Z}_p$-representations of G_∞. For an $\mathfrak{M} \in \mathrm{Mod}\ \mathrm{FI}^{\varphi}_{/\mathfrak{S}}$, one can associate a finite length $\mathbb{Z}_p$-representation of G_∞ by [Fon90, B.1.8]:

$$T_{\mathfrak{S}} : \quad \mathfrak{M} \to \mathrm{Hom}_{\mathfrak{S},\varphi}(\mathfrak{M}, \mathfrak{S}^{\mathrm{ur}}[1/p]/\mathfrak{S}^{\mathrm{ur}}).$$

In [Fon90, § B.1.8.4] and [Fon90, § A.1.2], Fontaine has proved that the functor $T_{\mathfrak{S}} : \mathrm{Mod}\ \mathrm{FI}^{\varphi}_{/\mathfrak{S}} \to \mathrm{Rep}_{\mathrm{tor}}(G_\infty)$ is an *exact* functor. If $\mathfrak{M} \simeq \bigoplus_{i=1}^{m} \mathfrak{S}/p^{n_i}\mathfrak{S}$ as finite $\mathfrak{S}$-modules, then

$$T_{\mathfrak{S}}(\mathfrak{M}) \simeq \bigoplus_{i=1}^{m} \mathbb{Z}/p^{n_i}\mathbb{Z}$$

as finite $\mathbb{Z}_p$-modules. As the consequence, if $\mathfrak{M} \in \mathrm{Mod}^{\varphi}_{/\mathfrak{S}}$ is a finite free $\mathfrak{S}$-module with rank d, and we define

$$T_{\mathfrak{S}}(\mathfrak{M}) = \mathrm{Hom}_{\mathfrak{S},\varphi}(\mathfrak{M}, \mathfrak{S}^{\mathrm{ur}}),$$

then $T_{\mathfrak{S}}(\mathfrak{M})$ is a continuous finite free $\mathbb{Z}_p$-representation of G_∞ with $\mathbb{Z}_p$-rank d.

As in [Bre98c] or [Kis, § 1.1], we define a functor $\mathcal{M}_{\mathfrak{S}} : \mathrm{Mod}\ \mathrm{FI}^{\varphi}_{/\mathfrak{S}} \to {}'\mathrm{Mod}^{\varphi}_{/S}$ as follows. We have a map of $W(k)$-algebra $\mathfrak{S} \to S$ given by $u \mapsto u$, so we regard S as an $\mathfrak{S}$-algebra. We will denote by φ the map $\mathfrak{S} \hookrightarrow S$ obtained by composing this map with φ on $\mathfrak{S}$. Given an $\mathfrak{M} \in \mathrm{Mod}\ \mathrm{FI}^{\varphi}_{/\mathfrak{S}}$, set $\mathcal{M} = \mathcal{M}_{\mathfrak{S}}(\mathfrak{M}) := S \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$.

One has the map $1 \otimes \varphi : S \otimes_{\varphi,\mathfrak{S}} \mathfrak{M} \to S \otimes_{\mathfrak{S}} \mathfrak{M}$. Set

$$\mathrm{Fil}^r \mathcal{M} = \{y \in \mathcal{M} \mid (1 \otimes \varphi)(y) \in \mathrm{Fil}^r S \otimes_{\mathfrak{S}} \mathfrak{M} \subset S \otimes_{\mathfrak{S}} \mathfrak{M}\}$$

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and define $\varphi_r : \mathrm{Fil}^r \mathcal{M} \to \mathcal{M}$ as the composite

$$\mathrm{Fil}^r \mathcal{M} \xrightarrow{1\otimes\varphi} \mathrm{Fil}^r S \otimes_{\mathfrak{S}} \mathfrak{M} \xrightarrow{\varphi_r\otimes 1} S \otimes_{\varphi,\mathfrak{S}} \mathfrak{M} = \mathcal{M}.$$

This gives $\mathcal{M}$ the structure of an object in $'\mathrm{Mod}^{\varphi}_{/S}$. We have the following result similar to Lemma 2.2.1 in [Bre98c] and Proposition 1.1.11 in [Kis].

Proposition 3.3.1 (Breuil, Kisin). *The functor $\mathcal{M}_{\mathfrak{S}} : \mathrm{Mod\ FI}^{\varphi}_{/\mathfrak{S}} \to {}'\mathrm{Mod}^{\varphi}_{/S}$ defined above induces an exact and fully faithful functor $\mathcal{M}_{\mathfrak{S}} : \mathrm{Mod\ FI}^{\varphi}_{/\mathfrak{S}} \to \mathrm{Mod\ FI}^{\varphi}_{/S}$. This functor is an equivalence of categories between the full subcategories consisting of objects killed by p.*

Proof. Lemma 2.2.1 in [Bre98c] and Proposition 1.1.11 in [Kis] proved the case $r = 1$. The idea of the proof can be easily extended for $0 \leqslant r \leqslant p-2$. In particular, the equivalence of subcategories consisting of p-torsion objects is again (almost) *verbatim* the proof of Theorem 4.1.1 in [Bre99a]. □

Corollary 3.3.2. *The functor $\mathcal{M}_{\mathfrak{S}} : \mathrm{Mod\ FI}^{\varphi}_{/\mathfrak{S}} \to {}'\mathrm{Mod}^{\varphi}_{/S}$ induces an exact and fully faithful functor $\mathcal{M}_{\mathfrak{S}} : \mathrm{Mod}^{\varphi}_{/\mathfrak{S}} \to \mathrm{Mod}^{\varphi}_{/S}$.*

Remark 3.3.3. In fact, the functor $\mathcal{M}_{\mathfrak{S}}$ can be proved to be an equivalence [CL07].

Note that A_{cris} is an object in $'\mathrm{Mod}^{\varphi}_{/S}$ by defining $\varphi_r := \varphi/p^r$ on $\mathrm{Fil}^r A_{\mathrm{cris}}$. For any $\mathcal{M} \in \mathrm{Mod}^{\varphi}_{/S}$, one can define a finite free continuous $\mathbb{Z}_p$-representation of G_∞:

$$T_{\mathrm{cris}} : \mathcal{M} \to \mathrm{Hom}_{'\mathrm{Mod}^{\varphi}_{/S}}(\mathcal{M}, A_{\mathrm{cris}}) \tag{3.3.1}$$

as in [Bre99a, § 2.3.1]. Let $\mathfrak{M} \in \mathrm{Mod}^{\varphi}_{/\mathfrak{S}}$ and $\mathcal{M} = \mathcal{M}_{\mathfrak{S}}(\mathfrak{M}) \in \mathrm{Mod}^{\varphi}_{/S}$. For any $f \in T_{\mathfrak{S}}(\mathfrak{M}) = \mathrm{Hom}_{\mathfrak{S},\varphi}(\mathfrak{M}, \mathfrak{S}^{\mathrm{ur}})$, consider the natural embedding $\iota : \mathfrak{S}^{\mathrm{ur}} \hookrightarrow A_{\mathrm{cris}}$. It is easy to check that $\varphi(\iota \circ f) \in T_{\mathrm{cris}}(\mathcal{M}) = \mathrm{Hom}_{'\mathrm{Mod}^{\varphi}_{/S}}(\mathcal{M}, A_{\mathrm{cris}})$. Therefore, we get a natural map $\mathrm{Hom}_{\mathfrak{S},\varphi}(\mathfrak{M}, \mathfrak{S}^{\mathrm{ur}}) \to \mathrm{Hom}_{'\mathrm{Mod}^{\varphi}_{/S}}(\mathcal{M}_{\mathfrak{S}}(\mathfrak{M}), A_{\mathrm{cris}})$.

Lemma 3.3.4. *The natural map $T_{\mathfrak{S}}(\mathfrak{M}) \to T_{\mathrm{cris}}(\mathcal{M}_{\mathfrak{S}}(\mathfrak{M}))$ defined above is an isomorphism of finite free $\mathbb{Z}_p$-representations of G_∞.*

Proof. It suffices to show that, for any $\mathfrak{M} \in \mathrm{Mod\ FI}^{\varphi}_{/\mathfrak{S}}$, the natural map

$$\mathrm{Hom}_{\mathfrak{S},\varphi}(\mathfrak{M}, \mathfrak{S}^{\mathrm{ur}}[1/p]/\mathfrak{S}^{\mathrm{ur}}) \to \mathrm{Hom}_{'\mathrm{Mod}^{\varphi}_{/S}}(\mathcal{M}_{\mathfrak{S}}(\mathfrak{M}), A_{\mathrm{cris}}[1/p]/A_{\mathrm{cris}}) \tag{3.3.2}$$

is an isomorphism of finite $\mathbb{Z}_p[G_\infty]$-modules. Note that the left-hand side of (3.3.2) is an exact functor on $\mathrm{Mod\ FI}^{\varphi}_{/\mathfrak{S}}$. The right-hand side is also an exact functor from the facts that $\mathcal{M}_{\mathfrak{S}}$ is exact (Proposition 3.3.1) and $\mathrm{Ext}^1_{'\mathrm{Mod}^{\varphi}_{/S}}(\mathcal{M}, A_{\mathrm{cris}}[1/p]/A_{\mathrm{cris}}) = 0$ for any $\mathcal{M} \in \mathrm{Mod\ FI}^{\varphi}_{/S}$ (Lemma 2.3.1.3 in [Bre99a]). Thus by the standard *dévissage*, it suffices to prove (3.3.2) for the case that p kills $\mathfrak{M}$, and this is Proposition 4.2.1 in [Bre99b]. □

3.4 G_∞-stable $\mathbb{Z}_p$-lattices in a semi-stable Galois representation

A (φ, N)-module over $\mathfrak{S}$ is a finite free φ-module $\mathfrak{M} \in \mathrm{Mod}^{\varphi}_{/\mathfrak{S}}$, equipped with a linear endomorphism $N : \mathfrak{M}/u\mathfrak{M} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to \mathfrak{M}/u\mathfrak{M} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ such that $N\varphi = p\varphi N$. We denote by $\mathrm{Mod}^{\varphi,N}_{/\mathfrak{S}}$ the category of (φ, N)-module over $\mathfrak{S}$, and by $\mathrm{Mod}^{\varphi,N}_{/\mathfrak{S}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ the associated isogeny category.[1] The following theorem is one of the main results (cf. Corollary 1.3.15) in [Kis06].

[1]Recall that, if $\mathcal{C}$ is an additive category, then the associated isogeny category $\mathcal{D}$ has the same objects and $\mathrm{Hom}_{\mathcal{D}}(A, B) = \mathrm{Hom}_{\mathcal{C}}(A, B) \otimes_{\mathbb{Z}} \mathbb{Q}$ for all objects A and B.

Theorem 3.4.1 (Kisin). *There exists a fully faithful $\otimes$-functor Θ from the category of positive weakly admissible filtered (φ, N)-modules $\mathrm{MF}^{\mathrm{w}}(\varphi, N)$ to $\mathrm{Mod}^{\varphi,N}_{/\mathfrak{S}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$.*

Let $\mathfrak{M} \in \mathrm{Mod}^{\varphi,N}_{/\mathfrak{S}}$ and $M = \mathfrak{M} \otimes_{\mathfrak{S}} \mathcal{O}$. Then there exists a $D \in \mathrm{MF}^{\mathrm{w}}(\varphi, N)$ such that $\mathfrak{M} = \Theta(D)$ if and only if there exists a differential operator N_∇ on M such that $(M, \varphi, N_\nabla) \in \mathrm{Mod}^{\varphi,N_\nabla}_{/\mathcal{O}}$, $D(M) \simeq D$ in $\mathrm{MF}(\varphi, N)$ and $N_\nabla \bmod u = N$ on $\mathfrak{M}/u\mathfrak{M} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. Such N_∇ (if it exists) is necessarily unique.

Remark 3.4.2. (1) The above theorem is valid without any restriction of the maximal Hodge–Tate weight. Here we only consider the case of Hodge–Tate weights in $\{0, \dots, r\}$ with $r \leqslant p-2$.

(2) The second paragraph of the above theorem is not the same as that of Corollary 1.3.15 in [Kis06]. However, they are equivalent (see Lemma 1.3.10 and Lemma 1.3.13 in [Kis06]), and our description of Theorem 3.4.1 will be more convenient.

Furthermore, Kisin proved (cf. Proposition 2.1.5 in [Kis06]) that there exists a canonical bijection (without restriction of maximal Hodge–Tate weights)

$$\eta : T_{\mathfrak{S}}(\mathfrak{M}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \xrightarrow{\sim} V_{\mathrm{st}}(D), \tag{3.4.1}$$

which is compatible with the action of G_∞ on the two sides. For our purpose to connect strongly divisible lattices, we reconstruct (3.4.1) in a slightly different way.

