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Schubert cycles, differential forms and $\mathcal{D}$-modules on varieties of unseparated flags

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Abstract. Proper homogeneous G-spaces (where G is semisimple algebraic group) over positive characteristic fields k can be divided into two classes, the first one being the flag varieties G/P and the second one consisting of varieties of unseparated flags (proper homogeneous spaces not isomorphic to flag varieties as algebraic varieties). In this paper we compute the Chow ring of varieties of unseparated flags, show that the Hodge cohomology of usual flag varieties extends to the general setting of proper homogeneous spaces and give an example showing (by geometric means) that the D-affinity of Beilinson and Bernstein fails for varieties of unseparated flags.

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Let G be a semisimple and simply connected algebraic group over an algebraically closed field k of characteristic $p > 0$. Projective homogeneous G-spaces can be divided into isomorphism classes of algebraic varieties. Contrary to the behavior over the complex numbers, there are several isomorphism classes not containing the flag varieties G/P, where P is a parabolic subgroup. We call a projective homogeneous G-space X a variety of unseparated flags (vuf) if it does not admit an isomorphism to a flag variety. A typical example of a vuf is the divisor $X \subseteq \mathbb{P}^n \times \mathbb{P}^n$ defined by the equation $x_0 y_0^p + \cdots x_n y_n^p = 0$, where $p \geqslant n+1$. Now X is a projective homogeneous space for $\mathrm{SL}_{n+1}(k)$ and using standard exact sequences one may prove that $\Gamma(X, \omega_X^{-1}) = 0$. Unlike flag varieties, vufs do not have vanishing cohomology for effective line bundles (Kempf vanishing). Moreover there are vufs not satisfying Kodaira's vanishing theorem [12] (by the result of Deligne, Illusie and Raynaud ([3], Corollaire 2.8), these do not admit a flat lift to $\mathbb{Z}$). In general vufs can be realized as the G-orbit of the highest weight line in $\mathbb{P}(L(\lambda))$, where $L(\lambda)$ denotes the modular simple G-representation of a certain highest weight λ [11].

In this paper we show that the Hodge cohomology of flag varieties carries over to vufs in a natural way and give examples showing that the $\mathcal{D}$-affinity of Beilinson and Bernstein breaks down in the general setting of projective homogeneous G-spaces. If X is a vuf, we prove that

$$\mathrm{H}^i(X, \Omega_X^j) = 0, \tag{$*$}$$

if $i \neq j$ and that the cycle map

$$\mathrm{CH}^i(X) \otimes_{\mathbb{Z}} k \to \mathrm{H}^i(X, \Omega_X^i)$$

is an isomorphism. A vuf is isomorphic to a coset space $G/\widetilde{P}$, where $\widetilde{P}$ is non-reduced subgroup scheme containing a Borel subgroup. There is a purely inseparable finite morphism

$$f\colon\ G/P \to G/\widetilde{P}$$

from a flag variety G/P, where $P = \widetilde{P}_{\mathrm{red}}$. A natural $\mathbb{Z}$-basis of the Chow ring $\mathrm{CH}(G/P)$ is given by the Schubert cycles $\{[X(w)]\}$ in G/P. Using f^*, $\mathrm{CH}(G/\widetilde{P})$ gets identified with a subring of $\mathrm{CH}(G/P)$ with $\mathbb{Z}$-basis $\{p^{v_w}[X(w)]\}$ for certain $v_w \geqslant 0$.

In most cases there is a nice smooth fibration $G/\widetilde{P} \to G/Q$, where Q is a parabolic subgroup. This fibration is the basis of our proof. For $X = G/Q$ the result (*) is well-known by a representation theoretic proof of Marlin [13]. Marlin's proof was originally in characteristic zero but using the linkage principle for cohomology groups of line bundles on G/B, Andersen showed [1] that it carries over to prime characteristic (see also [10], II.6.18). Using cell decompositions and the flat $\mathbb{Z}$-liftings of flag varieties, Srinivas has also obtained this result [14].

