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Uniform Diophantine approximation and run-length function in continued fractions

BO TAN† and QING-LONG ZHOU‡

† *School of Mathematics and Statistics, Huazhong University of Science and Technology, 430074 Wuhan, PR China*
(e-mail: tanbo@hust.edu.cn)
‡ *School of Mathematics and Statistics, Wuhan University of Technology, 430070 Wuhan, PR China*
(e-mail: zhouql@whut.edu.cn)

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Abstract. We study the multifractal properties of the uniform approximation exponent and asymptotic approximation exponent in continued fractions. As a corollary, we calculate the Hausdorff dimension of the uniform Diophantine set

$$\mathcal{U}(\hat{\nu}) = \{x \in [0, 1) \colon \text{for all } N \gg 1, \text{ there exists } n \in [1, N], \\ \text{such that } |T^n(x) - y| < |I_N(y)|^{\hat{\nu}}\}$$

for a class of quadratic irrational numbers $y \in [0, 1)$. These results contribute to the study of the uniform Diophantine approximation, and apply to investigating the multifractal properties of run-length function in continued fractions.

Key words: uniform Diophantine approximation, continued fractions, run-length function
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1. *Introduction*

1.1. *Uniform Diophantine approximation.*

The classical metric Diophantine approximation is concerned with the question of how well a real number can be approximated by rationals. A qualitative answer is provided by the fact that the set of rationals is dense in the reals. Dirichlet pioneered the quantitative study by showing that, for any $x \in \mathbb{R}$ and $Q > 1$, there exists $(p, q) \in \mathbb{Z} \times \mathbb{N}$ such that

$$|qx - p| \leq \frac{1}{Q} \quad \text{and} \quad q < Q. \tag{1.1}$$

The result serves as a start point of the metric theory in Diophantine approximation. An easy application yields the following corollary: for any $x \in \mathbb{R}$, there exists infinitely many $(p, q) \in \mathbb{Z} \times \mathbb{N}$ such that

$$|qx - p| \leq \frac{1}{q}.$$

This corollary claims that $|qx - p|$ is small compared with q, while Dirichlet's original theorem in equation (1.1) provides a uniform estimate of $|qx - p|$ in terms of Q. These two kinds of approximations are referred to as uniform approximation and asymptotic, respectively. See [28] for more of an account on the related subject.

In this article, we are interested in the numbers which are approached in a uniform or asymptotic way by an orbit (in a dynamical system) with a prescribed speed. Let (X, T, μ) be a measure-preserving dynamical system, where (X, d) is a metric space, $T : X \to X$ is a Borel transformation, and μ is a T-invariant Borel probability measure on X. As is well known, Birkhoff's ergodic theorem [29] implies that, in an ergodic dynamical system, for almost all $y \in X$, the set

$$\left\{x \in X : \liminf_{n\to\infty} d(T^n(x), y) = 0\right\}$$

is of full μ-measure. The result, which gives a qualitative characterization of the distributions of the T-orbits in X, can be regarded as a counterpart of the density property of rational numbers in the reals. It leads naturally to the quantitative study of the distributions of the T-orbits.

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The shrinking target problem in dynamical system (X, T) aims at a quantitative study of Birkhoff's ergodic theorem, which investigates the set

$$W_y(T, \psi) = \{x \in X : d(T^n(x), y) < \psi(n) \text{ for infinitely many } n \in \mathbb{N}\},$$

where $\psi : \mathbb{N} \to \mathbb{R}$ is a positive function such that $\psi(n) \to 0$ as $n \to \infty$ and $y \in X$. Hill and Velani [13] studied the Hausdorff dimension of the set

$$\{x \in X : d(T^n(x), y) < e^{-\tau n} \text{ for infinitely many } n \in \mathbb{N}\}$$

in the system (X, T) with T an expanding rational map of degree greater than or equal to 2 and X the corresponding Julia set, where $\tau > 0$. See [26] for more information.

Representations of real numbers are often induced by dynamical systems or algorithms, and thus the related Diophantine approximation problems are in the nature of dynamical system, fractal geometry, and number theory. An active topic of research lies in studying the approximation of real numbers in dynamical systems by the orbits of the points. Recently, many researchers have studied the Hausdorff dimension of the set $W_y(T, \psi)$ in the corresponding dynamical system under different expansions, and obtained many significant results [19, 24, 25, 27]. Marked by the famous mass transfer principle established by Beresnevich and Velani [2], studies on the asymptotic approximation properties of orbits in dynamical systems are relatively mature. However, there are few results on the uniform approximation properties of orbits.