Let $D \in \mathrm{MF}^{\mathrm{w}}(\varphi, N)$ be a weakly admissible filtered (φ, N)-module under our Assumption 2.3.1, $\mathfrak{M} = \Theta(D)$ and $(M, \varphi, N_\nabla) \in \mathrm{Mod}^{\varphi,N_\nabla}_{/\mathcal{O}}$ as in Theorem 3.4.1. Let $\mathcal{D} = \mathcal{D}(D)$ (recall that $\mathcal{D}(D) := S \otimes_{W(k)} D$ in § 2.2). By Corollary 3.2.3, we have $\mathcal{D} = S_{K_0} \otimes_{\varphi,\mathcal{O}} M = S_{K_0} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M} = \mathcal{M}_{\mathfrak{S}}(\mathfrak{M}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$, where $\mathcal{M}_{\mathfrak{S}}$ is the functor defined in Corollary 3.3.2. Then we have a natural map of $\mathbb{Z}_p[G_\infty]$-modules

$$\mathrm{Hom}_{\mathfrak{S},\varphi}(\mathfrak{M}, \mathfrak{S}^{\mathrm{ur}}) \xrightarrow{\sim} \mathrm{Hom}_{'\mathrm{Mod}^{\varphi}_{/S}}(\mathcal{M}_{\mathfrak{S}}(\mathfrak{M}), A_{\mathrm{cris}}) \hookrightarrow \mathrm{Hom}_{'\mathrm{Mod}^{\varphi}_{/S}}(\mathcal{D}, B^+_{\mathrm{cris}}). \tag{3.4.2}$$

The first map is an isomorphism by Lemma 3.3.4. Recall that

$$V_{\mathrm{st}}(\mathcal{D}) = \mathrm{Hom}_{'\mathrm{Mod}^{\varphi,N}_{/S}}(\mathcal{D}, \widehat{A_{\mathrm{st}}}[1/p]).$$

The canonical projection $\widehat{A_{\mathrm{st}}} \to A_{\mathrm{cris}}$ defined by sending $\gamma_i(X)$ to 0 induces a natural map:

$$\mathrm{Hom}_{'\mathrm{Mod}^{\varphi,N}_{/S}}(\mathcal{D}, \widehat{A_{\mathrm{st}}}[1/p]) \to \mathrm{Hom}_{'\mathrm{Mod}^{\varphi}_{/S}}(\mathcal{D}, B^+_{\mathrm{cris}}). \tag{3.4.3}$$

We claim that the above map is a bijection. Let us accept the claim and postpone the proof in Lemma 3.4.3. Recall that Theorem 2.2.1 has shown that there exists a canonical isomorphism $V_{\mathrm{st}}(\mathcal{D}) \simeq V_{\mathrm{st}}(D)$ as $\mathbb{Q}_p$-representations of G. Therefore, combining (3.4.2) and (3.4.3), we have a natural injection

$$\eta : T_{\mathfrak{S}}(\mathfrak{M}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \hookrightarrow V_{\mathrm{st}}(D)$$

of $\mathbb{Q}_p[G_\infty]$-modules and thus $\dim_{\mathbb{Q}_p}(V_{\mathrm{st}}(D)) \geqslant \mathrm{rank}_{\mathfrak{S}}(\mathfrak{M}) = \dim_{K_0}(D)$. An elementary argument [CF00, Proposition 4.5] showed that weak admissibility of D implies that $\dim_{\mathbb{Q}_p}(V_{\mathrm{st}}(D))$ has to be $\dim_{K_0}(D)$. Hence the map η is a bijection.

Lemma 3.4.3. *The natural map defined in (3.4.3) is a bijection.*

Proof. We follow the idea of Lemma 2.3.1.1 in [Bre99a]. For any $f \in \mathrm{Hom}_{'\mathrm{Mod}^{\varphi,N}_{/S}}(\mathcal{D}, \widehat{A_{\mathrm{st}}}[1/p])$, let f_0 be its image of the map in (3.4.3). For any $x \in D$ where $\mathcal{D} = D \otimes_{W(k)} S$, since $N^i(x) = 0$ for i big enough, we can easily check that

$$f(x) = \sum_{i=0}^{\infty} f_0(N^i(x))\gamma_i(\log(1+X)), \tag{3.4.4}$$

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where $\gamma_i(x) = x^i/i!$ is the standard divided power. So if $f_0 = 0$, we have $f = 0$ because D generates $\mathcal{D}$. Thus (3.4.3) is injective. To prove the surjectivity, let $f_0 \in \mathrm{Hom}_{'\mathrm{Mod}^{\varphi}_{/S}}(\mathcal{D}, B^+_{\mathrm{cris}})$. For any $y \in \mathcal{D}$, define

$$f(y) = \sum_{i=0}^{\infty} f_0(N^i(y))\gamma_i(\log(1+X)).$$

To see that f is well defined, note that $f(y)$ converges in $B^+_{\mathrm{cris}}[\![X]\!]$, and if $x \in D$ then $f(x)$ converges in $\widehat{A_{\mathrm{st}}}[1/p]$ because $N^i(x) = 0$ for i big enough. By a standard computation, we can easily check that $f : \mathcal{D} \to B^+_{\mathrm{cris}}[\![X]\!]$ is S-linear. Therefore $f : \mathcal{D} \to \widehat{A_{\mathrm{st}}}[1/p]$ is well defined. It suffices to check that f preserves Frobenius, monodromy and filtration. Since f_0 preserves all these structures, it is a straightforward calculation to check that f preserves Frobenius, monodromy and filtration, combining with the facts that $\varphi(\log(1+X)) = p\log(1+X)$, $N(\log(1+X)) = 1$, $N^j(\mathrm{Fil}^i\mathcal{D}) \subset \mathrm{Fil}^{i-j}\mathcal{D}$ and $\log(1+X) \in \mathrm{Fil}^1\widehat{A_{\mathrm{st}}}$. □

Remark 3.4.4. (1) Let $V_{\mathrm{cris}}(\mathcal{D}) := \mathrm{Hom}_{'\mathrm{Mod}^{\varphi}_{/S}}(\mathcal{D}, B^+_{\mathrm{cris}})$. The above lemma gives a natural transformation which makes the following diagram commutative.

$$\begin{tikzcd}
\mathcal{MF}(\varphi, N) \arrow[r, "V_{\mathrm{cris}}"] \arrow[d, equal] & \mathrm{Rep}_{\mathbb{Q}_p}(G_\infty) \\
\mathcal{MF}(\varphi, N) \arrow[r, "V_{\mathrm{st}}"] & \mathrm{Rep}_{\mathbb{Q}_p}(G) \arrow[u] \arrow[ul, Rightarrow]
\end{tikzcd}$$

(2) From the above proof, we see that the lemma is always valid without any restriction of the maximal Hodge–Tate weight.

One advantage of using (φ, N)-module over $\mathfrak{S}$ is that we can classify all G_∞-stable $\mathbb{Z}_p$-lattices inside semi-stable representations. Let $\mathrm{Rep}_{\mathbb{Z}_p}(G_\infty)$ denote the category of continuous finite free $\mathbb{Z}_p$-representations of G_∞.

Lemma 3.4.5 (Kisin).

(1) *Let V be a semi-stable representation with Hodge–Tate weights in $\{0, \ldots, r\}$. For any G_∞-stable $\mathbb{Z}_p$-lattice $T \subset V$, there always exists an $\mathfrak{N} \in \mathrm{Mod}^{\varphi}_{/\mathfrak{S}}$ such that $T_{\mathfrak{S}}(\mathfrak{N}) \simeq T$.*

(2) *The functor $T_{\mathfrak{S}} : \mathrm{Mod}^{\varphi}_{/\mathfrak{S}} \to \mathrm{Rep}_{\mathbb{Z}_p}(G_\infty)$ is fully faithful.*

Proof. These are easy consequences of Lemma 2.1.15 and Proposition 2.1.12 in [Kis06]. We remark that the lemma is valid without restriction of r. □

Recall that $\widetilde{\mathrm{Mod}}^{\varphi}_{/S}$ denotes the category of quasi-strongly divisible lattices of weight r. Let $\mathcal{M} \in \widetilde{\mathrm{Mod}}^{\varphi}_{/S}$ be a quasi-strongly divisible lattice. By Definition 2.3.3, there exists a $\mathcal{D} \in \mathcal{MF}^{\mathrm{w}}(\varphi, N)$ such that $\mathcal{M} \subset \mathcal{D}$ and $\mathcal{D} \simeq \mathcal{D}(D)$ with D weakly admissible. Let $V := V_{\mathrm{st}}(\mathcal{D})$ be the semi-stable Galois representation. Then we can associate a G_∞-stable $\mathbb{Z}_p$-lattice in V as the following:

$$\mathcal{M} \mapsto T_{\mathrm{cris}}(\mathcal{M}) = \mathrm{Hom}_{'\mathrm{Mod}^{\varphi}_{/S}}(\mathcal{M}, A_{\mathrm{cris}}) \hookrightarrow \mathrm{Hom}_{'\mathrm{Mod}^{\varphi}_{/S}}(\mathcal{D}, B^+_{\mathrm{cris}}) \simeq V_{\mathrm{st}}(\mathcal{D}) = V.$$

Recall that the isomorphism $V_{\mathrm{st}}(\mathcal{D}) \xrightarrow{\sim} \mathrm{Hom}_{'\mathrm{Mod}^{\varphi}_{/S}}(\mathcal{D}, B^+_{\mathrm{cris}})$ has been established in Lemma 3.4.3. Therefore T_{cris} induces a functor from $\widetilde{\mathrm{Mod}}^{\varphi}_{/S}$ to $\mathrm{Rep}^{\mathrm{st}}_{\mathbb{Z}_p}(G_\infty)$, where $\mathrm{Rep}^{\mathrm{st}}_{\mathbb{Z}_p}(G_\infty)$ denotes the category of G_∞-stable $\mathbb{Z}_p$-lattices in semi-stable Galois representations with Hodge–Tate weights in $\{0, \ldots, r\}$.

Proposition 3.4.6. *The functor T_{cris} induces an anti-equivalence between $\widetilde{\mathrm{Mod}}^{\varphi}_{/S}$ and $\mathrm{Rep}^{\mathrm{st}}_{\mathbb{Z}_p}(G_\infty)$.*

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Proof. We first prove the essential surjectivity of the functor. Let $\mathfrak{M} = \Theta(D)$ as in Theorem 3.4.1 and $\mathcal{D} = \mathcal{D}(D)$. By Corollary 3.2.3 and Theorem 3.4.1, we see that $\mathcal{D} = \mathfrak{M} \otimes_{\mathfrak{S},\varphi} S_{K_0}$ and $T_{\mathfrak{S}}(\mathfrak{M})$ is a G_∞-stable $\mathbb{Z}_p$-lattice in V via η. Suppose that $T \subset V$ is a G_∞-stable $\mathbb{Z}_p$-lattice. Then by Lemma 3.4.5, there exists an $\mathfrak{N} \in \mathrm{Mod}^{\varphi}_{/\mathfrak{S}}$, such that $T \simeq T_{\mathfrak{S}}(\mathfrak{N})$. We claim that $\mathfrak{M} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \simeq \mathfrak{N} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. In fact, since $T_{\mathfrak{S}}(\mathfrak{M})$ and $T_{\mathfrak{S}}(\mathfrak{N})$ are G_∞-stable $\mathbb{Z}_p$-lattices in V, there exist G_∞-equivariant maps $f : T_{\mathfrak{S}}(\mathfrak{M}) \to T_{\mathfrak{S}}(\mathfrak{N})$ and $g : T_{\mathfrak{S}}(\mathfrak{N}) \to T_{\mathfrak{S}}(\mathfrak{M})$ such that $f \circ g = p^n \mathrm{Id}$. By full faithfulness of $T_{\mathfrak{S}}$, there exists $F : \mathfrak{N} \to \mathfrak{M}$ and $G : \mathfrak{M} \to \mathfrak{N}$ such that $G \circ F = p^n \mathrm{Id}$. Hence the claim follows. Now put $\mathcal{N} = \mathcal{M}_{\mathfrak{S}}(\mathfrak{N})$. We see that $\mathcal{N}$ is a quasi-strongly divisible lattice in $\mathcal{D}$, and, by Lemma 3.3.4, $T_{\mathrm{cris}}(\mathcal{N}) = T$. This proves that the functor is essential surjective. Let $\mathcal{M}, \mathcal{N} \in \widetilde{\mathrm{Mod}}^{\varphi}_{/S}$ and $f : T_{\mathrm{cris}}(\mathcal{N}) \to T_{\mathrm{cris}}(\mathcal{M})$ a morphism of $\mathbb{Z}_p[G_\infty]$-module. From the above proof, there exist $\mathfrak{M}, \mathfrak{N} \in \mathrm{Mod}^{\varphi}_{/\mathfrak{S}}$ such that $T_{\mathfrak{S}}(\mathfrak{M}) = T_{\mathrm{cris}}(\mathcal{M})$ and $T_{\mathfrak{S}}(\mathfrak{N}) = T_{\mathrm{cris}}(\mathcal{N})$. Since $T_{\mathfrak{S}}$ is fully faithful (Lemma 3.4.5(2)), there exists $\mathfrak{f} : \mathfrak{M} \to \mathfrak{N}$ a morphism in $\mathrm{Mod}^{\varphi}_{/\mathfrak{S}}$ such that $T_{\mathfrak{S}}(\mathfrak{f}) = f$. Then by Corollary 3.3.2 and Lemma 3.3.4, we have $T_{\mathrm{cris}}(\mathcal{M}_{\mathfrak{S}}(\mathfrak{f})) = f$. It suffices to show that $\mathcal{M} = \mathcal{M}_{\mathfrak{S}}(\mathfrak{M})$ and $\mathcal{N} = \mathcal{M}_{\mathfrak{S}}(\mathfrak{N})$. Therefore, we reduce the proof to the following lemma. $\square$

LEMMA 3.4.7. *Fix a $\mathcal{D} \in \mathcal{MF}^{\mathrm{w}}(\varphi, N)$. Let $\mathcal{M}$, $\mathcal{M}'$ be two quasi-strongly divisible lattices contained in $\mathcal{D}$. If $T_{\mathrm{cris}}(\mathcal{M}) = T_{\mathrm{cris}}(\mathcal{M}')$ then $\mathcal{M} = \mathcal{M}'$.*

We postpone our proof of this lemma till after Lemma 5.3.1.