The linkage principle is however not available for vufs and vufs do not in general admit flat $\mathbb{Z}$-liftings. Our proof is an induction using the structure of vufs, filtrations of differentials coming from the above fibrations and the Leray spectral sequence.

In the last section of the paper we give examples of vufs with an effective divisor D such that $\mathrm{H}^i(U, \mathcal{O}_U) \neq 0$, where $U = X - \mathrm{Supp}D$ and $i > 0$. These are (using [8]) examples of projective homogeneous G-spaces, which are not $\mathcal{D}$-affine in the sense of Beilinson and Bernstein [2]. For flag varieties in prime characteristic it is still though (as pointed out by Haastert in [8]) an intriguing question whether $\mathcal{D}$-affinity holds. The paper is organized as follows

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1. Preliminaries

Let k be an algebraically closed field of prime characteristic p. We will only consider schemes and morphisms over k. Let $X(A) = \mathrm{Mor}_k(\mathrm{Spec}\, A, X)$ denote the set of A-points of X, where A is a k-algebra. Let G be a simply connected and semisimple algebraic group. We will assume that $p > 3$ if G has a component of type G_2 and $p > 2$ if G has a component of type B_n, C_n or F_4.

1.1. G-SPACES

A G-space is an algebraic variety X endowed with a morphism $G \times X \to X$ inducing an action of $G(A)$ on $X(A)$ for all k-algebras A. A G-space X is called homogeneous if the action $G(k) \times X(k) \to X(k)$ on k-points is transitive. A k-point $x \in X(k)$ gives a natural morphism $G \to X$. The fiber product $G_x = G \times_X \mathrm{Spec}(k)$ is easily seen to be a closed subgroup scheme of G. It is called the stabilizer group scheme of x.

1.2. THE FROBENIUS SUBCOVER

An algebraic variety X gives rise to a new algebraic variety $X^{(n)}$ with the same underlying point space as X, but where the k-multiplication is twisted via the ring homomorphism: $a \mapsto \sqrt[p^n]{a}$. The n-th order Frobenius homomorphism induces a natural morphism $F^n_X : X \to X^{(n)}$. As X is reduced, $\mathcal{O}_{X^{(n)}}$ can be identified with the k-subalgebra of p^n-th powers of regular functions on X. We call $X^{(n)}$ the nth Frobenius subcover of X. Recall that X is said to be defined over $\mathbb{Z}/p$ if there exists an $\mathbb{Z}/p$-variety X', such that $X \cong X' \times_{\mathrm{Spec}\, \mathbb{Z}/p} \mathrm{Spec}\, k$. If X is defined over $\mathbb{Z}/p$, then X is isomorphic to $X^{(n)}$ (the isomorphism is given locally by $f \otimes a \mapsto f \otimes a^{p^n}$, where $a \in k$).

1.3. THE FROBENIUS KERNEL AND HOMOGENEOUS $G^{(n)}$-SPACES

Now $G^{(n)}$ is an algebraic group of the same type as G and $F_G^n\colon G \to G^{(n)}$ is a homomorphism of algebraic groups. The kernel of F_G^n is called the nth Frobenius kernel of G and denoted G_n.

Let X be a homogeneous G-space and x a closed point of X. If $G_n \subseteq G_x$, then X is a homogeneous $G/G_n \cong G^{(n)}$-space with stabilizer group scheme $G_x^{(n)}$.

1.4. HOMOGENEOUS SPACES FOR UNIPOTENT GROUPS

Recall the following nice result from Demazure–Gabriel [6]. If U is a connected, unipotent and smooth algebraic group, then U has a central composition series

$$U = U_0 \supseteq U_1 \supseteq U_2 \supseteq \dots$$

with quotients $U_i/U_{i+1} \cong \mathbb{G}_a$ [6], pp. 530–535), where $\mathbb{G}_a$ denotes the affine line $\mathbb{A}_k^1$ with $+$. This implies ([6], IV.3.16 Corollaire) that if V is a closed subgroup scheme of U of codimension n, then U/V is isomorphic to affine n-space $\mathbb{A}_k^n$ as schemes.

