*Glasgow Math. J.* **46** (2004) 205–210. © 2004 Glasgow Mathematical Journal Trust.
DOI: 10.1017/S001708950300168X. Printed in the United Kingdom

# ASCENT AND DESCENT OF GORENSTEIN PROPERTY

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(Received 23 January, 2003; accepted 23 June, 2003)

**Abstract.** Let $A$ be a commutative noetherian local ring, $I$ an ideal of $A$, and $B = A/I$. Assume that the André-Quillen homology functors $H_n(A, B, -) = 0$ for all $n \geq 3$. Then $A$ is Gorenstein if and only if $B$ is.

2000 *Mathematics Subject Classification.* 13H10, 13D03.

Let $f : A \to B$ be a surjective homomorphism of noetherian local commutative rings. If Ker $(f)$ is generated by a regular sequence, it is well known that $A$ is complete intersection (resp. Gorenstein, Cohen-Macaulay) if and only if $B$ is. Another family of homomorphisms under which these properties ascend and descend is the one of local flat (non surjective) homomorphisms: $B$ is complete intersection (resp. Gorenstein, Cohen-Macaulay) if and only if $A$ and $B \otimes_A k$ are, where $k$ is the residue field of $A$.

Ascent and descent of these properties was studied, mainly by L. L. Avramov and H.-B. Foxby, for a family of homomorphisms generalizing the two cases above: homomorphisms of finite flat dimension (see e.g. [**4**], [**6**], [**7**], and, in some sense, for a larger family of homomorphisms [**5**]).

Here we consider a different class of homomorphisms. Let $H_n(A, B, -)$ be the André-Quillen homology functors [**1**], [**15**]. If $f : A \to B$ is a surjective homomorphism of noetherian local rings, then Ker$(f)$ is generated by a regular sequence if and only if $H_n(A, B, -) = 0$ for all $n \geq 2$. The class of homomorphisms considered in this paper is the one satisfying $H_n(A, B, -) = 0$ for all $n \geq 3$. In some sense it is related to complete intersection rings as $H_n(A, B, -) = 0$ for all $n \geq 2$ is related to regular rings: if $B = k$ is the residue field of $A$, we have [**1**, 6.26, 6.27]

$$\begin{aligned} H_n(A, k, -) = 0 \quad &\text{for all} \quad n \geq 2 \Leftrightarrow \text{if } A \text{ is regular} \\ H_n(A, k, -) = 0 \quad &\text{for all} \quad n \geq 3 \Leftrightarrow A \text{ is complete intersection.} \end{aligned}$$

Moreover, this is a natural class of surjective homomorphisms under which the complete intersection property ascends and descends. So we may ask if the same is valid for Gorenstein and Cohen-Macaulay properties. On the other hand, this class of homomorphisms generalizes the one whose kernel is generated by a regular sequence in a very different way that the homomorphisms of finite flat dimension: if $H_n(A, B, -) = 0$ for all $n \geq 3$ and $B$ is of finite flat dimension over $A$, then Ker$(f)$ is generated by a regular sequence [**1**, 17.2]; moreover, it is easy to see that if $A$ and $B$ are complete intersection rings then $H_n(A, B, -) = 0$ for all $n \geq 3$.

* Partially supported by Xunta de Galicia PGIDT99PXI20702B.

If $f: A \to B$ is a surjective homomorphism of noetherian local rings with $H_n(A, B, -) = 0$ for all $n \geq 3$, we prove in this paper that $A$ is Gorenstein if and only if $B$ is, and if $A$ is Cohen-Macaulay then $B$ is Cohen-Macaulay. The main ingredients of the proof are:

a) A result of A. Blanco, J. Majadas and A. G. Rodicio [**11**] characterizing this class of homomorphisms in terms of the Koszul homology of the kernel ideal.

b) A relativization of a characterization, by Avramov and Golod, of Gorenstein rings in terms of the Koszul complex of the maximal ideal [**9**]. Once we get the adequate notion of Poincaré algebra in our context, our proof of this relativization follows closely [**9**], with a little more length, due to the non rigidity of $\mathrm{Ext}_A(B, A)$ when $B$ is not a field. In fact, the proof in [**9**] of the absolute case shows that to deduce Gorensteiness it suffices the injectivity of $\Delta_1$ (see below for the definition of $\Delta_i$), whereas in our case we need to assume the bijectivity of all $\Delta_i$.