We may summarize our discussion in this subsection into the follow commutative diagram.

$$\begin{tikzcd}
\mathrm{Mod}^{\varphi}_{/\mathfrak{S}} \arrow[r, hook, "\mathcal{M}_{\mathfrak{S}}"] \arrow[rr, hook, bend left, "T_{\mathfrak{S}}"] & \mathrm{Mod}^{\varphi}_{/S} \arrow[r, "T_{\mathrm{cris}}"] & \mathrm{Rep}_{\mathbb{Z}_p}(G_\infty) \\
 & \widetilde{\mathrm{Mod}}^{\varphi}_{/S} \arrow[u, hook] \arrow[r, "T_{\mathrm{cris}}", "\sim"'] & \mathrm{Rep}^{\mathrm{st}}_{\mathbb{Z}_p}(G_\infty) \arrow[u, hook]
\end{tikzcd}$$

3.5 Full faithfulness of T_{st}

Now suppose that T is a G*-stable* $\mathbb{Z}_p$-lattice in a semi-stable Galois representation V. By Proposition 3.4.6, there exists a quasi-strongly divisible lattice $\mathcal{M}$ in $\mathcal{D}$ such that $T_{\mathrm{cris}}(\mathcal{M}) = T|_{G_\infty}$ and there exists an $\mathfrak{M} \in \mathrm{Mod}^{\varphi}_{/\mathfrak{S}}$ such that $\mathcal{M} = \mathcal{M}_{\mathfrak{S}}(\mathfrak{M})$.

PROPOSITION 3.5.1. *Let the notation be as the above. If $N(\mathcal{M}) \subset \mathcal{M}$, then $(\mathcal{M}, \varphi, \mathrm{Fil}^r \mathcal{M}, N)$ is a strongly divisible lattice in $\mathcal{D}$ and $T_{\mathrm{st}}(\mathcal{M}) = T$.*

Proof. Clearly $\mathcal{M}$ is a strongly divisible lattice in $\mathcal{D}$. It suffices to prove that $T_{\mathrm{st}}(\mathcal{M}) = T$. By Proposition 2.3.4,

$$T_{\mathrm{st}}(\mathcal{M}) = \mathrm{Hom}_{'\mathrm{Mod}^{\varphi,N}_{/S}}(\mathcal{M}, \widehat{A_{\mathrm{st}}}) \subset V_{\mathrm{st}}(\mathcal{D}) \simeq V_{\mathrm{st}}(D) = V$$

is a G-stable $\mathbb{Z}_p$-lattice. As in (3.4.3), the canonical projection $\widehat{A_{\mathrm{st}}} \to A_{\mathrm{cris}}$ defined by sending $\gamma_i(X) \to 0$ induces a natural map

$$T_{\mathrm{st}}(\mathcal{M}) = \mathrm{Hom}_{'\mathrm{Mod}^{\varphi,N}_{/S}}(\mathcal{M}, \widehat{A_{\mathrm{st}}}) \to \mathrm{Hom}_{'\mathrm{Mod}^{\varphi}_{/S}}(\mathcal{M}, A_{\mathrm{cris}}) = T_{\mathrm{cris}}(\mathcal{M}). \tag{3.5.1}$$

Then we have the following commutative diagram.

$$\begin{tikzcd}
\mathrm{Hom}_{{}'\mathrm{Mod}^{\varphi,N}_{/S}}(\mathcal{M}, \widehat{A}_{\mathrm{st}}) \arrow[r, hook] \arrow[d, "(3.5.1)"] & \mathrm{Hom}_{{}'\mathrm{Mod}^{\varphi,N}_{/S}}(\mathcal{D}, \widehat{A}_{\mathrm{st}}[1/p]) \arrow[d, "(3.4.3)", "\wr"'] \\
\mathrm{Hom}_{{}'\mathrm{Mod}^{\varphi}_{/S}}(\mathcal{M}, A_{\mathrm{cris}}) \arrow[r, hook] \arrow[d, equal] & \mathrm{Hom}_{{}'\mathrm{Mod}^{\varphi}_{/S}}(\mathcal{D}, B^{+}_{\mathrm{cris}}) \arrow[d, equal] \\
T \arrow[r, hook] & V
\end{tikzcd}$$

Thus it suffices to show that (3.5.1) is an isomorphism of $\mathbb{Z}_p$-modules. This has been proved in [Bre99a, § 2.3.1]. □

Corollary 3.5.2. *The functor T_{st} in Conjecture 1.0.1 (Breuil's conjecture) is fully faithful.*

Proof. Let $\mathcal{M}$, $\mathcal{M}'$ be strongly divisible lattices, $\mathcal{D} = \mathcal{M} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$, $\mathcal{D}' = \mathcal{M}' \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ and $T_{\mathrm{st}}(\mathcal{M})$, $T_{\mathrm{st}}(\mathcal{M}')$ be G-stable $\mathbb{Z}_p$-lattices in $V_{\mathrm{st}}(\mathcal{D})$, $V_{\mathrm{st}}(\mathcal{D}')$ respectively. Suppose that $f : T_{\mathrm{st}}(\mathcal{M}) \to T_{\mathrm{st}}(\mathcal{M}')$ is a morphism of $\mathbb{Z}_p[G]$-modules. Tensoring by $\mathbb{Q}_p$, there exists an $\mathfrak{f} : \mathcal{D}' \to \mathcal{D}$ such that $V_{\mathrm{st}}(\mathfrak{f}) = f \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. It suffices to show that $\mathfrak{f}(\mathcal{M}') \subset \mathcal{M}$. Select an n such that $p^n\mathfrak{f}(\mathcal{M}') \subset \mathcal{M}$. Then $\mathfrak{g} := p^n\mathfrak{f}$ is a morphism of strongly divisible lattices and $T_{\mathrm{st}}(\mathfrak{g}) = p^n f$. Note that (3.5.1) is an isomorphism of $\mathbb{Z}_p[G_\infty]$-modules. So if $\mathfrak{g}$ is regarded as a morphism of quasi-strongly divisible lattices, we have $T_{\mathrm{cris}}(\mathfrak{g}) = T_{\mathrm{st}}(\mathfrak{g}) = p^n f$. On the other hand, by Proposition 3.4.6, T_{cris} is fully faithful, and there exists a morphism $\mathfrak{g}' : \mathcal{M}' \to \mathcal{M}$ in $\mathrm{Mod}^{\varphi}_{/S}$ such that $T_{\mathrm{cris}}(\mathfrak{g}') = f$. Therefore $p^n\mathfrak{g}' = \mathfrak{g} = p^n\mathfrak{f}$. Then $\mathfrak{f} = \mathfrak{g}'$ and $\mathfrak{f}(\mathcal{M}') = \mathfrak{g}'(\mathcal{M}') \subset \mathcal{M}$. □

Also we reduce the proof of the essential surjectivity of T_{st} to the following lemma.

Lemma 3.5.3. *With notation as above, if T is G-stable then $N(\mathcal{M}) \subset \mathcal{M}$.*

We will devote the next two sections to prove this lemma. Combining with Proposition 3.5.1, Corollary 3.5.2 and Proposition 3.4.6, we prove the Main Theorem (Theorem 2.3.5) and the following result.

Theorem 3.5.4. *The functor T_{cris} induces an anti-equivalence between the category of quasi-strongly divisible lattices of weight r and the category of G_∞-stable $\mathbb{Z}_p$-lattices inside semi-stable Galois representations with Hodge–Tate weights in $\{0, \ldots, r\}$. Furthermore, a quasi-strongly divisible lattice $\mathcal{M}$ is strongly divisible if and only if $T_{\mathrm{cris}}(\mathcal{M})$ is G-stable.*

4. Cartier dual and a theorem to connect $\mathcal{M}$ with $T_{\mathrm{cris}}(\mathcal{M})$

In this section, we extend a theorem of Faltings (cf. [Fal99, Theorem 5]) to a more general setting to connect filtered φ-modules over S with their associated $\mathbb{Z}_p$-representations of G_∞. This theorem is one of the technical keys to prove Lemma 3.5.3. For this purpose, we need a more explicit structure of $\mathrm{Fil}^r\mathcal{M}$ and a notion of Cartier dual for $\mathcal{M} \in \mathrm{Mod}^{\varphi}_{/S}$. Luckily, such a Cartier dual is available from the thesis of Caruso [Car05]. In the following two sections, we always regard $W(k)[u]$ and S as subrings of A_{cris} via $u \mapsto [\underline{\pi}]$, and denote the identity matrix by I.

4.1 Structure of filtration of quasi-strongly divisible lattice

Lemma 4.1.1. *Let A be a $d \times d$ matrix with coefficients in $W(k)[u]$. Suppose that there exists matrices B' and C with coefficients in S and $\mathrm{Fil}^p S$ respectively such that $AB' = E(u)^r I + C$. Then the following hold.*

(1) *There exists a matrix B with coefficients in S such that $AB = E(u)^r I$.*

(2) *Let $a_i \in A_{\mathrm{cris}}$ for $i = 1, \ldots, d$. If $(a_1, \ldots, a_d)A$ is in $\mathrm{Fil}^r A_{\mathrm{cris}}$, then there exist $b_i \in A_{\mathrm{cris}}$ and $c_i \in \mathrm{Fil}^p A_{\mathrm{cris}}$ for $i = 1, \ldots, d$ such that*

$$(a_1, \ldots, a_d) = (b_1, \ldots, b_d)B + (c_1, \ldots, c_d).$$

Proof. Note that, for any $f \in S$, we can always write $f = f_0 + f_1$ with $f_0 \in W(k)[u]$ and $f_1 \in \mathrm{Fil}^p S$. So $B' = B_0 + B_1$ with the coefficients of B_0 in $W(k)[u]$ and the coefficients of B_1 in $\mathrm{Fil}^p S$. Therefore, $E(u)^r I = AB_0 + C_1$ with the coefficients of C_1 in $W(k)[u] \cap \mathrm{Fil}^p S = E(u)^p W(k)[u]$. Thus $C_1 = E(u)^p C_2$ with the coefficients of C_2 in $W(k)[u]$. Now we have $E(u)^r I = AB_0 + E(u)^p C_2$. Since $E(u)^n \to 0$ p-adically in S when $n \to \infty$, so $I - E(u)^{p-r} C_2$ is invertible. Thus we obtain

$$E(u)^r I = AB_0 (I - E(u)^{p-r} C_2)^{-1}. \tag{4.1.1}$$

Let $B = B_0 (I - E(u)^{p-r} C_2)^{-1}$ and we settle part (1).