2. On the structure of vufs

Fix a projective homogeneous G-space X and let $x \in G(k)$. By Borel's fixed point theorem the stabilizer group scheme $\widetilde{P} = G_x$ contains a Borel subgroup B. We will identify X with $G/\widetilde{P}$. Let $P = \widetilde{P}_{\text{red}}$. Notice that P is a parabolic subgroup of G, since the reduced part of a subgroup scheme over a perfect field is a subgroup. Let T be a maximal torus in B and $R = R(T, G)$ the root system of G wrt. T. Let the negative roots R^- be the roots of B and denote by $S \subseteq R^+$ the simple roots. Let $x_\alpha\colon \mathbb{G}_a \to G$ denote the root homomorphism corresponding to the root $\alpha \in R$. The root subgroup $U_\alpha \subseteq G$ is $x_\alpha(\mathbb{G}_a)$. When $\alpha \in S$ we let $P(\alpha)$ denote the maximal parabolic subgroup not containing U_α. Any parabolic subgroup $P \supseteq B$ in G is an intersection of maximal parabolic subgroups.

2.1. PARABOLIC SUBGROUP SCHEMES

Keeping the assumption on p and using the notation above, we can now state the following

THEOREM 1. *Suppose that $P = P(\alpha_1)\cap\ldots\cap P(\alpha_r)$ for simple roots $\alpha_1, \dots, \alpha_r \in S$. Then there are unique integers $n_1, \dots, n_r$ such that*

$$\widetilde{P} = G_{n_1}P(\alpha_1) \cap \cdots \cap G_{n_r}P(\alpha_r).$$

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Proof. ([11], 3.1). □

Let $\mu(\widetilde{P}) = \max\{n \in \mathbb{N}_{\geqslant 0} | G_n P \subseteq \widetilde{P}\}$. If $\mu(\widetilde{P}) = m > 0$, then $Q = \widetilde{P}^{(m)}$ is a parabolic subgroup scheme of $G^{(m)}$ such that

$$Q = G^{(m)}_{n_1 - m} P(\alpha_1)^{(m)} \cap \cdots \cap G^{(m)}_{n_r - m} P(\alpha_r)^{(m)}$$

and $\mu(Q) = 0$.

We let $U_{\alpha,n} = x_\alpha((\mathbb{G}_a)_n)$, with the p-adic convention that $(\mathbb{G}_a)_\infty = \mathbb{G}_a$ (and $(\mathbb{G}_a)_0 = 0$). Now if H is a connected subgroup scheme of G, then $H \cap U_\alpha = U_{\alpha,n}$ for some $n \in \mathbb{Z} \cup \{\infty\}$ and to H we associate (following Wenzel [15]) a W-function $n_H\colon R^+ \to \mathbb{Z} \cup \{\infty\}$ given by

$$n_H(\alpha) = n, \quad if\ H \cap U_\alpha = U_{\alpha,n}.$$

If $\beta \in R$ is a root we let $\operatorname{Supp}\beta$ denote the simple roots occuring with non-zero coefficients when β is expressed in the basis S. We have the following [9]

