# A NOTE ON THE FRACTIONAL CALCULUS

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We prove the following theorem, which settles a conjecture made by J. S. Lew (2, p. 379).

**Theorem.** *Let $E$ be the space $L^p[0, 1]$ ($1 \leqslant p < \infty$), or $C[0, 1]$. Then there is precisely one family $(I_\alpha)_{\alpha>0}$ of operators on $E$ satisfying the following conditions*:

(i) $(I_1 f)(x) = \int_0^x f(t)\,dt$ *for all* $f \in E$ *and* $x \in [0, 1]$,

(ii) $I_\alpha I_\beta = I_{\alpha+\beta}$ *for all* $\alpha, \beta > 0$,

(iii) $I_\alpha \geqslant 0$ *for all* $\alpha > 0$ (i.e. $f \in E$ *and* $f \geqslant 0 \Rightarrow I_\alpha f \geqslant 0$),

(iv) $\alpha \mapsto I_\alpha$ *is a continuous map of* $(0, \infty)$ *into* $\mathscr{L}(E)$ *for some Hausdorff topology on* $\mathscr{L}(E)$, *weaker than the norm topology.*

*That family is given by the Riemann–Liouville formula*

$$(I_\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x - t)^{\alpha-1} f(t)\,dt. \tag{1}$$

**Proof.** The family of operators defined by (1) certainly satisfies these conditions, the continuity demanded in (iv) holding for the norm topology on $\mathscr{L}(E)$. The proof of the uniqueness is given over the complex field, since the result for real scalars may be deduced from the complex case.

The convolution of $f, g \in L^1[0, 1]$ is given a.e. by

$$(f * g)(x) = \int_0^x f(x - t)g(t)\,dt \quad (0 \leqslant x \leqslant 1).$$

Equipped with $*$, $L^1[0, 1]$ becomes a commutative Banach algebra, and $E$ a Banach $L^1[0, 1]$ module. A theorem of Titchmarsh [1, p. 39] shows that if $f_1, \ldots, f_m \in L^1[0, 1]$, then $f_1 * \cdots * f_m = 0$ if and only if there are $\lambda_k \in [0, 1]$ ($1 \leqslant k \leqslant m$) such that $\Sigma_{k=1}^m \lambda_k = 1$ and such that $f_k$ vanishes a.e. on $[0, \lambda_k]$.

For $g \in L^1[0, 1]$ and $f \in E$, set $R_g f = g * f$. Then $R_g \in \mathscr{L}(E)$ and $\|R_g\| \leqslant \|g\|_1$. Titchmarsh's Theorem shows that $R_g$ is injective if $g$ vanishes a.e. on no interval $[0, \lambda]$ with $\lambda > 0$, and that $g \mapsto R_g$ is an injective map of $L^1[0, 1]$ into $\mathscr{L}(E)$. For $\alpha \in \mathbf{R}$, $\alpha > 0$, and for $x \in (0, 1]$ define

$$g_\alpha(x) = \frac{1}{\Gamma(\alpha)} x^{\alpha-1}, \tag{2}$$

and write $R_\alpha = R_{g_\alpha}$. Thus $(R_\alpha)_{\alpha>0}$ is the family of operators defined by the right hand member of (1).

Observe first that if $T \in \mathscr{L}(E)$ and if $TR_1 = R_1 T$, then for $k = 1, 2, \ldots$ we have $R_1^k = R_k$, and hence $T$ commutes with $R_g$ whenever $g$ is the restriction to $[0, 1]$ of a polynomial. Density of such functions in $L^1[0, 1]$ and continuity of $g \mapsto R_g$ entail that

$$TR_1 = R_1 T \Rightarrow TR_g = R_g T \quad \text{for all } g \in L^1[0, 1].$$

We now show that if $T \in \mathscr{L}(E)$, $m \in \{1, 2, 3, \ldots\}$ and $T^m = R_1$, then there is a complex $m$-th root of unity $\eta$ such that $T = \eta R_{1/m}$. Suppose first that $T = R_g$ for some $g \in L^1[0, 1]$. We have $T^m = R_g^m = R_1 = R_{1/m}^m$. Hence $0 = R_g^m - R_{1/m}^m = R_h$, where $h = (g - \eta_1 g_{1/m}) * \cdots * (g - \eta_m g_{1/m})$ and $\eta_1, \ldots, \eta_m$ are the complex $m$-th roots of unity. Thus $h = 0$ a.e. and hence either (i) $g$ agrees a.e. on $[0, 1]$ with $\eta_k g_{1/m}$ for some $k$ or else (ii) $g$ agrees a.e. with at least two $\eta_k g_{1/m}$ on some interval $[0, \lambda]$, $\lambda > 0$. But $g_{1/m}$ vanishes a.e. on no such interval, and so the first situation must obtain, as claimed.

Now we relax the hypothesis $T = R_g$. Since $T^m = R_1$, $T$ and $R_1$ commute, and hence $S = R_1 T$ commutes with $R_g$ for all $g \in L^1[0, 1]$. Put $g = T1$. Then for all $f \in E$,

$$Sf = TR_1 f = TR_f 1 = R_f g = R_g f,$$

so that $S = R_g$. Also $S^m = T^m R_1^m = R_{m+1} = (R_{1+(1/m)})^m$. The above argument shows that $g = \eta g_{1+(1/m)}$ for some complex $m$-th root of unity $\eta$, and so

$$R_1 T = S = R_g = \eta R_1 R_{1/m}.$$

This implies that $T = \eta R_{1/m}$, since $R_1$ is injective.

We have thus shown that the conditions $T^m = R_1$ and $T \geq 0$ are satisfied for a unique $T \in \mathscr{L}(E)$, namely $T = R_{1/m}$. An obvious argument shows that if $(I_\alpha)_{\alpha > 0} \subset \mathscr{L}(E)$ satisfies (i), (ii) and (iii), then $I_\alpha = R_\alpha$ for all positive rational $\alpha$, and condition (iv) implies this equality for all positive real $\alpha$, as required.

The theorem also holds when $E$ is the space of all locally $p$-th power integrable functions ($1 \leq p < \infty$) [resp. $C^k$ functions ($0 \leq k \leq \infty$), resp. distributions) on $\mathbf{R}$ whose supports are bounded on the left, provided the range of integration is $(-\infty, x)$ in (i) and (1). (In the distribution case these integrals are interpreted as the convolutions $g_1 * f$ and $g_\alpha * f$ where $g_\alpha(x)$ is given by (2) for $x > 0$ and set equal to zero for $x \leq 0$.) The proof in these cases is similar, except that instead of Titchmarsh's theorem, the fact is used that each of these spaces forms an integral domain with respect to convolution (3, p. 173).

## REFERENCES

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