For part (2), write $(a_1, \ldots, a_d) = (b'_1, \ldots, b'_d) + (c_1, \ldots, c_d)$ with $b'_i \in W(R)$ and $c_i \in \mathrm{Fil}^p A_{\mathrm{cris}}$ for $i = 1, \ldots, d$. It suffices to prove that there exists $b_i \in A_{\mathrm{cris}}$ such that $(b'_1, \ldots, b'_d) = (b_1, \ldots, b_d)B$. Note that

$$(a_1, \ldots, a_d)A = (b'_1, \ldots, b'_d)A + (c_1, \ldots, c_d)A \in \mathrm{Fil}^r A_{\mathrm{cris}}.$$

Then $(b'_1, \ldots, b'_d)A \in \mathrm{Fil}^r A_{\mathrm{cris}} \cap W(R) = E(u)^r W(R)$. So there exists $b_i \in W(R)$ such that $(b'_1, \ldots, b'_d)A = E(u)^r (b_1, \ldots, b_d)$. Multiplying by B on both sides, we get $(b'_1, \ldots, b'_d)AB = E(u)^r (b_1, \ldots, b_d)B$. Finally, $(b'_1, \ldots, b'_d) = (b_1, \ldots, b_d)B$ as required. $\square$

Proposition 4.1.2. *Let $\mathcal{M} \in \mathrm{Mod}^{\varphi}_{/S}$. There exists $\alpha_1, \ldots, \alpha_d \in \mathrm{Fil}^r \mathcal{M}$ such that:*

(1) $\mathrm{Fil}^r \mathcal{M} = \bigoplus_{i=1}^d S\alpha_i + (\mathrm{Fil}^p S)\mathcal{M}$;

(2) $E(u)^r \mathcal{M} \subseteq \bigoplus_{i=1}^d S\alpha_i$ *and* $(\varphi_r(\alpha_1), \ldots, \varphi_r(\alpha_d))$ *is a basis of* $\mathcal{M}$.

Proof. Considering $\mathcal{M}/p\mathcal{M}$, by Proposition 2.2.1.3 in [Bre99a], $\mathcal{M}/p\mathcal{M}$ has a 'base adaptée', i.e. there exist a basis $(e_1, \ldots, e_d)$ of $\mathcal{M}$ and $\alpha_1, \ldots, \alpha_d \in \mathrm{Fil}^r \mathcal{M}$ such that

$$\mathrm{Fil}^r \mathcal{M}/p\,\mathrm{Fil}^r \mathcal{M} = \bigoplus_{i=1}^d S_1 \bar{\alpha}_i + \mathrm{Fil}^p S_1 (\mathcal{M}/p\mathcal{M}) \tag{4.1.2}$$

such that $(\bar{\alpha}_1, \ldots, \bar{\alpha}_d) = (u^{r_1}\bar{e}_1, \ldots, u^{r_d}\bar{e}_d)$ with $0 \leqslant r_i \leqslant er$, where $S_1 = S/pS$ and $\bar{\alpha}_i$, $\bar{e}_i$ is the image of α_i, e_i in $\mathcal{M}/p\mathcal{M}$ respectively. Let

$$\widetilde{\mathcal{M}} = \bigoplus_{i=1}^d S\alpha_i + (\mathrm{Fil}^p S)\mathcal{M}.$$

Then $\widetilde{\mathcal{M}} \subset \mathrm{Fil}^r \mathcal{M}$. We claim that the natural map

$$f : \widetilde{\mathcal{M}}/\mathrm{Fil}^p S\mathcal{M} \to \mathrm{Fil}^r \mathcal{M}/\mathrm{Fil}^p S\mathcal{M}$$

is surjective. To see the claim, note that $S/\mathrm{Fil}^p S \xrightarrow{\sim} W(k)[u]/(E(u)^p)$ is Noetherian. By Nakayama's lemma, it suffices to show that $f \bmod p$ is a surjection. Note that

$$\mathrm{Fil}^r \mathcal{M}/\mathrm{Fil}^p S\mathcal{M} \bmod p = (\mathrm{Fil}^r \mathcal{M})_1/(\mathrm{Fil}^p S\mathcal{M})_1,$$

where $(\mathrm{Fil}^r \mathcal{M})_1 = \mathrm{Fil}^r \mathcal{M}/p\,\mathrm{Fil}^r \mathcal{M}$ and $(\mathrm{Fil}^p S\mathcal{M})_1 = \mathrm{Fil}^p S\mathcal{M}/p\,\mathrm{Fil}^p S\mathcal{M}$. By (4.1.2), we see that $f \bmod p$ is surjective and thus prove the claim. Then

$$\mathrm{Fil}^r \mathcal{M} = \widetilde{\mathcal{M}} = \bigoplus_{i=1}^d S\alpha_i + (\mathrm{Fil}^p S)\mathcal{M}. \tag{4.1.3}$$

Let $(\alpha_1, \dots, \alpha_d) = (e_1, \dots, e_d)A$ where A is a $d \times d$ matrix with coefficients in S. Write $A = A_0 + A_1$ with the coefficients of A_0 in $W(k)[u]$ and the coefficients of A_1 in $\mathrm{Fil}^p S$. Replacing $(\alpha_1, \dots, \alpha_d)$ by $(e_1, \dots, e_d)A_0$, we can always assume that the coefficients of A are in $W(k)[u]$. By (4.1.3), there exist $d \times d$ matrices B', C with coefficients in S, $\mathrm{Fil}^p S$ respectively such that $E(u)^r I = AB' + C$. Then by Lemma 4.1.1, there exists a B with coefficients in S such that $AB = E(u)^r I$. Therefore $E(u)^r \mathcal{M} \subset \bigoplus_{i=1}^d S\alpha_i$. Since $\varphi_r(\mathrm{Fil}^r \mathcal{M})$ generates $\mathcal{M}$ and one always has $p | \varphi_r(\mathrm{Fil}^p S)$, we see that $(\varphi_r(\alpha_1), \dots, \varphi_r(\alpha_d))$ is a basis of $\mathcal{M}$. □

Let $\mathcal{D} \in \mathcal{MF}(\varphi, N)$ be a filtered (φ, N)-module over S. Following [Bre97, § 3], we define

$$\mathrm{Fil}^r(A_{\mathrm{cris}} \otimes_S \mathcal{D}) = \sum_{i=0}^{r} \mathrm{Im}(\mathrm{Fil}^{r-i} A_{\mathrm{cris}} \otimes_S \mathrm{Fil}^i \mathcal{D}), \tag{4.1.4}$$

where $\mathrm{Im}(\mathrm{Fil}^{r-i} A_{\mathrm{cris}} \otimes_S \mathrm{Fil}^i \mathcal{D})$ is the image of $\mathrm{Fil}^{r-i} A_{\mathrm{cris}} \otimes_S \mathrm{Fil}^i \mathcal{D}$ in $A_{\mathrm{cris}} \otimes_S \mathcal{D}$. We also define $\mathrm{Fil}^r(A_{\mathrm{cris}} \otimes_S \mathcal{M}) = \mathrm{Fil}^r(A_{\mathrm{cris}} \otimes_S \mathcal{D}) \cap (A_{\mathrm{cris}} \otimes_S \mathcal{M})$.

Corollary 4.1.3. *With the notation as in Proposition 4.1.2, we have*

$$\mathrm{Fil}^r(A_{\mathrm{cris}} \otimes_S \mathcal{M}) = \bigoplus_{i=1}^{d} A_{\mathrm{cris}} \otimes \alpha_i + \mathrm{Fil}^p A_{\mathrm{cris}} \otimes_S \mathcal{M}.$$

Proof. Since we always have $\mathrm{Fil}^{r-i} S \cdot \mathrm{Fil}^i \mathcal{D} \subset \mathrm{Fil}^r \mathcal{D}$, it is easy to see that

$$\mathrm{Fil}^r(A_{\mathrm{cris}} \otimes_S \mathcal{D}) = A_{\mathrm{cris}} \otimes_S \mathrm{Fil}^r \mathcal{D}.$$

Then the corollary follows the fact that $\mathrm{Fil}^i \mathcal{M} = \mathrm{Fil}^i \mathcal{D} \cap \mathcal{M}$. □

By the above corollary, we can $\varphi_{A_{\mathrm{cris}}}$-semi-linearly extend φ_r of $\mathcal{M}$ to

$$\varphi_r : \mathrm{Fil}^r(A_{\mathrm{cris}} \otimes_S \mathcal{M}) \to A_{\mathrm{cris}} \otimes_S \mathcal{M}$$

and we see that $(A_{\mathrm{cris}} \otimes_S \mathcal{M}, \mathrm{Fil}^r(A_{\mathrm{cris}} \otimes_S \mathcal{M}), \varphi_r)$ is an object in $'\mathrm{Mod}^{\varphi}_{/S}$.

4.2 Cartier dual on $\mathrm{Mod}^{\varphi}_{/S}$

In this subsection, we recall the construction of Cartier dual on $\mathrm{Mod}^{\varphi}_{/S}$ from [Car05]. Let $\mathcal{M} \in \mathrm{Mod}^{\varphi}_{/S}$. Define $\mathcal{M}^* := \mathrm{Hom}_S(\mathcal{M}, S)$,

$$\mathrm{Fil}^r \mathcal{M}^* := \{f \in \mathcal{M}^* \mid f(\mathrm{Fil}^r \mathcal{M}) \subset \mathrm{Fil}^r S\}$$

and

$$\varphi_r : \mathrm{Fil}^r \mathcal{M}^* \to \mathcal{M}^*, \quad \text{for all } x \in \mathrm{Fil}^r \mathcal{M},\ \varphi_r(f)(\varphi_r(x)) = \varphi_r(f(x)).$$

Note that $\varphi_r(f)$ is well defined because $\varphi_r(\mathrm{Fil}^r \mathcal{M})$ generates $\mathcal{M}$.

Theorem 4.2.1 (Caruso). *The functor $\mathcal{M} \to \mathcal{M}^*$ induces an exact anti-equivalence on* $\mathrm{Mod}^{\varphi}_{/S}$ *and* $(\mathcal{M}^*)^* = \mathcal{M}$.

Proof. Proposition V 3.3.1 in [Car05] proved the theorem on the category of *strongly divisible lattices*. The same proof also works on $\mathrm{Mod}^{\varphi}_{/S}$ if we ignore monodromy. □

Example 4.2.2. Let S^* be the Cartier dual of S. Then S^* is the S-rank-1 quasi-strongly divisible lattice with $\mathrm{Fil}^r S^* = S$ and $\varphi_r(1) = 1$.

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4.3 Application to Galois representations

Let $\mathcal{M} \in \mathrm{Mod}^{\varphi}_{/S}$ and $\mathcal{M}^*$ be its Cartier dual. The canonical perfect pairing $\mathcal{M} \times \mathcal{M}^* \to S$ in the construction of Cartier dual is compatible with filtration and Frobenius on both sides. Taking the Cartier dual on both sides, and noting that $(\mathcal{M}^*)^* \simeq \mathcal{M}$ by Theorem 4.2.1, we have a map

$$i : S^* \to \mathcal{M}^* \times (\mathcal{M}^*)^* \simeq \mathcal{M}^* \times \mathcal{M}.$$

Since i is compatible with filtration and Frobenius, i induces a pairing

$$\tilde{i} : \mathrm{Hom}_{'\mathrm{Mod}^{\varphi}_{/S}}(\mathcal{M}, A_{\mathrm{cris}}) \times \mathrm{Hom}_{'\mathrm{Mod}^{\varphi}_{/S}}(\mathcal{M}^*, A_{\mathrm{cris}}) \to \mathrm{Hom}_{'\mathrm{Mod}^{\varphi}_{/S}}(S^*, A_{\mathrm{cris}}). \tag{4.3.1}$$

Lemma 4.3.1. *The above pairing induces a perfect paring of $\mathbb{Z}_p$-representations of G_∞:*

$$T_{\mathrm{cris}}(\mathcal{M}) \times T_{\mathrm{cris}}(\mathcal{M}^*) \to T_{\mathrm{cris}}(S^*) \simeq \mathbb{Z}_p(r). \tag{4.3.2}$$

Proof. It suffices to show that the pairing (4.3.2) is perfect by modulo p. The proof of this assertion is contained in the proof of Theorem V 4.3.1 in [Car05]. Although the hypotheses of Theorem V 4.3.1 require $er < p-1$, the statement is always valid for any e if we only consider the pairing induced by filtered φ-modules over S killed by p, as explained in Caruso's remark in the end of the proof. $\square$

We use A^*_{cris} to denote A_{cris} with noncanonical filtration $\mathrm{Fil}^r A^*_{\mathrm{cris}} = A_{\mathrm{cris}}$ and Frobenius $\varphi_r(1) = 1$.

Lemma 4.3.2. *There are natural isomorphisms of $\mathbb{Z}_p[G_\infty]$-modules:*

$$\mathrm{Hom}_{A_{\mathrm{cris}},\mathrm{Fil}^r,\varphi}(A^*_{\mathrm{cris}}, A_{\mathrm{cris}} \otimes_S \mathcal{M}^*) \simeq \mathrm{Fil}^r(A_{\mathrm{cris}} \otimes_S \mathcal{M}^*)^{\varphi_r=1} \simeq \mathrm{Hom}_{'\mathrm{Mod}^{\varphi}_{/S}}(\mathcal{M}, A_{\mathrm{cris}}).$$

Proof. While the first isomorphism is totally trivial to check, the second isomorphism needs some arguments. Let $\alpha_1, \dots, \alpha_d \in \mathrm{Fil}^r\mathcal{M}$ constructed in Proposition 4.1.2, $(e_1, \dots, e_d) = (\varphi_r(\alpha_1), \dots, \varphi_r(\alpha_d))$ a basis of $\mathcal{M}$ and $(e_1^*, \dots, e_d^*)$ the dual basis. Write $(\alpha_1, \dots, \alpha_d) = (e_1, \dots, e_d)A$ where A is a $d \times d$ matrix with coefficients in S. By the argument after formula (4.1.3), we may assume that all the coefficients of A are in $W(k)[u]$. By Lemma 4.1.1, there exists a matrix B with coefficients in S such that $AB = BA = E(u)^r I$. Put $(\alpha_1^*, \dots, \alpha_d^*) = (e_1^*, \dots, e_d^*)B^{\mathrm{t}}$ (here t means transpose). It is easy to check that $\alpha_i^* \in \mathrm{Fil}^r\mathcal{M}^*$ for $i = 1, \dots, d$.