PROPOSITION 1. *Let $\widetilde{P} = G_{n_1} P(\alpha_1) \cap \ldots \cap G_{n_r} P(\alpha_r)$ be a parabolic subgroup scheme and let $n = n_{\widetilde{P}}$. Then $n(\alpha_1) = n_1, \ldots, n(\alpha_r) = n_r$ and if $\alpha \in S \setminus \{\alpha_1, \ldots, \alpha_r\}$ then $n(\alpha) = \infty$. The function n is uniquely determined by its values on S in the sense that for $\beta \in R^+$ we have*

$$n(\beta) = \min\{n(\alpha) | \alpha \in \operatorname{Supp}\beta\}.$$

2.2. Cell decompositions of vufs

A Schubert cell in G/P is a B-orbit. Schubert cells in G/P are parametrized by the finitely many T-fixed points in G/P, which again are given by representatives of cosets in the Weyl group W. Let $C(w)$ be a Schubert cell in G/P. Since T fixes a point, $C(w)$ is a homogeneous U-space, where U denotes the unipotent radical of B. By applying the U-equivariant map

$$f\colon G/P \to G/\widetilde{P}$$

to $C(w)$ we get a homogeneous U-space $\widetilde{C(w)} = f(C(w))$, which by Section 1.4 is an affine space. A Schubert variety $X(w)$ in G/P is the closure of a Schubert cell $C(w)$. We denote the scheme theoretic image $f(X(w))$ by $\widetilde{X(w)}$ and call it a Schubert variety in $G/\widetilde{P}$. In this way we get a cell decomposition

$$X = X_0 \supseteq X_1 \supseteq X_2 \supseteq \cdots$$

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of $X = G/\widetilde{P}$, where X_i is the union of the codimension i Schubert varieties in X and $X_i \setminus X_{i+1}$ is a union of affine spaces. From this it follows that CH(X) is generated by $[\widetilde{X(w)}]$ as an abelian group ([7], Example 1.9.1) .

2.3. The Chow ring of a vuf

If H is a T-invariant subgroup scheme of G, we let $R(H,T)$ denote the roots of H. First assume that $\widetilde{P}_{\text{red}} = P = B$. In this case we have $\widetilde{B} = B\widetilde{U}$, where $\widetilde{U}$ is an infinitesimal subgroup scheme of U^+ (the opposite unipotent radical of B), such that $\widetilde{B} \cap U^+ = \widetilde{U}$. Let $w \in W$ and by abuse of notation, let wB denote the T-fixed point corresponding to w. Then $C(w) = UwB$ and $\widetilde{C(w)} = Uw\widetilde{B}$. Let $C(w) = U/N_w$ and $\widetilde{C(w)} = U/\widetilde{N_w}$, where $N_w \subseteq \widetilde{N_w}$ are the stabilizers of wB under the action of U. Now $\widetilde{N_w}$ is a product of $U_{\alpha,n}$. Denote by M_w the part of $\widetilde{N_w}$ with $0 \leqslant n < \infty$. Then $R(M_w,T) = \{\alpha \in R^+ | w^{-1}(\alpha) \in R(\widetilde{U},T)\} = \{\alpha \in R^+ | a \in w(R(\widetilde{U},T))\} = R^+ \cap w(R(\widetilde{U},T))$. Now let $d_w = \Sigma_{w(\alpha)<0} n(\alpha)$ where the sum is over $\alpha \in R^+$. By computing on the level of Schubert cells we get that the induced morphism $X(w) \to \widetilde{X(w)}$ is of degree p^{d_w}. In the general case where $\widetilde{P}_{\text{red}}$ is some parabolic subgroup P, where $R(P,T) = R_I$ for $I \subseteq S$, we let

$$d_w = \sum_{\substack{\alpha \in R^+ \setminus R_I \\ w(\alpha)<0}} n(\alpha),$$

for a representative w of a coset of W_I in W. Now for the Schubert variety $X(w) \subseteq G/P$, the degree of $X(w) \to \widetilde{X(w)}$ is p^{d_w}. If f is $G/P \to G/\widetilde{P}$, then $f_*[X(w)] = p^{d_w}[\widetilde{X(w)}]$ and by the projection formula ([7], Proposition 8.3 (c)) one gets $f^*[\widetilde{X(w)}] = p^{d_{w_0}-d_w}[X(w)]$. By letting $v_w = d_{w_0} - d_w$, we get that CH$(G/\widetilde{P})$ gets identified via the ring homomorphism