We want to point out two cases where our results are already known:

- The case where the kernel is a principal ideal $(x)$ was obtained in [**18**] (in fact under the (a priori weaker) condition that the annihilator $(0 : x)$ is a free $B$-module).
- The case where $A$ is a supplemented $B$-algebra (i.e., the homomorphism $f: A \to B$ has a ring homomorphism section). In this case it is easy to show that Gorenstein and Cohen-Macaulay properties ascend and descend (this is essentially done in [**3**, Proposition 3]): we may assume that $A$ is complete [**1**, 10.18]. Let $B \to R \to A$ be a Cohen factorization [**8**], i.e., $R$ is a noetherian local ring, $B \to R$ is a local flat homomorphism with regular closed fiber $R \otimes_B k$, and $R \to A$ is surjective. We have exact sequences [**1**, 5.1]

$$0 = H_3(A, B, k) \to H_2(B, A, k) \to H_2(B, B, k) = 0$$
$$0 = H_2(B, A, k) \to H_2(R, A, k) \to H_1(B, R, k)$$

Since $H_1(B, R, k) = H_1(k, R \otimes_B k, k) = H_2(R \otimes_B k, k, k) = 0$ [**1**, 4.54, 5.1, 6.26], we have $H_2(R, A, k) = 0$ and so Ker $(R \to A)$ is generated by a regular sequence [**1**, 6.25]. Therefore $A$ is Gorenstein (resp. Cohen-Macaulay) if and only if $R$ is if and only if $B$ is.

Definition 1. Let $B$ be a noetherian local ring, and

$$H = \bigoplus_{i=0}^{n} H_i$$

a graded (anti) commutative $B$-algebra of finite type. We say that $H$ is a Poincaré $B$-algebra if:

i) $H_0 = B$;

ii) $\mathrm{Ext}_B^q(H_i, B) = 0$ for $0 < i < n$ for all $q > 0$;

iii) $H_n$ is a free $B$-module;

iv) The canonical homomorphisms induced by multiplication

$$\Delta_i : H_{n-i} \to \mathrm{Hom}_B(H_i, H_n)$$

are all isomorphisms $0 \leq i \leq n$.

Note that from the isomorphism $\Delta_n$, since $H_n$ is a free $B$-module and $H_0 = B$, $H_n$ is free of rank 1.

The graded algebras that we are going to consider are Koszul homology algebras associated to a set of generators of $A$. For the definition and basic results on the Koszul complex, see [**17**, Chapitre IV.A)] or [**12**, Section 1.6].

LEMMA 2. *Let $A$ be a noetherian local ring, $I$ an ideal of $A$, and $B = A/I$. Let $E$ be the Koszul complex associated to a finite set of generators of $I$. Then the fact that $H(E)$ is a Poincaré $B$-algebra does not depend on the choice of the (finite) set of generators of $I$.*

*Proof.* If $I = (x_1, \ldots, x_r) = (x_1, \ldots, x_r, y_1, \ldots, y_s)$, let $E(x)$, $E(x, y)$ the Koszul complexes associated to $x_1, \ldots, x_r$, and to $x_1, \ldots, x_r, y_1, \ldots, y_s$ resp. Then we have isomorphisms [**12**, 1.6.21]

$$H_p(x, y; A) = \bigoplus_{u+v=p} \wedge^u_B(B^s) \otimes_B H_v(x; A).$$

compatible with the algebra structures. Having in mind the isomorphisms (since $\wedge^u_B(B^s)$ is $B$-free of finite type)

$$\begin{aligned}&\mathrm{Hom}_B\big(\wedge^u_B(B^s), \wedge^s_B(B^s)\big) \otimes_B \mathrm{Hom}_B(H_v(x; A), H_n(x; A))\\ &\quad = \mathrm{Hom}_B\big(\wedge^u_B(B^s) \otimes_B H_v(x; A), \wedge^s_B(B^s) \otimes_B H_n(x; A)\big)\end{aligned}$$

we deduce that $H(x; A)$ is a Poincaré $B$-algebra if and only if $H(x, y; A)$ is. If $y_1, \ldots, y_s$ and $x_1, \ldots, x_r$ are two sets of generators of $I$, compare $H(x; A)$ with $H(x, y; A)$ and this one with $H(y; A)$ □

The following proposition is [**9**, Proposition 2] (see also [**12**, 3.4.6]).