Forgetting filtration and Frobenius structure for a while, since $\mathcal{M}$ is S-finite free, we can identify $A_{\mathrm{cris}} \otimes_S \mathcal{M}^*$ with $\mathrm{Hom}_S(\mathcal{M}, A_{\mathrm{cris}})$ by sending $\sum_{i=1}^d a_i \otimes e_i^*$ to $\sum_{i=1}^d a_i e_i^*$. For any

$$f \in \mathrm{Fil}^r(A_{\mathrm{cris}} \otimes_S \mathcal{M}^*) = A_{\mathrm{cris}} \otimes_S \mathrm{Fil}^r\mathcal{M}^* \quad \text{(Corollary 4.1.3)},$$

write $f = \sum_i a_i \otimes f_i$ with $a_i \in A_{\mathrm{cris}}$ and $f_i \in \mathrm{Fil}^r\mathcal{M}^*$. Then for any $x \in \mathrm{Fil}^r\mathcal{M}$, $f(x) = \sum_i a_i f_i(x) \in \mathrm{Fil}^r S \cdot A_{\mathrm{cris}} \subset \mathrm{Fil}^r A_{\mathrm{cris}}$. That is, f is a map from $\mathcal{M}$ to A_{cris} preserving filtration. On the other hand, let f be an S-linear map from $\mathcal{M}$ to A_{cris} preserving filtration. Then $f(\alpha_i) \in \mathrm{Fil}^r A_{\mathrm{cris}}$ for all $i = 1, \dots, d$. Denote $a_i = f(e_i)$, $i = 1, \dots, d$. We have $(a_1, \dots, a_d)A \in \mathrm{Fil}^r A_{\mathrm{cris}}$ where A is the matrix constructed in the first paragraph. By Lemma 4.1.1, we have

$$(a_1, \dots, a_d) = (b_1, \dots, b_d)B + (c_1, \dots, c_d)$$

with $b_i \in A_{\mathrm{cris}}$ and $c_i \in \mathrm{Fil}^p A_{\mathrm{cris}}$ for $i = 1, \dots, d$. So we have

$$f = \sum_{i=1}^d a_i e_i^* = \sum_{i=1}^d b_i \alpha_i^* + \sum_{i=1}^d c_i e_i^* \in \mathrm{Fil}^r(A_{\mathrm{cris}} \otimes_S \mathcal{M}^*).$$

Therefore, we have that $f \in \mathrm{Hom}_S(\mathcal{M}, A_{\mathrm{cris}})$ preserves filtration if and only if $f \in \mathrm{Fil}^r(A_{\mathrm{cris}} \otimes_S \mathcal{M}^*)$. Now suppose that $f \in \mathrm{Hom}_S(\mathcal{M}, A_{\mathrm{cris}})$ also preserves Frobenius, that is, $f(\varphi_r(x)) = \varphi_r(f(x))$ for all $x \in \mathrm{Fil}^r\mathcal{M}$. Then

$$\varphi_r(f)(e_i) = \varphi_r(f)(\varphi_r(\alpha_i)) = \varphi_r(f(\alpha_i)) = f(\varphi_r(\alpha_i)) = f(e_i), \quad \forall\, i = 1, \dots, d.$$

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Therefore, $\varphi_r(f) = f$. On the other hand, if $f \in \mathrm{Fil}^r(A_{\mathrm{cris}} \otimes_S \mathcal{M}^*)^{\varphi_r=1}$, reversing the above argument shows that $f \in \mathrm{Hom}_{{}'\mathrm{Mod}^{\varphi}_{/S}}(\mathcal{M}, A_{\mathrm{cris}})$. □

By the above lemma, we get

$$T_{\mathrm{cris}}(\mathcal{M}) \simeq \mathrm{Fil}^r(A_{\mathrm{cris}} \otimes_S \mathcal{M}^*)^{\varphi_r=1} \hookrightarrow A_{\mathrm{cris}} \otimes_S \mathcal{M}^*. \tag{4.3.3}$$

So we also have $T_{\mathrm{cris}}(\mathcal{M}^*) \hookrightarrow A_{\mathrm{cris}} \otimes_S \mathcal{M}$.

From now on, we choose a generator t of $(\mathrm{Fil}^1 A_{\mathrm{cris}})^{\varphi_1=1}$ to identify $T_{\mathrm{cris}}(S^*)$ with $\mathbb{Z}_p(1)$. We will use a specific generator in § 5 and still denote it by t. See the discussion after Lemma 5.1.2.

Corollary 4.3.3. *The following diagram commutes.*

$$\begin{tikzcd} T_{\mathrm{cris}}(\mathcal{M}) \times T_{\mathrm{cris}}(\mathcal{M}^*) \arrow[r, hook] \arrow[d, "(4.3.2)"] & A_{\mathrm{cris}} \otimes_S \mathcal{M}^* \times A_{\mathrm{cris}} \otimes_S \mathcal{M} \arrow[d] \\ \mathbb{Z}_p(r) \arrow[r, "1 \mapsto t^r"] & A_{\mathrm{cris}} \end{tikzcd} \tag{4.3.4}$$

Here the top row is induced by (4.3.3) and the right column is induced by the canonical pairing $\mathcal{M} \times \mathcal{M}^* \to S$.

Proof. This follows from the fact that (4.3.2) is induced by taking the dual of the canonical pairing $\mathcal{M} \times \mathcal{M}^* \to S$. □

Now we can construct the following theorem to compare $\mathcal{M} \otimes_S A_{\mathrm{cris}}$ with $T^{\vee}_{\mathrm{cris}}(\mathcal{M}) \otimes_{\mathbb{Z}_p} A_{\mathrm{cris}}$.

Theorem 4.3.4. *There exist A_{cris}-linear injections*

$$\iota^* : T^{\vee}_{\mathrm{cris}}(\mathcal{M})(r) \otimes_{\mathbb{Z}_p} A^*_{\mathrm{cris}} \to A_{\mathrm{cris}} \otimes_S \mathcal{M}, \quad \iota : A_{\mathrm{cris}} \otimes_S \mathcal{M} \to T^{\vee}_{\mathrm{cris}}(\mathcal{M}) \otimes_{\mathbb{Z}_p} A_{\mathrm{cris}},$$

such that ι and ι^* are compatible with G_∞-actions, Frobenius and filtration. Furthermore, $\iota \circ \iota^* = \mathrm{Id} \otimes t^r$.

Remark 4.3.5. (1) Suppose that $\mathcal{M}$ is further a strongly divisible lattice. Let $\mathcal{D} = \mathcal{M} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ and $D \in \mathrm{MF}^{\mathrm{w}}(\varphi, N)$ such that $\mathcal{D} = \mathcal{D}(D)$. In [Bre97], Breuil extended the classical isomorphism

$$D \otimes_{K_0} B_{\mathrm{st}} \simeq V^{\vee}_{\mathrm{st}}(D) \otimes_{\mathbb{Q}_p} B_{\mathrm{st}}$$

to the $\widehat{B_{\mathrm{st}}}$-version: $\iota_S : \mathcal{D} \otimes_S \widehat{B_{\mathrm{st}}} \simeq V^{\vee}_{\mathrm{st}}(\mathcal{D}) \otimes_{\mathbb{Q}_p} \widehat{B_{\mathrm{st}}}$ where $\widehat{B_{\mathrm{st}}} := \widehat{A_{\mathrm{st}}}[1/p, 1/t]$. Note that B_{st} is a $\widehat{B_{\mathrm{st}}}$-algebra after modulo X. It is not hard to see that $\iota_S \otimes_{\widehat{B_{\mathrm{st}}}} B_{\mathrm{st}} \simeq \iota \otimes_{A_{\mathrm{cris}}} B_{\mathrm{st}}$. Therefore, ι may be seen as an integral version of ι_S.

(2) There exists a geometric interpretation of the above theorem. Conjecturally, the log-crystalline cohomology $\mathcal{M}$ of a scheme X over $\mathcal{O}_K$ (with some hypotheses) satisfies the axioms of strongly divisible modules, whereas the étale cohomology is closely related to $T_{\mathrm{cris}}(\mathcal{M})$. In the above situation, the morphism in Theorem 4.3.4 should correspond to an integral version of period isomorphism between these cohomologies. See [Bre02, § 4] for the exposé of this direction.

(3) If $\mathcal{M}$ comes from an $\mathfrak{M} \in \mathrm{Mod}^{\varphi}_{/\mathfrak{S}}$, i.e. $\mathcal{M} = \mathcal{M}_{\mathfrak{S}}(\mathfrak{M})$, then we have a similar result to the above theorem without restriction of r; see [Liu, § 5.3] for details.

Proof of Theorem 4.3.4. We use the same idea as the proof of Theorem 5(ii) in [Fal99]. First an easy computation shows that $T_{\mathrm{cris}}(\mathcal{M}) = \mathrm{Hom}_{A_{\mathrm{cris}}, \mathrm{Fil}^r, \varphi}(A_{\mathrm{cris}} \otimes_S \mathcal{M}, A_{\mathrm{cris}})$. Then we get a map

$$\tilde{\iota} : T_{\mathrm{cris}}(\mathcal{M}) \times A_{\mathrm{cris}} \otimes_S \mathcal{M} \to A_{\mathrm{cris}}. \tag{4.3.5}$$

Therefore, we get a natural map

$$\iota : A_{\mathrm{cris}} \otimes_S \mathcal{M} \to T^{\vee}_{\mathrm{cris}}(\mathcal{M}) \otimes_{\mathbb{Z}_p} A_{\mathrm{cris}},$$

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and it is easy to check that ι preserves G_∞-actions, Frobenius and filtration. On the other hand, by (4.3.3) and Lemma 4.3.1, we get

$$\iota^* : T_{\mathrm{cris}}(\mathcal{M}^*) \otimes_{\mathbb{Z}_p} A^*_{\mathrm{cris}} = T^\vee_{\mathrm{cris}}(\mathcal{M})(r) \otimes_{\mathbb{Z}_p} A^*_{\mathrm{cris}} \hookrightarrow A_{\mathrm{cris}} \otimes_S \mathcal{M},$$

and Lemma 4.3.2 shows that the above map is compatible with G_∞-actions, Frobenius and filtration. Combining ι^* with (4.3.5), it suffices to show that the following diagram commutes.

$$\begin{tikzcd}
T_{\mathrm{cris}}(\mathcal{M}) \times T_{\mathrm{cris}}(\mathcal{M}^*) \otimes_{\mathbb{Z}_p} A^*_{\mathrm{cris}} \arrow[r, "\mathrm{Id}\times\iota^*"] \arrow[d, "(4.3.2)\otimes\mathrm{Id}"] & T_{\mathrm{cris}}(\mathcal{M}) \times A_{\mathrm{cris}} \otimes_S \mathcal{M} \arrow[d, "(4.3.5)"] \\
\mathbb{Z}_p(r) \otimes_{\mathbb{Z}_p} A^*_{\mathrm{cris}} \arrow[r, "1\mapsto t^r"] & A_{\mathrm{cris}}
\end{tikzcd}$$

Note that we have an injection $T_{\mathrm{cris}}(\mathcal{M}) \hookrightarrow A_{\mathrm{cris}} \otimes_S \mathcal{M}^*$ by (4.3.3). So the commutativity of the above diagram follows the commutativity of diagram (4.3.4), and this is proved in Corollary 4.3.3. $\square$

Let $\alpha_1, \dots, \alpha_d \in \mathrm{Fil}^r \mathcal{M}$ as in Proposition 4.1.2 and $e_1, \dots, e_d \in \mathcal{M}$ a basis of $\mathcal{M}$. Let $\mathfrak{e}_1, \dots, \mathfrak{e}_d$ be a basis of $T^\vee_{\mathrm{cris}}(\mathcal{M})$. By Theorem 4.3.4, we have

$$\iota(\alpha_1, \dots, \alpha_d) = (\mathfrak{e}_d, \dots, \mathfrak{e}_d) C,$$

where C is a $d \times d$ matrix with coefficients in $\mathrm{Fil}^r A_{\mathrm{cris}}$.