$$f^*\colon \text{CH}(G/\widetilde{P}) \to \text{CH}(G/P),$$

with the subring of CH(G/P) generated by $\{p^{v_w}[X(w)]\}$. In particular it follows from the fact that $\{[X(w)]\}$ is a $\mathbb{Z}$-basis of CH(G/P) [5], that $\{p^{v_w}[X(w)]\}$ is a $\mathbb{Z}$-basis of CH$(G/\widetilde{P})$. So CH$(G/\widetilde{P})$ and CH(G/P) are isomorphic as abelian groups (but not as rings in general). It would be interesting to find an algebraic description of the ring CH$(G/\widetilde{P})$ like the one in [4].

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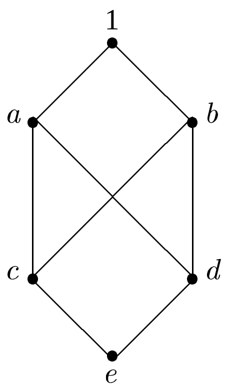
EXAMPLE 1. Let $G = \mathrm{SL}_3(k)$ and $S = \{\alpha, \beta\}$. Let $\widetilde{B}$ be the parabolic subgroup scheme given by $n(\alpha) = 0$ and $n(\beta) = 1$. The Chow ring $\mathrm{CH}(G/B)$ is a free $\mathbb{Z}$-module with basis $1, a, b, c, d, e$ corresponding to the Schubert cycles



where $1 = [X(w_0)]$, $a = [X(w_0 s_\alpha)]$, $b = [X(w_0 s_\beta)]$, $c = [X(s_\alpha)]$, $d = [X(s_\beta)]$, $e = [\{*\}]$, with relations $a^2 = d$, $b^2 = c$, $ab = c + d$, $ac = e$, $bd = e$ (all other products are 0). Now the Chow ring $\mathrm{CH}(G/\widetilde{B})$ is the subring with $\mathbb{Z}$-basis 1, a, $p \cdot b$, $p \cdot c$, d, $p \cdot e$.

3. Hodge cohomology

In the case of flag varieties we have the following result [13, 1]:

THEOREM 2. *Let P be a parabolic subgroup of G and let $X = G/P$. Then* $\mathrm{H}^i(X, \Omega_X^j) = 0$, *if* $i \neq j$ *and* $\mathrm{H}^i(X, \Omega_X^i)$ *is the trivial G-representation of dimension* $\mathrm{rk}\,\mathrm{CH}^i(X)$.

3.1. Filtrations of Ω_X^n

If $f\colon X \to Y$ is a smooth morphism, there is a short exact sequence

$$0 \to f^*\Omega_Y \to \Omega_X \to \Omega_{X/Y} \to 0.$$

This induces a filtration $\mathcal{F}_0 \supseteq \mathcal{F}_1 \supseteq \cdots$ of $\Omega_X^n = \wedge^n \Omega_X$. The associated graded $\mathcal{O}_X$-module to this filtration is

$$\mathrm{Gr}\,\Omega_X^n = \bigoplus_i \mathrm{Gr}^i \Omega_X^n = \bigoplus_i \mathcal{F}_i/\mathcal{F}_{i+1} \cong \bigoplus_{i=0}^n f^*\Omega_Y^i \otimes \Omega_{X/Y}^{n-i}.$$

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3.2. HOMOGENEOUS $G^{(m)}$-SPACES

Let $\widetilde{P}$ be a parabolic subgroup scheme of G. Suppose that $m = \mu(\widetilde{P})$. Then there is a G-equivariant diagram

$$\begin{array}{ccc} G & \longrightarrow & G/\widetilde{P} \\ \big\downarrow & & \Big\| \\ G^{(m)} & \longrightarrow & G^{(m)}/Q \end{array},$$

where $Q = \widetilde{P}^{(m)} \subseteq G^{(m)}$ and $\mu(Q) = 0$.

