ON PRODUCTS OF IDEMPOTENT MATRICES

by J. A. ERDOS

In [1], J. M. Howie considered the semigroup of transformations of sets and proved (Theorem 1) that every transformation of a finite set which is not a permutation can be written as a product of idempotents. In view of the analogy between the theories of transformations of finite sets and linear transformations of finite dimensional vector spaces, Howie's theorem suggests a corresponding result for matrices. The purpose of this note is to prove such a result.

THEOREM. *Every singular square matrix can be written as a product of idempotent matrices.*

Proof. If E is an idempotent matrix and P is non-singular, then $P^{-1}EP$ is also idempotent and hence it is sufficient to prove that every singular matrix is similar to a product of idempotents. Two separate proofs of this are given. The first is an induction proof for matrices that are similar to triangular matrices and hence holds in general only for matrices over an algebraically closed field. For a proof valid for matrices over any field an explicit construction is given in II. It is clear that II is sufficient to establish the result but as it relies on the " rational canonical form " and as I is so simple, it is of interest to include both proofs.

I. *Proof for matrices that are similar to triangular matrices.*

We proceed by induction on the order of the matrix, and make the induction hypothesis that the theorem is valid for such matrices of order $n-1$. Suppose that a matrix A is of order n. As A is singular it is similar to an upper triangular matrix whose first column is zero and so we may partition this matrix as

$$\begin{bmatrix} 0 & X & \alpha \\ 0 & T & Y \\ 0 & 0 & \lambda \end{bmatrix},$$

where T is of order $n-2$. If T is singular, so is

$$\begin{bmatrix} T & Y \\ 0 & \lambda \end{bmatrix}$$

and by the induction hypothesis,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & T & Y \\ 0 & 0 & \lambda \end{bmatrix}$$

can be written as a product of idempotents. But

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & T & Y \\ 0 & 0 & \lambda \end{bmatrix} \begin{bmatrix} 0 & X & \alpha \\ 0 & I & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & X & \alpha \\ 0 & T & Y \\ 0 & 0 & \lambda \end{bmatrix}$$

and the postmultiplying matrix is idempotent. So it is sufficient to consider the case when T is non-singular.

Let $Z = XT^{-1}$, and let

$$B = \begin{bmatrix} 0 & X & 0 \\ 0 & T & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then by the induction hypothesis B can be written as a product of idempotents. We now have two cases.

(i) $ZY = \alpha$. Let

$$E = \begin{bmatrix} 0 & Z & 0 \\ 0 & I & 0 \\ 1 & -Z & 1 \end{bmatrix},$$

$$F = \begin{bmatrix} 0 & Z & \lambda+\alpha \\ 0 & I & Y \\ 0 & 0 & 0 \end{bmatrix}.$$

Then a direct calculation shows that E and F are idempotents and, using $X = ZT$, that

$$EFB = A.$$

(ii) $ZY - \alpha = k \neq 0$. Let

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & I & 0 \\ -\lambda/k & (\lambda/k)Z & 0 \end{bmatrix},$$

$$F = \begin{bmatrix} 1 & 0 & \alpha \\ 0 & I & Y \\ 0 & 0 & 0 \end{bmatrix}.$$

Again E and F are idempotents and

$$EFB = A.$$

For the order 2 case, we have

$$\begin{bmatrix} 0 & \alpha \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \lambda/\alpha & 0 \end{bmatrix} \begin{bmatrix} 1 & \alpha \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

if $\alpha \neq 0$ and

$$\begin{bmatrix} 0 & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & \lambda \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix};$$

so the theorem is valid for matrices of order 2.
    The result follows.

II. *Proof for the general case.*
    It is well known (see for example [2], Section III), that any matrix is similar to a matrix of the form

$$A = \operatorname{diag}\{C_1, C_2, ..., C_k\},$$

where C_i is a square matrix of order n_i of the form

$$\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & 1 \\ a_i & b_i & c_i & \cdots & z_i \end{bmatrix}.$$

As A is singular, we can arrange that C_k is singular and then $a_k = 0$. Note that if $n_i = 1$, C_i reduces to $[a_i]$. This reduction can be carried out over any field.
    Let E_0 be the matrix

$$\begin{bmatrix} 0 & 0 \\ 0 & I_{n-1} \end{bmatrix}$$

and, for $1 \leq i \leq n-1$, let E_i be

$$\begin{bmatrix} I_{i-1} & 0 & 0 \\ 0 & \begin{matrix} 1 & 1 \\ 0 & 0 \end{matrix} & 0 \\ 0 & 0 & I_{n-i-1} \end{bmatrix},$$

where I_t denotes the identity matrix of order t. Then premultiplication of any matrix by E_i adds the $(i+1)$th row to the ith row and reduces the $(i+1)$th row to zero. E_i is clearly idempotent. Let

$$N = E_{n-1}E_{n-2} \ldots E_1 E_0.$$

Then it is easily seen that

$$N = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & & \\ 0 & 0 & 0 & \ldots & 1 \\ 0 & 0 & 0 & \ldots & 0 \end{bmatrix}.$$

For $i \leq k$, let $r_i = \sum_{j=1}^{i-1} n_j$ and, for $i < k$, let

$$F_i = \begin{bmatrix} I_{r_i} & 0 & 0 \\ & \begin{matrix} 1 & 0 & & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & & & \\ 0 & 0 & \ldots & 1 & 0 \\ a_i & b_i & \ldots & z_i & 0 \end{matrix} & 0 \\ 0 & 0 & I_{n-r_{i+1}-1} \end{bmatrix}.$$

Then a calculation shows that, for each i, F_i is idempotent and that, if

$$B = NF_1F_2\ldots F_{k-1},$$

then B coincides with A except in the last row, the last row of B being zero.

Now let

$$G = \begin{bmatrix} I_{r_k} & 0 \\ 0 & \begin{matrix} 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 1 & 0 & \ldots & 0 & 0 \\ \vdots & & & & & \\ 0 & 0 & 0 & \ldots & 1 & 0 \\ b_k & c_k & d_k & \ldots & z_k & 0 \end{matrix} \end{bmatrix}.$$

Then, since $a_k = 0$, $GB = A$ and G is an idempotent. Therefore A is the product

$$GE_{n-1}E_{n-2}\ldots E_1E_0F_1F_2\ldots F_{k-1}$$

and the theorem is proved.

J. A. ERDOS

REFERENCES

1. J. M. Howie, The subsemigroup generated by the idempotents of a full transformation semi-group, *J. London Math. Soc.* **41** (1966), 707–716.

2. B. L. van der Waerden, *Modern algebra*, Vol. II (New York, 1950).

UNIVERSITY OF GLASGOW
GLASGOW, W.2
PRESENT ADDRESS: KING'S COLLEGE, LONDON, W.C.2