LEMMA 4.3.6. *There exists a $d \times d$ matrix C' with coefficients in A_{cris} such that the coefficients of $C'C - t^r I$ are all in $\mathrm{Fil}^p A_{\mathrm{cris}}$.*

Proof. Forgetting G_∞-actions, Frobenius and filtration structures, we may identify $T^\vee_{\mathrm{cris}}(\mathcal{M}) \otimes_{\mathbb{Z}_p} A_{\mathrm{cris}}$ with $T^\vee_{\mathrm{cris}}(\mathcal{M})(r) \otimes_{\mathbb{Z}_p} A^*_{\mathrm{cris}}$ as finite free A_{cris}-modules. In particular, we regard $(\mathfrak{e}_1, \dots, \mathfrak{e}_d)$ as a basis of $T^\vee_{\mathrm{cris}}(\mathcal{M})(r)$. Then $\iota^* \circ \iota$ makes sense and $\iota^* \circ \iota = t^r \otimes \mathrm{Id}$ by Theorem 4.3.4. Therefore, we get

$$t^r(\alpha_1, \dots, \alpha_d) = \iota^* \circ \iota(\alpha_1, \dots, \alpha_d) = \iota^*(\mathfrak{e}_1, \dots, \mathfrak{e}_d) C. \tag{4.3.6}$$

Note that $\mathrm{Fil}^r A^*_{\mathrm{cris}} = A_{\mathrm{cris}}$, so that $(\mathfrak{e}_1, \dots, \mathfrak{e}_d) \in \mathrm{Fil}^r(T^\vee_{\mathrm{cris}}(\mathcal{M})(r) \otimes_{\mathbb{Z}_p} A^*_{\mathrm{cris}})$, and then $\iota^*(\mathfrak{e}_1, \dots, \mathfrak{e}_d)$ is in $\mathrm{Fil}^r(\mathcal{M} \otimes_S A_{\mathrm{cris}})$. By Corollary 4.1.3, we have

$$\iota^*(\mathfrak{e}_1, \dots, \mathfrak{e}_d) = (\alpha_1, \dots, \alpha_d) C' + (e_1, \dots, e_d) D, \tag{4.3.7}$$

where $e_1, \dots, e_d$ is a basis of $\mathcal{M}$, and C' and D are $d \times d$ matrices with coefficients in A_{cris} and $\mathrm{Fil}^p A_{\mathrm{cris}}$ respectively. Write $(\alpha_1, \dots, \alpha_d) = (e_1, \dots, e_d) A$ with A a $d \times d$ matrix. Combining (4.3.6) and (4.3.7), we have

$$t^r A = AC'C + DC.$$

By Proposition 4.1.2, there exists a $d \times d$ matrix B with coefficients in S such that $AB = BA = E(u)^r I$, so we get $E(u)^r(t^r I - C'C) = BDC$. Note that the coefficients of C and D are in $\mathrm{Fil}^r A_{\mathrm{cris}}$ and $\mathrm{Fil}^p A_{\mathrm{cris}}$ respectively. Thus the coefficients of $E(u)^r(t^r I - C'C)$ are in $\mathrm{Fil}^{r+p} A_{\mathrm{cris}}$. By Lemma 3.2.2, the coefficients of $C'C - t^r I$ are all in the $\mathrm{Fil}^p A_{\mathrm{cris}}$. $\square$

5. The proof of Lemma 3.5.3

In this section, we will show how to recover monodromy N on $\mathcal{M}$ by the G-action on T and then prove Lemma 3.5.3. Recall that T is a G-stable $\mathbb{Z}_p$-lattice in a semi-stable p-adic Galois representation V, $\mathcal{M} = \mathcal{M}_{\mathfrak{S}}(\mathfrak{M})$ the quasi-strongly divisible lattice such that $T_{\mathrm{cris}}(\mathcal{M}) = T|_{G_\infty}$ (Proposition 3.4.6) and $\mathcal{D} := \mathcal{M} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \in \mathcal{MF}^{\mathrm{w}}(\varphi, N)$ satisfying $V \simeq V_{\mathrm{st}}(\mathcal{D})$. We first construct a G-action on $A_{\mathrm{cris}} \otimes_S \mathcal{D}$ by using N on $\mathcal{D}$.

5.1 G-action on $A_{\mathrm{cris}} \otimes_S \mathcal{D}$

We already have a natural semi-linear G_∞-action on $A_{\mathrm{cris}} \otimes_S \mathcal{D}$ induced from the G_∞-action on A_{cris}. We extend this to a G-action by using N on $\mathcal{D}$. For any $\sigma \in G$, recall that $\underline{\epsilon}(\sigma) = \sigma([\underline{\pi}])/[\underline{\pi}]$. For any $a \otimes x \in A_{\mathrm{cris}} \otimes_S \mathcal{D}$, define

$$\sigma(a \otimes x) = \sum_{i=0}^{\infty} \sigma(a)\gamma_i(-\log(\underline{\epsilon}(\sigma))) \otimes N^i(x), \tag{5.1.1}$$

where $\gamma_i(x) = x^i/i!$ is the standard divided power. Note that if $\sigma \in G_\infty$, then $\log(\underline{\epsilon}(\sigma)) = 0$ and $\sigma(a \otimes x) = \sigma(a) \otimes x$. Thus G-action defined above (if it is well defined) is compatible with the natural G_∞-action on $A_{\mathrm{cris}} \otimes_S \mathcal{D}$.

Lemma 5.1.1. *The above action is a well defined A_{cris}-semi-linear G-action on $A_{\mathrm{cris}} \otimes_S \mathcal{D}$ and compatible with Frobenius and filtration.*

Proof. In fact, this result has been explicitly or nonexplicitly used in several papers, e.g. [Fal99, § 4]. To see that the series on the right side of (5.1.1) converges, note that $\mathcal{D} = D \otimes_{W(k)} S$ and N is nilpotent on D. It suffices to show that $\gamma_i(-\log(\underline{\epsilon}(\sigma))) \to 0$ when $i \to \infty$. This is a well-known result. See for example, [Fon94a, § 5.2.4].

For any $f(u) \in S$, $x \in \mathcal{D}$ and σ, $\tau \in G$, we need to check that:

(1) $\sigma(1 \otimes f(u)x) = \sigma(f([\underline{\pi}]) \otimes x) = f(\sigma([\underline{\pi}])) \otimes \sigma(x)$;

(2) $\sigma(\tau(1 \otimes x)) = (\sigma \circ \tau)(1 \otimes x)$;

(3) the G-action preserves filtration and commutes with φ.

It is fairly standard direct calculations to check these equations combined with the facts that $\mathrm{Fil}^1 S \cdot N(\mathrm{Fil}^i \mathcal{D}) \subset \mathrm{Fil}^i \mathcal{D}$, $\log(\underline{\epsilon}(\sigma)) \in \mathrm{Fil}^1 A_{\mathrm{cris}}$ and $N\varphi = p\varphi N$ in $\mathcal{D}$. $\square$

One the other hand, given the G-action on $A_{\mathrm{cris}} \otimes_S \mathcal{D}$ defined via (5.1.1), we want to define a certain logarithm of the G-action to recover N. (We should be careful at this point because the G-action is not linear.) A technical result is needed to define such a logarithm.

For any field extension F over $\mathbb{Q}_p$, denote $F_{p^\infty} = \bigcup_{n=1}^\infty F(\zeta_{p^n})$ with ζ_{p^n} a p^nth primitive root of unity. Thus $K_{\infty,p^\infty} = \bigcup_{n=1}^\infty K(\sqrt[p^n]{\pi}, \zeta_{p^n})$ is Galois. So we have the following field extensions.

$$\begin{tikzcd}[column sep=small] & K_{\infty,p^\infty} \arrow[dl, dash] \arrow[dr, dash] & \\ K_\infty \arrow[dr, dash] & & K_{p^\infty} \arrow[dl, dash, "H_K"] \\ & K & \end{tikzcd}$$

Let $H_K = \mathrm{Gal}(K_{p^\infty}/K) \subset \mathrm{Gal}(\mathbb{Q}_{p,p^\infty}/\mathbb{Q}_p) \simeq \mathbb{Z}_p^\times$. So H_K may be identified as a closed subgroup of $\mathbb{Z}_p^\times$.

Lemma 5.1.2. *The following hold:*

(1) $K_{p^\infty} \cap K_\infty = K$;

(2) $\mathrm{Gal}(K_{\infty,p^\infty}/K_\infty) \simeq H_K$ *and* $\mathrm{Gal}(K_{\infty,p^\infty}/K_{p^\infty}) \simeq \mathbb{Z}_p(1)$;

(3) $\mathrm{Gal}(K_{\infty,p^\infty}/K) = \mathrm{Gal}(K_{\infty,p^\infty}/K_{p^\infty}) \rtimes \mathrm{Gal}(K_{\infty,p^\infty}/K_\infty) \simeq \mathbb{Z}_p(1) \rtimes H_K$, *and* H_K *acts on* $\mathbb{Z}_p(1)$ *by the cyclotomic character.*

Proof. We only need to prove part (1). For any $n \geqslant 0$, let $F_n = K(\pi_n) \cap K_{p^\infty}$ and denote $K(\pi_n)$ by K_n. We prove that $F_n = K$ by an induction on n. The case $n = 0$ is trivial. Now suppose that $F_n = K$ and $F_{n+1} \neq K$. We first show that $\zeta_p \in K$. Note that

$$[F_{n+1} \cdot K_n : K_n] \mid [K_{n+1} : K_n] = p \text{ and } F_{n+1} \cdot K_n \neq K_n.$$

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We have $[F_{n+1}\cdot K_n : K_n] = p$ and $F_{n+1}\cdot K_n = K_{n+1}$. Since F_{n+1} is abelian over K and $F_{n+1}\cap K_n = K$, K_{n+1}/K_n is Galois and $\mathrm{Gal}(K_{n+1}/K_n) \simeq \mathrm{Gal}(F_{n+1}/K)$. Let $\sigma \in \mathrm{Gal}(K_{n+1}/K_n)$ be a nontrivial element; then $\sigma(\pi_{n+1})/\pi_{n+1} \in K_{n+1}$ is a nontrivial pth root of unity. So $\zeta_p \in K_{n+1}$. Note that

$$[K_n(\zeta_p) : K_n] \leqslant p-1 \text{ and } [K_n(\zeta_p) : K_n] \mid [K_{n+1} : K_n] = p.$$

We have $K_n(\zeta_p) = K_n$ and $\zeta_p \in K_n$. By the induction that $F_n = K$, we get $\zeta_p \in K$.

Now $\mathrm{Gal}(K_{p^\infty}/K)$ is a closed subgroup of $\mathrm{Gal}(\mathbb{Q}_{p,p^\infty}/\mathbb{Q}_p(\zeta_p)) \simeq 1+p\mathbb{Z}_p$. Note that $p > 2$. By taking p-adic logarithm, we see that $1+p\mathbb{Z}_p \simeq \mathbb{Z}_p$ as pro-p-groups. Hence any closed subgroup of $1+p\mathbb{Z}_p$ has the form $1+p^n\mathbb{Z}_p$. Since $[F_{n+1} : K] = p$, there exists an m such that $\mathrm{Gal}(K_{p^\infty}/K) \simeq 1+p^m\mathbb{Z}_p \simeq \mathrm{Gal}(\mathbb{Q}_{p,p^\infty}/\mathbb{Q}_p(\zeta_{p^m}))$ and $\mathrm{Gal}(K_{p^\infty}/F_{n+1}) \simeq 1+p^{m+1}\mathbb{Z}_p \simeq \mathrm{Gal}(\mathbb{Q}_{p,p^\infty}/\mathbb{Q}_p(\zeta_{p^{m+1}}))$. Therefore $\zeta_{p^m} \in K$, $\zeta_{p^{m+1}} \notin K$ and $F_{n+1} = K(\zeta_{p^{m+1}})$. In particular, $\mathrm{Gal}(K_{n+1}/K_n) \simeq \mathrm{Gal}(K(\zeta_{p^{m+1}})/K(\zeta_{p^m})) \simeq \mathbb{Z}/p\mathbb{Z}$. Choose $\sigma \in \mathrm{Gal}(K_{n+1}/K_n)$ such that $\sigma(\zeta_{p^{m+1}}) = \zeta_p\zeta_{p^{m+1}}$. Then $\sigma(\pi_{n+1}) = \zeta_p^b\pi_{n+1}$ for some $b \in (\mathbb{Z}/p\mathbb{Z})^\times$. Write

$$\zeta_{p^{m+1}} = \sum_{i=0}^{p-1} a_i\pi_{n+1}^i \quad \text{with } a_i \in \mathcal{O}_{K_n}.$$

Then

$$\zeta_p\zeta_{p^{m+1}} = \sigma(\zeta_{p^{m+1}}) = \sigma\left(\sum_{i=0}^{p-1} a_i\pi_{n+1}^i\right) = \sum_{i=0}^{p-1} a_i\zeta_p^{bi}\pi_{n+1}^i.$$

Thus we have $a_0 = \zeta_p a_0$ and $a_0 = 0$. Then $\zeta_{p^{m+1}}$ is not a unit. This is a contradiction. Therefore F_{n+1} has to be K. $\square$

Remark 5.1.3. The above lemma fails if $p = 2$ in general. For example, let $K = \mathbb{Q}_2$ and $\pi = 2$. Then $\mathbb{Q}_2(\sqrt{2}) \subset \mathbb{Q}_2(\zeta_8)$. On the other hand, if $\mathbb{Q}_2(\zeta_4) \subset K$, then $\mathrm{Gal}(K_{2^\infty}/K) \subset \mathrm{Gal}(\mathbb{Q}_{2,2^\infty}/\mathbb{Q}_2(\zeta_4)) \simeq 1+4\mathbb{Z}_2$. The above strategy by p-adic logarithm also works here and we still have $K_{2^\infty} \cap K_\infty = K$.