Let (X, T) be an exponentially mixing system with respect to the probability measure μ, and let $\psi : \mathbb{N} \to \mathbb{R}$ be a positive function satisfying that $\psi(n) \to 0$ as $n \to \infty$. Kleinbock, Konstantous, and Richter [17] studied the Lebesgue measure of the set of real numbers $x \in X$ with the property that, for every sufficiently large integer N, there is an integer n with $1 \le n \le N$ such that the distance between $T^n(x)$ and a fixed y is at most $\psi(N)$, that is,

$$\begin{aligned}\mathcal{U}(\psi) = \{x \in [0, 1) :\ &\text{for all } N \gg 1, \text{ there exists } n \in [1, N],\\ &\text{such that } |T^n(x) - y| < \psi(N)\}.\end{aligned}$$

They gave the sufficient conditions for $\mathcal{U}(\psi)$ to be of zero or full measure. Although the Khintchine type 0-1 law of the set $\mathcal{U}(\psi)$ has not been established, the work has aroused the interest of researchers (see [9, 16, 18] for the related studies). Bugeaud and Liao [5] investigated the size of the set

$$\{x \in [0, 1) :\ \text{for all } N \gg 1, \text{ there exists } n \in [1, N], \text{ such that } T_\beta^n(x) < |I_N(0)|^{\hat{\nu}}\}$$

in β-dynamical systems from the perspective of Hausdorff dimension, where T_β is the β-transformation on $[0, 1)$ defined by $T_\beta(x) = \beta x \bmod 1$, $I_N(0)$ denotes the basic interval of order N which contains the point 0, and $\hat{\nu}$ is a non-negative real number. For more information related to the uniform approximation properties, see [15, 33] and the references therein.

In this paper, we shall investigate the uniform approximation properties of the orbits under the Gauss transformation.

The Gauss transformation $T : [0, 1) \to [0, 1)$ is defined as

$$T(0) = 0, \quad T(x) = \frac{1}{x} (\text{mod } 1) \quad \text{for } x \in (0, 1).$$

Additionally, each irrational number $x \in [0, 1)$ can be uniquely expanded into the following form:

$$x = \cfrac{1}{a_1(x) + \cfrac{1}{a_2(x) + \ddots + \cfrac{1}{a_n + T^n(x)}}} = \cfrac{1}{a_1(x) + \cfrac{1}{a_2(x) + \cfrac{1}{a_3(x) + \ddots}}}, \tag{1.2}$$

with $a_n(x) = \lfloor 1/(T^{n-1}(x)) \rfloor$, called the nth partial quotient of x (here $\lfloor \cdot \rfloor$ denotes the greatest integer less than or equal to a real number and T^0 denotes the identity map). For simplicity of notation, we write equation (1.2) as

$$x = [a_1(x), a_2(x), \ldots, a_n(x) + T^n(x)] = [a_1(x), a_2(x), a_3(x), \ldots]. \tag{1.3}$$

As was shown by Philipp [21], the system $([0, 1), T)$ is exponentially mixing with respect to the Gauss measure μ given by $d\mu = dx/(1 + x) \log 2$. Thus, the above result of [17] applies for the Gauss measure of the set $\mathcal{U}(\psi)$ in the system of continued fractions. In consequence, we shall focus on the size of $\mathcal{U}(\psi)$ in dimension.

The dimension of sets $\mathcal{U}(\psi)$ depend on the choice of the given point y. In this paper, we will consider a class of quadratic irrational numbers $y = (\sqrt{i^2 + 4} - i)/2 = [i, i, \ldots]$ with $i \in \mathbb{N}$, and calculate the Hausdorff dimension of the set

$$\begin{aligned}\mathcal{U}(\hat{\nu}) = \{x \in [0, 1) \colon \text{ for all } N \gg 1, \text{ there exists } n \in [1, N],& \\ \text{such that } |T^n(x) - y| < |I_N(y)|^{\hat{\nu}}\}&.\end{aligned}$$

For $\beta \in [0, 1]$, let $s(\beta, g(y))$ denote the solution of

$$P\bigg(T, -s\bigg(\log |T'| + \frac{\beta}{1-\beta} \log g(y)\bigg)\bigg) = 0,$$

where $P(T, \phi)$ is the pressure function with potential ϕ in the continued fraction system $([0, 1), T)$, T' is the derivative of T, and $\log g(y)$ is the limit $\lim_n \log q_n(y)/n$ which equals $\log((i + \sqrt{i^2 + 4})/2)$ by Lemma 2.1(3).