3.3. FIBRATIONS OF VUFS

In this section we prove:

THEOREM 3. *Let $X = G/\widetilde{P}$ be a projective homogeneous G-space, where $\widetilde{P}$ is a parabolic subgroup scheme of G. Then*

$$\mathrm{H}^i(X, \Omega_X^j) = 0, \quad \text{if } i \neq j.$$

The G-module $\mathrm{H}^i(X, \Omega_X^i)$ is trivial and the cycle map

$$\mathrm{CH}^i(X) \bigotimes_{\mathbb{Z}} k \to \mathrm{H}^i(X, \Omega_X^i)$$

is an isomorphism.

Proof. The result will follow from Theorem 2 using ([14], Applications 3.1) and Section 2.3 for the last part if we can prove that

$$\mathrm{H}^i(X, \Omega_X^j) = \mathrm{H}^i(G/P, \Omega_{G/P}^j)$$

as G-modules, where $P = \widetilde{P}_{\text{red}}$. This is done by induction on the rank of G (the rank one case being $X = \mathbb{P}^1$ and only one maximal parabolic subgroup). By Section 3.2 we may assume that $\mu(\widetilde{P}) = 0$, so that the minimal reduced parabolic subgroup Q containing $\widetilde{P}$ is $\neq G$. The inclusion $\widetilde{P} \subseteq Q$ gives a smooth morphism

$$f : X \to Y,$$

where $X = G/\widetilde{P}$ and $Y = G/Q$. If $\widetilde{P} = Q$ the result follows from Theorem 2. Using the projection formula, the E_2-term in the Leray spectral sequence for $\mathrm{Gr}^i\Omega_X^n$ (Section 3.1) is

$$\begin{aligned} E_2^{pq} &= \mathrm{H}^p(Y, R^q f_*(f^*\Omega_Y^i \otimes \Omega_{X/Y}^{n-i})) \\ &= \mathrm{H}^p(Y, \Omega_Y^i \otimes R^q f_* \Omega_{X/Y}^{n-i}). \end{aligned}$$

Now $R^q f_* \Omega_{X/Y}^{n-i}$ is a homogeneous bundle induced by the Q-representation

$$V = \mathrm{H}^q(Q/\widetilde{P}, \Omega_{Q/\widetilde{P}}^{n-i})$$

and since the reductive part of Q has rank less than G, it follows by induction that V is the trivial Q-representation

$$\mathrm{H}^q(Q/P, \Omega_{Q/P}^{n-i})$$

and that $E_2^{pq} \neq 0 \iff p = i$ and $q = n - i$. This means that

$$\mathrm{H}^j(X, f^*\Omega_X^i \otimes \Omega_{X/Y}^{n-i})$$

$$= \begin{cases} 0 & \text{if } j \neq n, \\ H^n(G/P, \Omega_{G/P}^i \otimes R^{n-1} f_* \Omega_{G/P/G/Q}^{n-i}) & \text{if } j = n. \end{cases}$$

In any case Ω_X^n has a filtration where the graded pieces has $H^j \neq 0 \iff j = n$ and H^n of a graded piece is a trivial G-representation. In conclusion $\mathrm{H}^i(X, \Omega_X^j) = 0$ if $i \neq j$ and $\mathrm{H}^i(X, \Omega_X^i)$ has the same filtration with trivial G-modules as $\mathrm{H}^i(G/P, \Omega_{G/P}^i)$. This shows that

$$\mathrm{H}^i(X, \Omega_X^i) = \mathrm{H}^i(G/P, \Omega_{G/P}^i)$$

as G-modules. □

4. $\mathcal{D}$-modules

In this section we give examples showing that there are projective homogeneous spaces in prime characteristic, which are not $\mathcal{D}$-affine.