Fix a topological generator τ of $\mathrm{Gal}(K_{\infty,p^\infty}/K_{p^\infty})$. The above lemma shows that $-\log(\underline{\epsilon}(\tau))$ is a generator of $(\mathrm{Fil}^1 A_{\mathrm{cris}})^{\varphi_1=1}$. So from now on, we fix $t := -\log(\underline{\epsilon}(\tau))$. Note that τ acts trivially on $\underline{\epsilon}(\tau)$, thus on t. Therefore, for any $n \geqslant 0$ and $x \in \mathcal{D}$, an easy induction on n shows that

$$(\tau-1)^n(x) = \sum_{m=n}^{\infty}\left(\sum_{i_1+\cdots+i_n=m,\, i_j\geqslant 1} \frac{m!}{i_1!\cdots i_n!}\right)\gamma_m(t)\otimes N^m(x). \tag{5.1.2}$$

In particular, $(\tau-1)^n(x) \in \mathrm{Fil}^n B^+_{\mathrm{cris}} \otimes_S \mathcal{D}$ and $((\tau-1)^n/n)(x) \to 0$ p-adically as $n \to \infty$ (in fact, it is easy to show that $\gamma_n(t)/n \to 0$ p-adically, see [Fon94a, § 5.2.4]). So we can define

$$\log(\tau)(x) = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{(\tau-1)^n}{n}(x) \tag{5.1.3}$$

and a direct computation shows that

$$\log(\tau)(x) = t\otimes N(x). \tag{5.1.4}$$

5.2 A $\mathbb{Q}_p$-version of Theorem 4.3.4

Let $D \in \mathrm{MF}^{\mathrm{w}}(\varphi, N)$ be a weakly admissible filtered (φ, N)-module and

$$\mathcal{D} = \mathcal{D}(D) := D\otimes_{W(k)} S \in \mathcal{MF}^{\mathrm{w}}(\varphi, N).$$

By Lemma 3.4.3, the map

$$V_{\mathrm{st}}(\mathcal{D}) = \mathrm{Hom}_{'\mathrm{Mod}_{/S}^{\varphi,N}}(\mathcal{D}, \widehat{A_{\mathrm{st}}}[1/p]) \to \mathrm{Hom}_{'\mathrm{Mod}_{/S}^{\varphi}}(\mathcal{D}, B^+_{\mathrm{cris}}) \tag{5.2.1}$$

induced by the canonical projection $\widehat{A_{\rm st}} \to A_{\rm cris}$ defined by sending $\gamma_i(X) \to 0$ is an isomorphism compatible with G_∞-action. On the other hand,

$$\mathrm{Hom}_{'\mathrm{Mod}^{\varphi}_{/S}}(\mathcal{D}, B^+_{\rm cris}) \simeq \mathrm{Hom}_{A_{\rm cris},\mathrm{Fil}^{\cdot},\varphi}(A_{\rm cris} \otimes_S \mathcal{D}, B^+_{\rm cris}). \tag{5.2.2}$$

By Lemma 5.1.1, we have a natural G-action on $A_{\rm cris} \otimes_S \mathcal{D}$ via (5.1.1). So there exists a G-action on the right side of (5.2.2) defined by

$$\sigma(f)(x) = \sigma(f(\sigma^{-1}(x))) \quad \text{for any } x \in A_{\rm cris} \otimes \mathcal{D}.$$

Combining (5.2.1) with (5.2.2) together, we have the next result.

Lemma 5.2.1. *The map*

$$V_{\rm st}(\mathcal{D}) = \mathrm{Hom}_{'\mathrm{Mod}^{\varphi,N}_{/S}}(\mathcal{D}, \widehat{A_{\rm st}}[1/p]) \to \mathrm{Hom}_{A_{\rm cris},\mathrm{Fil}^{\cdot},\varphi}(A_{\rm cris} \otimes_S \mathcal{D}, B^+_{\rm cris})$$

induced by (5.2.1) and (5.2.2) is a G-equivariant isomorphism.

Proof. Lemma 3.4.3 has proved that the above map is a $\mathbb{Q}_p$-linear bijection. So we only need to check the G-equivariance. For any $f \in \mathrm{Hom}_{'\mathrm{Mod}^{\varphi,N}_{/S}}(\mathcal{D}, \widehat{A_{\rm st}}[1/p])$, let $f_0 \in \mathrm{Hom}_{'\mathrm{Mod}^{\varphi}_{/S}}(\mathcal{D}, B^+_{\rm cris})$ be its image of the map defined in (5.2.1). It suffices to check, for any $x \in D$, $\sigma \in G$, that $\sigma(f)_0(x) = \sigma(f_0(\sigma^{-1}(x)))$. Using (3.4.4) and the fact that $\sigma(X) = \underline{\epsilon}(\sigma)X + \underline{\epsilon}(\sigma) - 1$, we have

$$\begin{aligned}\sigma(f(x)) &= \sum_{i\geqslant 0} \sigma(f_0(N^i(x)))\gamma_i(\log(1+\sigma(X))) \\ &= \sum_{i\geqslant 0} \sigma(f_0(N^i(x))) \sum_{j=0}^{i} \gamma_{i-j}(\log(\underline{\epsilon}(\sigma)))\gamma_j(\log(1+X)).\end{aligned}$$

Modulo X, we then get

$$\begin{aligned}\sigma(f)_0(x) &= \sum_{j\geqslant 0} \sigma(f_0(N^j(x)))\gamma_j(\log(\underline{\epsilon}(\sigma))) \\ &= \sigma\biggl(f_0\biggl(\sum_{j\geqslant 0} \gamma_j(\log(\sigma^{-1}\underline{\epsilon}(\sigma))) \otimes N^j(x)\biggr)\biggr) \\ &= \sigma(f_0(\sigma^{-1}(x))).\end{aligned}$$

□

Corollary 5.2.2. *The $B^+_{\rm cris}$-linear injections*

$$\iota \otimes_{\mathbb{Z}_p} \mathbb{Q}_p : A_{\rm cris} \otimes_S \mathcal{D} \to V^\vee_{\rm st}(\mathcal{D}) \otimes_{\mathbb{Z}_p} A_{\rm cris},$$

$$\iota^* \otimes_{\mathbb{Z}_p} \mathbb{Q}_p : V^\vee_{\rm st}(\mathcal{D})(r) \otimes_{\mathbb{Z}_p} A^*_{\rm cris} \to A_{\rm cris} \otimes_S \mathcal{D},$$

are compatible with G-actions, where ι and ι^* are constructed as in Theorem 4.3.4.

5.3 Proof of the Main Theorem

We use the same notation as in § 3.5 and Lemma 3.5.3. Recall that T is a G-stable $\mathbb{Z}_p$-lattice in a semi-stable p-adic Galois representation V, and $\mathcal{M}$ the quasi-strongly divisible lattice such that $T_{\rm cris}(\mathcal{M}) = T|_{G_\infty}$ (Proposition 3.4.6). Also recall that τ is the fixed topological generator of $\mathrm{Gal}(K_{\infty,p^\infty}/K_{p^\infty})$ discussed in § 5.1. We will use Lemma 4.3.6 and Corollary 5.2.2 to prove that N is stable on $\mathcal{M}$ in two steps. The first step is to show that $A_{\rm cris} \otimes_S \mathcal{M}$ is G-stable in $A_{\rm cris} \otimes_S \mathcal{D}$. More generally, we have the following lemma.

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Lemma 5.3.1. *We use the notation as in Theorem 4.3.4. Let $\mathcal{M}, \mathcal{M}' \in \mathrm{Mod}^{\varphi}_{/S}$. Suppose that we have the commutative diagram*

$$\begin{tikzcd} A_{\mathrm{cris}} \otimes_S \mathcal{M}' \arrow[r, "\iota_{\mathcal{M}'}"] \arrow[d, "\mathfrak{f}"] & T^{\vee}_{\mathrm{cris}}(\mathcal{M}') \otimes_{\mathbb{Z}_p} A_{\mathrm{cris}} \arrow[d, "f"] \\ A_{\mathrm{cris}} \otimes_S \mathcal{M} \arrow[r, "\iota_{\mathcal{M}}"] & T^{\vee}_{\mathrm{cris}}(\mathcal{M}) \otimes_{\mathbb{Z}_p} A_{\mathrm{cris}} \end{tikzcd} \tag{5.3.1}$$

where $\mathfrak{f}$ and f are A_{cris}-linear or τ-semi-linear morphisms compatible with Frobenius and filtration. Then we have that, if $p|f$, then $p|\mathfrak{f}$.

Proof. We only prove the lemma in the case that $\mathfrak{f}$ and f are A_{cris}-linear. The proof for the τ-semi-linear case is totally the same.

Let d' be the S-rank of $\mathcal{M}'$, and $\alpha'_1, \ldots, \alpha'_{d'} \in \mathrm{Fil}^r \mathcal{M}'$ such that $\varphi_r(\alpha'_1), \ldots, \varphi_r(\alpha'_{d'})$ is a basis of $\mathcal{M}'$. Since $\mathfrak{f}$ preserves filtration, $\mathfrak{f}(\alpha'_1, \ldots, \alpha'_{d'}) \in [\mathrm{Fil}^r(A_{\mathrm{cris}} \otimes_S \mathcal{M})]^d$. By Corollary 4.1.3, we have

$$\mathrm{Fil}^r(A_{\mathrm{cris}} \otimes_S \mathcal{M}) = \bigoplus_{i=1}^{d} A_{\mathrm{cris}} \otimes \alpha_i + \mathrm{Fil}^p A_{\mathrm{cris}} \otimes_S \mathcal{M} \tag{5.3.2}$$

with $(e_1, \ldots, e_d) = (\varphi_r(\alpha_1), \ldots, \varphi_r(\alpha_d))$ a basis of $\mathcal{M}$. Therefore there exist $d \times d'$ matrices X, W with coefficients in A_{cris}, $\mathrm{Fil}^p A_{\mathrm{cris}}$ respectively such that

$$\mathfrak{f}(\alpha'_1, \ldots, \alpha'_{d'}) = (\alpha_1, \ldots, \alpha_d)X + (e_1, \ldots, e_d)W. \tag{5.3.3}$$

We claim that the coefficients of X are in $\mathrm{Fil}^1 A_{\mathrm{cris}} + pA_{\mathrm{cris}}$.