THEOREM 1.1. *Given a non-negative real number $\hat{\nu}$, we have*

$$\dim_H \mathcal{U}(\hat{\nu}) = \begin{cases} s\bigg(\dfrac{4\hat{\nu}}{(1+\hat{\nu})^2}, \dfrac{i + \sqrt{i^2+4}}{2}\bigg) & \textit{if } 0 \le \hat{\nu} \le 1, \\ 0 & \textit{otherwise.} \end{cases}$$

Throughout the paper, $\dim_H$ *denotes the Hausdorff dimension of a set.*

We now turn to the discussion of two approximation exponents which are relevant to asymptotic/uniform Diophantine approximation. For $x \in [0, 1)$, we define the asymptotic approximation exponent of x by

$$\nu(x) = \sup\{\nu \ge 0 \colon |T^n(x) - y| < |I_n(y)|^{\nu} \text{ for infinitely many } n \in \mathbb{N}\}$$

and the uniform approximation exponent by

$$\begin{aligned}\hat{\nu}(x) = \sup\{\hat{\nu} \ge 0 \colon \text{ for all } N \gg 1,& \\ \text{there exists } n \in [1, N], \text{ such that } |T^n(x) - y| < |I_N(y)|^{\hat{\nu}}\}&,\end{aligned}$$

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where $I_N(y)$ denotes the basic interval of order N which contains y. The exponents $\nu(x)$ and $\hat{\nu}(x)$ are analogous to the exponents introduced in [1], see also [4, 5]. By the definitions of $\nu(x)$ and $\hat{\nu}(x)$, it is readily checked that $\hat{\nu}(x) \le \nu(x)$ for all $x \in [0, 1)$. Actually, applying Philipp's result [21], we deduce that $\nu(x) = 0$ for Lebesgue almost all $x \in [0, 1)$ (see Lemma 3.1). Li *et al* [19] studied the multifractal properties of the asymptotic exponent $\nu(x)$ and showed that for $0 \le \nu \le +\infty$,

$$\dim_H\{x \in [0, 1) \colon \nu(x) \ge \nu\} = s\bigg(\frac{\nu}{1+\nu}, \frac{i+\sqrt{i^2+4}}{2}\bigg). \tag{1.4}$$

We will denote by $E(\hat{\nu})$ the level set of the uniform approximation exponent:

$$E(\hat{\nu}) = \{x \in [0, 1) \colon \hat{\nu}(x) = \hat{\nu}\}.$$

THEOREM 1.2. *Given a non-negative real number $\hat{\nu}$, we have*

$$\dim_H E(\hat{\nu}) = \dim_H\{x \in [0, 1) \colon \hat{\nu}(x) \ge \hat{\nu}\} = \dim_H \mathcal{U}(\hat{\nu}).$$

Actually, Theorems 1.1 and 1.2 follow from the following more general result which gives the Hausdorff dimension of the set

$$E(\hat{\nu}, \nu) = \{x \in [0, 1) \colon \hat{\nu}(x) = \hat{\nu},\ \nu(x) = \nu\}.$$

THEOREM 1.3. *Given two non-negative real numbers $\hat{\nu}$ and ν with $\hat{\nu} \le \nu$, we have*

$$\dim_H E(\hat{\nu}, \nu) = \begin{cases} 1 & \text{if } \nu = 0, \\ s\bigg(\dfrac{\nu^2}{(1+\nu)(\nu-\hat{\nu})}, \dfrac{i+\sqrt{i^2+4}}{2}\bigg) & \text{if } 0 \le \hat{\nu} \le \dfrac{\nu}{1+\nu} < \nu \le \infty, \\ 0 & \text{otherwise.} \end{cases}$$

Here, we take $\nu^2/((1+\nu)(\nu-\hat{\nu})) = 1$ when $\nu = \infty$.

Let us make the following remarks regarding Theorems 1.1–1.3.