4.1. Differential operators

Let X be a smooth variety over k and let $\mathcal{D} = \mathcal{D}_X$ be the sheaf of differential operators on X. Denote by $\mathcal{M}(\mathcal{D})$ the category of left $\mathcal{D}$-modules, which are quasi-coherent as $\mathcal{O}_X$-modules (for the module structure coming from $\mathcal{O}_X \hookrightarrow \mathcal{D}$). We say that X is $\mathcal{D}$-affine if every $\mathcal{F} \in \mathcal{M}(\mathcal{D})$ is generated as a $\mathcal{D}$-module by its global sections and $\mathrm{H}^i(X, \mathcal{F}) = 0$, $i > 0$.

Remark 1. Let $M(D)$ denote the category of D-modules, where $D = \Gamma(X, \mathcal{D}_X)$. When X is $\mathcal{D}$-affine, the global section functor $\Gamma\colon \mathcal{M}(\mathcal{D}) \to M(D)$ is an equivalence of categories.

One remarkable fact is that flag varieties G/P in characteristic zero are $\mathcal{D}$-affine as proved by Beilinson and Bernstein [2]. In fact the only smooth complete varieties known to be $\mathcal{D}$-affine are flag varieties*. Haastert [8] has proved that $\mathcal{D}$-modules over full flag varieties in prime characteristics are generated by their global sections, which reduces the question of $\mathcal{D}$-affinity for flag varieties in prime characteristics to proving that $\mathcal{D}_{G/B}$ has vanishing higher cohomology. This seems to be a difficult problem known only in the special case SL_3/B.

4.2. Open affine immersions

If D is an effective divisor in X, then $j\colon U \hookrightarrow X$ is an affine morphism, where $U = X \setminus \mathrm{Supp} D$. This means that for any quasi-coherent sheaf $\mathcal{F}$ on U, we have $\mathrm{H}^i(U, \mathcal{F}) = \mathrm{H}^i(X, j_*\mathcal{F})$. If $\mathcal{F}$ is a $\mathcal{D}_U$-module, then $j_*\mathcal{F}$ is a $\mathcal{D}_X$-module, since there is an $\mathcal{O}_X$-algebra homomorphism $\mathcal{D}_X \hookrightarrow j_*\mathcal{D}_U$. In particular if X is $\mathcal{D}$-affine then every $\mathcal{D}_U$-module $\mathcal{F}$ on U has vanishing higher cohomology. One has the formula $j_*\mathcal{F} = \varinjlim_n \mathcal{F} \otimes \mathcal{O}(nD)$, so that

$$\mathrm{H}^i(U, \mathcal{F}) = \varinjlim_n \mathrm{H}^i(X, \mathcal{F} \otimes \mathcal{O}(nD))$$

4.3. The unseparated incidence variety

We review the definition of unseparated incidence varieties from [11].

Let $n > 1$ and $G = \mathrm{SL}_{n+1}(k)$. The natural action of G on $V = k^{n+1}$ makes $\mathbb{P}(V)$ and $\mathbb{P}(V^*)$ into homogeneous spaces for G. We fix points $x_1 \in \mathbb{P}(V)$ and $x_2 \in \mathbb{P}(V^*)$, such that $G_{x_1} = P_1$ and $G_{x_2} = P_2$ are appropriate parabolic subgroups containing the subgroup of upper triangular matrices B in G. The orbit Y of (x_1, x_2) in $\mathbb{P}(V) \times \mathbb{P}(V^*)$ is a projective homogeneous space for G isomorphic to G/P, where $P = P_1 \cap P_2$. Notice that the points of Y are just pairs of incident lines and hyperplanes and that $Y = Z(s)$, where s is the section $x_0y_0 + \cdots + x_ny_n$ of $\mathcal{O}(1) \times \mathcal{O}(1)$.