PROPOSITION 3. *Let $A$ be a noetherian local ring, $I$ an ideal of finite type of $A$ of grade $0$, and $B = A/I$. Let $E$ be the Koszul complex associated to a finite set of $n$ generators of $I$. For each $0 \leq i \leq n$, let*

$$\Delta_i : H_{n-i}(E) \to \mathrm{Hom}_B(H_i(E), H_n(E))$$

*be the homomorphism induced by the algebra structure on $H(E)$. Let $B_i \subset E_i$, $Z_i \subset E_i$, be the submodules of boundaries and cycles of $E$ respectively. There exists an exact sequence*

$$\begin{aligned}0 &\to \mathrm{Ext}^1_A(E_{i-1}/B_{i-1}, A) \to H_{n-i}(E) \xrightarrow{\Delta_i} \mathrm{Hom}_B(H_i(E), H_n(E))\\ &\to \mathrm{Ext}^1_A(B_{i-1}, A) \to \mathrm{Ext}^1_A(E_i/B_i, A) \to \mathrm{Ext}^1_A(H_i(E), A)\\ &\to \mathrm{Ext}^2_A(B_{i-1}, A) \to \ldots\end{aligned}$$

PROPOSITION 4. *Let $A$ be a noetherian local ring, $I$ an ideal of $A$, and $B = A/I$. Let $E$ be the Koszul complex associated to a finite set of $m$ generators of $I$. Let $n = m - \mathrm{grade}\, I$. The following are equivalent:*

i) *$H(E)$ is a Poincaré $B$-algebra;*

ii) *$\mathrm{Ext}^q_A(B, A) = 0$ for all $q \neq \mathrm{grade}\ I$, $\mathrm{Ext}^q_B(H_i(E), B) = 0$ for $0 < i < n$ for all $q > 0$, and $H_n(E)$ is a free $B$-module.*

*Proof.* First, we will see that we can assume $\mathrm{grade}\, I = 0$. If $\mathrm{grade}\, I = g > 0$, let $x_1, \ldots, x_g$ be a regular sequence in $I$. By Lemma 2 and its proof, the conditions i) and ii) of the proposition do not depend on the set of generators of $I$. Let then $I = (x_1, \ldots, x_g, y_1, \ldots, y_t)$ and let $E$ be the Koszul complex associated to this set of

https://doi.org/10.1017/S001708950300168X Published online by Cambridge University Press

generators of $I$. Let $A' = A/(x_1, \dots, x_g)$, $I' = I/(x_1, \dots, x_g)$. Let $E'$ be the Koszul complex over $A'$ associated to the set of generators $(y'_1, \dots, y'_t)$ of $I'$. We have $H(E) = H(E')$ [**12**, 1.6.13] and $\mathrm{Ext}^q_A(B, A) = \mathrm{Ext}^{q-g}_{A'}(B, A')$ for all $q$ [**16**] (or [**17**, p. IV-13]. Thus replacing $(A, I)$ by $(A', I')$, we can assume grade $I = 0$.

i) $\Rightarrow$ ii) By Proposition 3, if $\Delta_1$ is injective, we have $\mathrm{Ext}^1_A(B, A) = 0$. If $\Delta_2$ is injective, we have $\mathrm{Ext}^1_A(E_1/B_1, A) = 0$, and so, if moreover $\Delta_1$ is surjective, we obtain $\mathrm{Ext}^1_A(B_0, A) = 0$, i.e., $\mathrm{Ext}^2_A(B, A) = 0$.

Let $r \geq 3$ and assume we have $\mathrm{Ext}^j_A(B, A) = 0$ for all $1 \leq j \leq r-1$. Since $\mathrm{Ext}^q_B(H_i(E), B) = 0$ for all $q > 0$ and all $i$, and $\mathrm{Ext}^0_A(B, A) = \mathrm{Hom}_A(B, A) = H_n(E)$ is a free $B$-module by hypothesis, in the spectral sequence

$$E_2^{p,q} = \mathrm{Ext}^p_B\big(H_i(E), \mathrm{Ext}^q_A(B, A)\big) \Rightarrow \mathrm{Ext}^{p+q}_A(H_i(E), A)$$

we have $\mathrm{E}_2^{p,q} = 0$ if $1 \leq p+q \leq r-1$ and so $\mathrm{Ext}^j_A(H_i(E), A) = 0$ for $1 \leq j \leq r-1$. Therefore, from the exact sequences $0 \to H_i(E) \to E_i/B_i \to B_{i-1} \to 0$, $0 \to B_i \to E_i \to E_i/B_i \to 0$, we obtain $\mathrm{Ext}^q_A(B_i, A) = \mathrm{Ext}^{q+1}_A(B_{i-1}, A)$ for all $1 \leq q \leq r-2$. If $\Delta_r$ is injective, from Proposition 3 we deduce $\mathrm{Ext}^1_A(E_{r-1}/B_{r-1}, A) = 0$ and so, using that $\Delta_{r-1}$ is surjective, we obtain $\mathrm{Ext}^1_A(B_{r-2}, A) = 0$. Thus $\mathrm{Ext}^r_A(B, A) = \mathrm{Ext}^{r-1}_A(B_0, A) = \mathrm{Ext}^1_A(B_{r-2}, A) = 0$. This completes the induction step.