- Following the same line as the proofs of Theorems 1.1–1.3, these results remain valid for any quadratic irrational number y, see Remark 4.7 for more information.
- The fractal sets $\mathcal{U}(\hat{\nu})$, $E(\hat{\nu})$, and $E(\hat{\nu}, \nu)$ are not the so-called limsup sets, and thus we cannot obtain a natural covering to estimate the upper bound of the Hausdorff dimensions of the sets $\mathcal{U}(\hat{\nu})$ and $E(\hat{\nu}, \nu)$. To overcome this difficulty, we need a better understanding on the fractal structure of these sets; the previous work of Bugeaud and Liao [5] helps.

Combining equation (1.4) and Theorem 1.3, we obtain the dimension of the level set related to the asymptotic exponent $\nu(x)$.

COROLLARY 1.4. *Given a non-negative real number ν, we have*

$$\dim_H\{x \in [0, 1) \colon \nu(x) = \nu\} = s\bigg(\frac{\nu}{1+\nu}, \frac{i+\sqrt{i^2+4}}{2}\bigg).$$

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1.2. *Run-length function.* Applying the main ideas of the proofs of Theorems 1.1 and 1.3, we characterize the multifractal properties of run-length function in continued fractions.

The run-length function was initially introduced in a mathematical experiment of coin tossing, which counts the consecutive occurrences of 'heads' in n times trials. This function has been extensively studied for a long time. For $x \in [0, 1]$, let $r_n(x)$ be the dyadic run-length function of x, namely, the longest run of 0s in the first n digits of the dyadic expansion of x. Erdös and Rényi [7] did a pioneer work on the asymptotic behavior of $r_n(x)$: for Lebesgue almost all $x \in [0, 1]$,

$$\lim_{n\to\infty} \frac{r_n(x)}{\log_2 n} = 1.$$

Likewise, we define the run-length function in the continued fraction expansion: for $n \geq 1$, the nth maximal run-length function of x is defined as

$$R_n(x) = \max\{l \geq 1 \colon a_{i+1}(x) = \cdots = a_{i+l}(x) \text{ for some } 0 \leq i \leq n-l\}.$$

Wang and Wu [30] considered the metric properties of $R_n(x)$ and proved that

$$\lim_{n\to\infty} \frac{R_n(x)}{\log_{(\sqrt{5}+1)/2} n} = \frac{1}{2}$$

for almost all $x \in [0, 1)$. They also studied the following exceptional sets

$$F(\{\varphi(n)\}_{n=1}^{\infty}) = \left\{x \in [0, 1) \colon \lim_{n\to\infty} \frac{R_n(x)}{\varphi(n)} = 1\right\},$$
$$G(\{\varphi(n)\}_{n=1}^{\infty}) = \left\{x \in [0, 1) \colon \limsup_{n\to\infty} \frac{R_n(x)}{\varphi(n)} = 1\right\},$$

where $\varphi \colon \mathbb{N} \to \mathbb{R}^+$ is a non-decreasing function. They showed that:
(1) if $\lim_{n\to\infty}(\varphi(n+\varphi(n))/\varphi(n)) = 1$, then $\dim_H F(\{\varphi(n)\}_{n=1}^{\infty}) = 1$;
(2) if $\liminf_{n\to\infty}(\varphi(n)/n) = \beta \in [0, 1]$, then $\dim_H G(\{\varphi(n)\}_{n=1}^{\infty}) = s(\beta, (\sqrt{5}+1)/2)$.
In the study of Case (2), Wang and Wu studied essentially the Hausdorff dimension of the following set:

$$G(\beta) = \left\{x \in [0, 1) \colon \limsup_{n\to\infty} \frac{R_n(x)}{n} = \beta\right\}. \tag{1.5}$$

Replacing the limsup of the quantity $R_n(x)/n$ in equation (1.5) with liminf, we study the set

$$F(\alpha) = \left\{x \in [0, 1) \colon \liminf_{n\to\infty} \frac{R_n(x)}{n} = \alpha\right\},$$

and determine the Hausdorff dimension of the intersections of $F(\alpha) \cap G(\beta)$. As a corollary, we obtain the Hausdorff dimension of $F(\alpha)$.