To see the claim, applying $\iota_{\mathcal{M}}$ on both sides of (5.3.3), we have

$$\iota_{\mathcal{M}} \circ \mathfrak{f}(\alpha'_1, \ldots, \alpha'_{d'}) = \iota_{\mathcal{M}}(\alpha_1, \ldots, \alpha_d)X + \iota_{\mathcal{M}}(e_1, \ldots, e_d)W = (\mathfrak{e}_1, \ldots, \mathfrak{e}_d)(CX + W'),$$

where $\mathfrak{e}_1, \ldots, \mathfrak{e}_d$ is a basis of $T^{\vee}_{\mathrm{cris}}(\mathcal{M})$ as in Lemma 4.3.6 and C, W' are matrices with coefficients in A_{cris}, $\mathrm{Fil}^p A_{\mathrm{cris}}$ respectively such that $\iota_{\mathcal{M}}(\alpha_1, \ldots, \alpha_d) = (\mathfrak{e}_1, \ldots, \mathfrak{e}_d)C$ and $\iota_{\mathcal{M}}(e_1, \ldots, e_d)W = (\mathfrak{e}_1, \ldots, \mathfrak{e}_d)W'$. On the other hand, since diagram (5.3.1) is commutative and $p|f$, all the coefficients of $CX + W'$ are in pA_{cris}. By Lemma 4.3.6, there exists a matrix C' such that the coefficients of $C'C - t^r I$ are in $\mathrm{Fil}^p A_{\mathrm{cris}}$. Thus the coefficients of $t^r X$ are in $\mathrm{Fil}^p A_{\mathrm{cris}} + pA_{\mathrm{cris}}$. To show the claim, it suffices to show that if $x \in A_{\mathrm{cris}}$ and $t^r x \in pA_{\mathrm{cris}} + \mathrm{Fil}^p A_{\mathrm{cris}}$ then $x \in \mathrm{Fil}^1 A_{\mathrm{cris}} + pA_{\mathrm{cris}}$. Recall that $R = \varprojlim \mathcal{O}_{\bar{K}}/p$ constructed in § 2.2. For any $(a_i)_{i \geqslant 0} \in R$ with $a_i \in \mathcal{O}_{\bar{K}}/p$, let $\hat{a}_i \in \mathcal{O}_{\bar{K}}$ be a lift of a_i, then $a^{(0)} = \lim_{n \to \infty}(\hat{a}_n)^{p^n}$ is well defined and independent of the choice of $\hat{a}_i$. We define the valuation on R by $v_R((a_i)_{i \geqslant 0}) = v(a^{(0)})$ where $v(\cdot)$ is the standard valuation of $\mathcal{O}_{\bar{K}}$ (§§ 1.2.2 and 1.2.3 in [Fon94a]). Let $\mathrm{Fil}^i R$ be the image of $\mathrm{Fil}^i(W(R))$ under the reduction mod p. We see that $\mathrm{Fil}^1 R = \{x \in R \mid v_R(x) \geqslant 1\}$ and $A_{\mathrm{cris}}/(pA_{\mathrm{cris}} + \mathrm{Fil}^p A_{\mathrm{cris}}) \simeq R/\mathrm{Fil}^p R$. Let $\bar{x}$ and $\bar{t}$ be the image of x and t in $R/\mathrm{Fil}^p R$ respectively. Note that

$$v_R(\bar{t}) = v_R\left(\frac{\tau(\underline{\pi})}{\underline{\pi}} - 1\right) = \frac{p}{p-1}.$$

Since $\bar{t}^r \bar{x} \in \mathrm{Fil}^p R$, $v_R(\bar{t}^r \bar{x}) \geqslant p$. But

$$v_R(\bar{t}^r) = \frac{rp}{p-1} < p - 1$$

because $r \leqslant p - 2$. Therefore, $v_R(\bar{x}) \geqslant 1$ and $x \in \mathrm{Fil}^1 A_{\mathrm{cris}} \bmod p$.

Now since $\mathfrak{f}$ is compatible with Frobenius, by (5.3.3) we have

$$\begin{aligned} \mathfrak{f}((\varphi_r(\alpha'_1), \ldots, \varphi_r(\alpha'_{d'}))) &= \varphi_r((\alpha_1, \ldots, \alpha_d)X + (e_1, \ldots e_d)W) \\ &= (e_1, \ldots, e_d)\varphi(X) + \varphi(e_1, \ldots, e_d)\varphi_r(W). \end{aligned}$$

Since the coefficients of X are in $\mathrm{Fil}^1 A_{\mathrm{cris}} + pA_{\mathrm{cris}}$, we have $p|\varphi(X)$. Note that $p|\varphi_r(W)$ because the coefficients of W are in $\mathrm{Fil}^p A_{\mathrm{cris}}$. Finally, since $\varphi_r(\alpha'_1), \ldots, \varphi_r(\alpha'_{d'})$ is a basis of $\mathcal{M}'$, we get $p|\mathfrak{f}$. ☐

Proof of Lemma 3.4.7. It suffices to prove that $\mathcal{M}' \subset \mathcal{M}$. Choose the smallest integer n such that $p^n \mathcal{M}' \subset \mathcal{M}$. Then $p^n : \mathcal{M}' \to \mathcal{M}$ is a morphism in $\mathrm{Mod}^{\varphi}_{/S}$. Use Lemma 5.3.1 for $\mathfrak{f} = p^n$ and $f = p^n$. Then we see that n has to be 0. ☐

Combining Theorem 4.3.4 with Corollary 5.2.2, we have the commutative diagram

$$\begin{tikzcd} A_{\mathrm{cris}} \otimes_S \mathcal{D} \arrow[r, "\iota \otimes_{\mathbb{Z}_p} \mathbb{Q}_p"] & V^\vee_{\mathrm{st}}(\mathcal{D}) \otimes_{\mathbb{Z}_p} A_{\mathrm{cris}} \\ A_{\mathrm{cris}} \otimes_S \mathcal{M} \arrow[r, "\iota"] \arrow[u, hook] & T^\vee_{\mathrm{cris}}(\mathcal{M}) \otimes_{\mathbb{Z}_p} A_{\mathrm{cris}} \arrow[u, hook] \end{tikzcd} \tag{5.3.4}$$

where the top row map is compatible with G-action and the bottom row map is compatible with G_∞-action. We claim that $A_{\mathrm{cris}} \otimes_S \mathcal{M}$ is stable under G. To check that this, it suffices to check that $A_{\mathrm{cris}} \otimes_S \mathcal{M}$ is stable under τ. Since $T^\vee = T^\vee_{\mathrm{cris}}(\mathcal{M})$ is a G-stable $\mathbb{Z}_p$-lattice, we see that $T^\vee \otimes_{\mathbb{Z}_p} A_{\mathrm{cris}}$ is stable under τ. Choose n such that $p^n \tau(A_{\mathrm{cris}} \otimes_S \mathcal{M}) \subseteq A_{\mathrm{cris}} \otimes_S \mathcal{M}$. Now using Lemma 5.3.1 for $\mathfrak{f} = p^n \tau$ on $A_{\mathrm{cris}} \otimes_S \mathcal{M}$ and $f = p^n \tau$ on $T^\vee_{\mathrm{cris}}(\mathcal{M}) \otimes_{\mathbb{Z}_p} A_{\mathrm{cris}}$, we have $\tau(A_{\mathrm{cris}} \otimes_S \mathcal{M}) \subseteq A_{\mathrm{cris}} \otimes_S \mathcal{M}$.

Now we begin the second step to show that $\mathcal{M}$ is stable under N. By (5.1.4), for any $x \in \mathcal{M}$, we have $t \otimes N(x) = \log(\tau)(x)$. We claim that $t \otimes N(\mathcal{M}) \subset A_{\mathrm{cris}} \otimes_S \mathcal{M}$ by proving that $\log(\tau)(\mathcal{M}) \subset A_{\mathrm{cris}} \otimes_S \mathcal{M}$. It suffices to show that

$$\frac{(\tau - 1)^n}{n}(\mathcal{M}) \subset A_{\mathrm{cris}} \otimes_S \mathcal{M}$$

for all $n \geqslant p$. Let $(\alpha_1, \ldots, \alpha_d) \in \mathrm{Fil}^r \mathcal{M}$ constructed in Proposition 4.1.2, and $(e_1, \ldots, e_d) = (\varphi_r(\alpha_1), \ldots, \varphi_r(\alpha_d))$ a basis of $\mathcal{M}$. Using (5.1.2), we see that

$$(\tau - 1)^n(\alpha_1, \ldots, \alpha_d) \in [\mathrm{Fil}^n B^+_{\mathrm{cris}}(A_{\mathrm{cris}} \otimes_S \mathcal{M})]^d.$$

Since $\tau(\mathcal{M}) \subset (A_{\mathrm{cris}} \otimes_S \mathcal{M})$, we get

$$(\tau - 1)^n(\alpha_1, \ldots, \alpha_d) \in [\mathrm{Fil}^n A_{\mathrm{cris}}(A_{\mathrm{cris}} \otimes_S \mathcal{M})]^d.$$

Therefore, we obtain

$$(\tau - 1)^n(e_1, \ldots, e_d) = \varphi_r((\tau - 1)^n(\alpha_1, \ldots, \alpha_d)) \in [\varphi_r(\mathrm{Fil}^n A_{\mathrm{cris}}) \cdot \varphi(A_{\mathrm{cris}} \otimes_S \mathcal{M})]^d.$$

Now it suffices to check that, for any $n \geqslant p$ and $x \in \mathrm{Fil}^n A_{\mathrm{cris}}$, we have $\varphi_r(x)/n \in A_{\mathrm{cris}}$. We can further reduce the problem to check if $\varphi(E(u)^m)/p^r nm! \in S$ for all $m \geqslant n \geqslant p$. Note that $c_1 = \varphi(E(u))/p$ is a unit in S. So it is equivalent to show that $p^{m-r}/nm! \in \mathbb{Z}_p$ for all $m \geqslant n \geqslant p$ and we include the computation in the lemma below. Thus we prove the claim that $t \otimes N(x) \in A_{\mathrm{cris}} \otimes_S \mathcal{M}$.

Lemma 5.3.2. *If $m \geqslant n \geqslant p > 2$ and $r < p - 1$, then $m - r - v_p(nm!) \geqslant 0$.*

Proof. Since $n \geqslant p$, we have $v_p(n) \leqslant n/p \leqslant m/p$. Hence

$$d = m - v_p(nm!) \geqslant m - \frac{m}{p-1} - \frac{m}{p} = \frac{m(p^2 - 3p + 1)}{p(p-1)} \geqslant \frac{p^2 - 3p + 1}{p - 1} = p - 2 - \frac{1}{p-1}.$$

Since d is an integer, it follows that $d \geqslant p - 2 \geqslant r$. ☐

Finally, suppose that we have

$$N((e_1, \ldots, e_d)) = (e_1, \ldots, e_d)W$$

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with the coefficients of W in S_{K_0}. Select the smallest number n such that all the coefficients of p^nW are in S. Then $p^nN(\mathcal{M}) \subset \mathcal{M}$. Since $E(u)N(\mathrm{Fil}^r\mathcal{D}) \subset \mathrm{Fil}^r\mathcal{D}$, we have

$$E(u)p^nN((\alpha_1,\ldots,\alpha_d)) = (\alpha_1,\ldots,\alpha_d)X + (e_1,\ldots,e_d)Y \tag{5.3.5}$$

with the coefficients of X, Y in S, Fil^pS respectively. On the other hand, note that $t\otimes N(\mathcal{M}) \subset A_{\mathrm{cris}}\otimes_S\mathcal{M}$ and $t\otimes N(\mathrm{Fil}^r\mathcal{M}) \subset \mathrm{Fil}^r(A_{\mathrm{cris}}\otimes_S\mathcal{M})$ because there exists $\gamma \in A_{\mathrm{cris}}$ such that $t - E([\underline{\pi}])\gamma \in \mathrm{Fil}^pA_{\mathrm{cris}}$. We have

$$tN((\alpha_1,\ldots,\alpha_d)) = (\alpha_1,\ldots,\alpha_d)X' + (e_1,\ldots,e_d)Y' \tag{5.3.6}$$

with the coefficients of X', Y' in A_{cris}, $\mathrm{Fil}^pA_{\mathrm{cris}}$ respectively. Combining (5.3.5) with (5.3.6), we have

$$A(tX - E(u)p^nX') = tY - E(u)p^nY',$$

where $(\alpha_1,\ldots,\alpha_d) = (e_1,\ldots,e_d)A$. By Proposition 4.1.2, there exists a $d\times d$ matrix B with coefficients in S such that $BA = AB = E(u)^rI$, and we have

$$E(u)^r(tX - E(u)p^nX') = tBY - E(u)p^nBY'.$$

Note that the right-hand side is in $\mathrm{Fil}^1A_{\mathrm{cris}}\cdot\mathrm{Fil}^pA_{\mathrm{cris}}$. By Lemma 3.2.2, we get that $E(u)^{r-1}(tX - E(u)p^nX') \in \mathrm{Fil}^pA_{\mathrm{cris}}$. Modulo $\mathrm{Fil}^pA_{\mathrm{cris}} + pA_{\mathrm{cris}}$ both sides, we get the coefficients of $E(u)^{r-1}tX$ are in $\mathrm{Fil}^pA_{\mathrm{cris}} + pA_{\mathrm{cris}}$ (here we may assume that $n \geqslant 1$). Almost the same argument as in the proof of Lemma 5.3.1 shows that the coefficients of X are in $\mathrm{Fil}^1S + pS$.

Now consider the following:

$$\begin{aligned} c_1p^nN((e_1,\ldots,e_d)) &= c_1p^nN(\varphi_r(\alpha_1),\ldots,\varphi_d(\alpha_d)) \\ &= p^n\varphi_r(E(u)N((\alpha_1,\ldots,\alpha_d))) \\ &= \varphi_r((\alpha_1,\ldots,\alpha_d))\varphi(X) + \varphi((e_1,\ldots,e_d))\varphi_r(Y). \end{aligned}$$

But $p|\varphi(X)$ and $p|\varphi_r(Y)$ in A_{cris}. This contradicts the selection of n unless $n = 0$. That is, W has all its coefficients in S and then $N(\mathcal{M}) \subset \mathcal{M}$.

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Tong Liu tongliu@math.upenn.edu
Department of Mathematics, University of Pennsylvania, Philadelphia, PA 19104, USA