ii) $\Rightarrow$ i) Since $\mathrm{Ext}^0_A(B, A) = H_n(E)$ is a free $B$-module, $\mathrm{Ext}^q_B(H_i(E), B) = 0$ for all $q > 0$ and all $i$, and $\mathrm{Ext}^q_A(B, A) = 0$ for all $q > 0$, the spectral sequence

$$E_2^{p,q} = \mathrm{Ext}^p_B\big(H_i(E), \mathrm{Ext}^q_A(B, A)\big) \Rightarrow \mathrm{Ext}^{p+q}_A(H_i(E), A)$$

says that $\mathrm{Ext}^q_A(H_i(E), A) = 0$ for all $q > 0$ for all $i$. So from the exact sequences $0 \to H_i(E) \to E_i/B_i \to B_{i-1} \to 0$, $0 \to B_i \to E_i \to E_i/B_i \to 0$, $0 \to B_0 \to E_0 \to B \to 0$, and from the hypothesis $\mathrm{Ext}^q_A(B, A) = 0$ for all $q > 0$, we obtain, by recurrence on $r$, $\mathrm{Ext}^q_A(B_r, A) = 0$, and $\mathrm{Ext}^q_A(E_r/B_r, A) = 0$ for all $r \geq 0$ and all $q > 0$. So from the exact sequence of Proposition 3 with $i = r$ we deduce that $\Delta_r$ is an isomorphism for all $r \geq 0$. □

Corollary 5. *Let $A$ be a noetherian local ring, $I$ an ideal of $A$, and $B = A/I$. Let $E$ be the Koszul complex associated to a finite set of generators of $I$. Assume that $H(E)$ is a Poincaré $B$-algebra. Then*

i) *$A$ is Gorenstein if and only if $B$ is,*

ii) *If $A$ is Cohen-Macaulay, so is $B$.*

*Proof.* i) It follows from Proposition 4 and from the spectral sequence

$$E_2^{p,q} = \mathrm{Ext}^p_B\big(k, \mathrm{Ext}^q_A(B, A)\big) \Rightarrow \mathrm{Ext}^{p+q}_A(k, A)$$

where $k$ is the residue field of $A$ and $B$, since, with the notation as in the proof of Proposition 4, if $g = \mathrm{grade}\, I$, $\mathrm{Ext}^g_A(B, A) = \mathrm{Ext}^0_{A'}(B, A') = \mathrm{Hom}_{A'}(B, A') = H_n(E') = H_n(E)$ is a free $B$-module of rank 1.

ii) The same spectral sequence

$$E_2^{p,q} = \mathrm{Ext}^p_B\big(k, \mathrm{Ext}^q_A(B, A)\big) \Rightarrow \mathrm{Ext}^{p+q}_A(k, A)$$

gives an isomorphism $\mathrm{Ext}^p_B(k, B) = \mathrm{Ext}^{p+g}_A(k, A)$ for all $p$, and so depth $A = g + \mathrm{depth}\, B = \mathrm{grade}\, I + \mathrm{depth}\, B = \mathrm{ht}(I) + \mathrm{depth}\, B$, since $A$ is Cohen-Macaulay, and depth $A = \dim A = \mathrm{ht}(I) + \dim B$. Thus depth $B = \dim B$. □

COROLLARY 6. *Let $A$ be a noetherian local ring, $I$ an ideal of $A$, and $B = A/I$. Assume that the André-Quillen homology functors $H_n(A, B, -) = 0$ for all $n \geq 3$. Then*
i) *$A$ is Gorenstein if and only if $B$ is*
ii) *If $A$ is Cohen-Macaulay, so is $B$.*

*Proof.* Let $E$ be the Koszul complex associated to a finite set of generators of $I$. By [**11**, Corollary 3′], $H_1(E)$ is a free $B$-module and the canonical homomorphism $\wedge_B H_1(E) \to H(E)$ is an isomorphism. Therefore $H(E)$ is a Poincaré $B$-algebra, and Corollary 5 applies. □