Let X be the G-orbit of of $(x_1, F^r(x_2))$ in $\mathbb{P}(V) \times \mathbb{P}(V^*)^{(r)}$. Now $X \cong G/\widetilde{P}$, where $\widetilde{P} = P_1 \cap G_r P_2$. There is a natural equivariant morphism $\varphi\colon \mathbb{P}(V) \times \mathbb{P}(V^*) \to \mathbb{P}(V) \times \mathbb{P}(V^*)^{(r)}$ and X is the scheme theoretic image $\varphi(Y)$. The induced morphism $\varphi\colon Y \to X$ is the natural morphism $G/P \to G/\widetilde{P}$ given by the inclusion $P \subseteq \widetilde{P}$. One gets that X is the zero scheme of the section $\bar{s} = x_0^{p^r}\bar{y_0} + \cdots + x_n^{p^r}\bar{y_n}$ of $\mathcal{O}(p^r) \times \bar{\mathcal{O}}(1)$. Using the isomorphism from 1.2,

* Funch Thomsen has recently proved (*Bull. London Math. Soc.* 29 (1997) 317–321) that the $\mathcal{D}$-affinity of a smooth complete toric variety implies that it is a product of projective spaces.

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$X = Z(\bar{s})$ is isomorphic to its scheme theoretic image $Z(\widetilde{s}) \subseteq \mathbb{P}(V) \times \mathbb{P}(V^*)$, where $\widetilde{s} = x_0^{p^r} y_0 + \cdots x_n^{p^r} y_n$ is a section of $\mathcal{O}(p^r) \times \mathcal{O}(1)$.

Let $a, b \in \mathbb{Z}$. The restriction to Y of the line bundle $\mathcal{O}(a) \times \mathcal{O}(b)$ on $\mathbb{P}^n \times \mathbb{P}^n$ will be denoted $L(a, b)$. The restriction to X of the line bundle $\mathcal{O}(a) \times \bar{\mathcal{O}}(b)$ on $\mathbb{P}^n \times (\mathbb{P}^n)^{(r)}$ will be denoted $L(a, \bar{b})$. Notice that the isomorphism in 1.2 maps $L(a, \bar{b})$ to $L(a, b)$. For $a, b \geqslant 0$ and the effective line bundle $L = L(a, \bar{b})$ on X we get:

$$\begin{aligned} \mathrm{H}^{n-1}(X, L) &\cong \mathrm{H}^n(\mathbb{P}^n, \mathcal{O}(a - p^r)) \otimes \mathrm{H}^0(\mathbb{P}^n, \mathcal{O}(b-1)) \\ &\cong \mathrm{H}^0(\mathbb{P}^n, \mathcal{O}(p^r - a - n - 1)) \otimes \mathrm{H}^0(\mathbb{P}^n, \mathcal{O}(b-1)). \end{aligned}$$

4.4. NON $\mathcal{D}$-AFFINITY

Let X be the unseparated incidence variety $G/\widetilde{P}$, where $G = \mathrm{SL}_{n+1}(k)$ and $\widetilde{P} = P_1 \cap G_r P_2$. Assume that $p^r \geqslant n + 1$ (this ensures precisely that X is not Fano). Let $X(\omega_n) \subseteq G/P$ be the codimension one Schubert variety coming from pulling back the hyperplane in G/P_2. Let D be the image of $X(\omega_n)$ in $G/\widetilde{P}$. Then $\mathcal{O}_X(D) = L(0, \bar{1})$ and if $U = X \setminus \mathrm{Supp} D$, we get

$$\begin{aligned} \mathrm{H}^{n-1}(U, \mathcal{O}_U) &\cong \varinjlim_m \mathrm{H}^{n-1}(X, L(0, \bar{m})) \\ &\cong \varinjlim_m \mathrm{H}^0(\mathbb{P}^n, \mathcal{O}(p^r - n - 1)) \otimes \mathrm{H}^0(\mathbb{P}^n, \mathcal{O}(m-1)) \\ &\neq 0. \end{aligned}$$

By Section 4.2 this means that X is not $\mathcal{D}$-affine.

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