REMARK 7. Let $A$ be a noetherian local ring, $I$ an ideal of $A$, and $B = A/I$. Let $E$ be the Koszul complex associated to a finite set of generators of $I$. If $A$ and $B$ are Gorenstein, then $\mathrm{Ext}_A^q(B, A) = 0$ for all $q \neq \mathrm{grade}\, I$, $H_n(E) = \mathrm{Hom}_A(B, A)$ is a free $B$-module of rank 1, but the condition $\mathrm{Ext}_B^q(H_i(E), B) = 0$ for $0 < i < n$ for all $q > 0$ does not hold in general:

i) $\mathrm{Ext}_A^q(B, A) = 0$ for all $q \neq \mathrm{grade}\, I$. This follows, replacing $A$ by $A'$ as in the proof of Proposition 4, from [**2**, 4.20, 4.12]

ii) $H_n(E) = \mathrm{Hom}_A(B, A)$ is a free $B$-module of rank 1. In effect, if $I$ contains a regular element, $\mathrm{Hom}_A(B, A) = 0$. If not, $\mathrm{grade}\, I = 0$ and since $A$ is Cohen-Macaulay, $\dim A = \mathrm{ht}(I) + \dim B = \mathrm{grade}\, I + \dim B = \dim B$. Since $\mathrm{Ext}_A^q(B, A) = 0$ for all $q > 0$ by *i*), we have a spectral sequence

$$E_2^{p,q} = \mathrm{Ext}_A^p\big(\mathrm{Ext}_B^q(k, B), A\big) \Rightarrow \mathrm{Tor}_{q-p}^B(\mathrm{Hom}_A(B, A), k)$$

where $k$ is the residue field of $A$, which is convergent since $A$ is local Gorenstein. As $\mathrm{Ext}_B^q(k, B) = 0$ for $q \neq \dim B$ and $= k$ for $q = \dim B$, and the same holds for $\mathrm{Ext}_A^p(k, A)$, the spectral sequence gives $\mathrm{Tor}_t^B(\mathrm{Hom}_A(B, A), k) = k$ for $t = 0$ and is equal to 0 for $t > 0$. Hence $\mathrm{Hom}_A(B, A)$ is a free $B$-module of rank 1.

iii) We cannot deduce the condition $\mathrm{Ext}_B^q(H_i(E), B) = 0$ for $0 < i < n$ for all $q > 0$. In fact, in this case, this condition is equivalent to the Cohen-Macaulayness of the $B$-modules $H_i(E)$ (it is said that $I$ is a strongly Cohen-Macaulay ideal; see [**14**]), since $\mathrm{Ext}_B^q(H_i(E), B) = 0$ for all $q > 0 \Leftrightarrow \mathrm{depth}\, H_i(E) = \mathrm{depth}\, B = \dim B$ [**2**, 4.20, 4.12], and $\dim \mathrm{H}_i(E) = \dim B$ (see [**14**, Remark 1.3], sketch of proof: for the last non-vanishing Koszul homology module $H_{n-g}(E)$ is easy. Then, for the others, localize at associated prime ideals of $H_{n-g}(E)$ and use the rigidity of Koszul homology). In fact, under this additional hypothesis of a strongly Cohen-Macaulay ideal $I$, the Poincaré duality was already proved by J. Herzog (see [**10**, Proposition 2.3]).

REMARK 8. Our results give some (little) evidence on a conjecture of Rodicio (an analogue of the theorem of Ferrand-Vasconcelos in "higher dimension"), which says that $H_n(A, B, -) = 0$ for all $n \geq 3$ if and only if the complete intersection dimension of the $A$-module $B$ is finite and $H_1(E)$ is a free $B$-module (see [**19**, Conjecture 11]. The unproved part of the conjecture is that if $H_n(A, B, -) = 0$ for all $n \geq 3$ then the complete intersection dimension of $B$ is finite. We deduce from Proposition 4 that if $H_n(A, B, -) = 0$ for all $n \geq 3$ then the Gorenstein dimension of $B$ over $A$ is finite. For, if $A'$ is as in the proof of Proposition 4, $G\text{-}\dim_A B < \infty$ ($G\text{-}\dim_A B$ denotes the Gorenstein dimension of the $A$-module $B$, see [**2**]) if and only if $G\text{-}\dim_{A'} B < \infty$ [**2**, 4.33]. And the condition ii) of Proposition 4 says that $\mathrm{Ext}_{A'}^q(B, A') = 0$ for all $q > 0$ and $\mathrm{Hom}_{A'}(B, A') = H_n(E) = B$. Therefore $B$ is reflexive as an $A'$-module (see e.g. [**13**, 1.1.9]) and $G\text{-}\dim_{A'} B = 0$ [**2**, 3.8(C)].

In fact, with the terminology of [**5**], having in mind also the proof of Corollary 5 and [**1**, 5.27], we have proven that if $H_n(A, B, -) = 0$ for all $n \geq 3$ then $A \to B$ is quasi-Gorenstein.

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