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THEOREM 1.5. *For $\alpha, \beta \in [0, 1]$ with $\alpha \le \beta$, we have*

$$\dim_H(F(\alpha) \cap G(\beta)) = \begin{cases} 1 & \text{if } \beta = 0, \\ s\left(\dfrac{\beta^2(1-\alpha)}{\beta-\alpha}, \dfrac{\sqrt{5}+1}{2}\right) & \text{if } 0 \le \alpha \le \dfrac{\beta}{1+\beta} < \beta \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

THEOREM 1.6. *For $\alpha \in [0, 1]$, we have*

$$\dim_H F(\alpha) = \begin{cases} s\left(4\alpha(1-\alpha), \dfrac{\sqrt{5}+1}{2}\right) & \text{if } 0 \le \alpha \le \dfrac{1}{2}, \\ 0 & \text{otherwise.} \end{cases}$$

Throughout the paper, we use the following notation:

- $y = (\sqrt{i^2+4} - i)/2 = [i, i, \ldots]$ with $i \in \mathbb{N}$;
- $\tau(i) = (i + \sqrt{i^2+4})/2$, $\zeta(i) = (i - \sqrt{i^2+4})/2$;
- $\xi = \nu^2/(1+\nu)(\nu - \hat{\nu})$ with $0 \le \hat{\nu} < \nu$.

2. *Preliminaries*

2.1. *Properties of continued fractions.* This section is devoted to recalling some elementary properties in continued fractions. For more information on the continued fraction expansion, the readers are referred to [**12**, **14**, **22**]. We also introduce some basic techniques for estimating the Hausdorff dimension of a fractal set (see [**8**, **23**]).

For any irrational number $x \in [0, 1)$ with continued fraction expansion in equation (1.3), we write $p_n(x)/q_n(x) = [a_1(x), \ldots, a_n(x)]$ and call it the nth convergent of x. With the conventions $p_{-1}(x) = 1$, $q_{-1}(x) = 0$, $p_0(x) = 0$, and $q_0(x) = 1$, we know that $p_n(x)$ and $q_n(x)$ satisfy the recursive relations [**14**]:

$$\begin{aligned} p_{n+1}(x) &= a_{n+1}(x)p_n(x) + p_{n-1}(x), \\ q_{n+1}(x) &= a_{n+1}(x)q_n(x) + q_{n-1}(x), \quad n \ge 0. \end{aligned} \tag{2.1}$$

Clearly, $q_n(x)$ is determined by $a_1(x), \ldots, a_n(x)$, so we also write $q_n(a_1(x), \ldots, a_n(x))$ instead of $q_n(x)$. We write a_n and q_n in place of $a_n(x)$ and $q_n(x)$ for simplicity when no confusion can arise.

LEMMA 2.1. [**14**] *For $n \ge 1$ and $(a_1, \ldots, a_n) \in \mathbb{N}^n$, we have:*

(1) $q_n \ge 2^{(n-1)/2}$ *and* $\prod_{k=1}^n a_k \le q_n \le \prod_{k=1}^n (a_k + 1)$;

(2) *for any* $k \ge 1$,

$$1 \le \frac{q_{n+k}(a_1, \ldots, a_n, a_{n+1}, \ldots, a_{n+k})}{q_n(a_1, \ldots, a_n)q_k(a_{n+1}, \ldots, a_{n+k})} \le 2;$$

(3) *if* $a_1 = a_2 = \cdots = a_n = i$, *then*

$$\frac{(\tau(i))^n}{2} \le q_n(i, \ldots, i) = \frac{(\tau(i))^{n+1} - (\zeta(i))^{n+1}}{\tau(i) - \zeta(i)} \le 2(\tau(i))^n.$$

Proof. For the convenience of readers, we give the proof.

(1) By the recursive relations in equation (2.1), we readily check that

$$\prod_{k=1}^{n} a_k \le q_n \le \prod_{k=1}^{n} (a_k + 1).$$

Since $a_n \ge 1$ for $n \ge 1$, we have

$$1 = q_0 \le q_1 < q_2 < \cdots q_{n-1} < q_n.$$

By induction, $q_n \ge 2^{(n-1)/2}$ for all $n \ge 1$; similarly $p_n \ge 2^{(n-1)/2}$.

(2) Induction on k: assuming that

$$1 \le \frac{q_{n+k}(a_1, \ldots, a_n, a_{n+1}, \ldots, a_{n+k})}{q_n(a_1, \ldots, a_n) q_k(a_{n+1}, \ldots, a_{n+k})} \le 2$$

holds for all $k \in \{1, \ldots, m\}$, we prove that the above inequality holds for $k = m + 1$. Indeed, this is the case because

$$\begin{aligned}
&q_{n+m+1}(a_1, \ldots, a_n, a_{n+1}, \ldots, a_{n+m+1}) \\
&\quad = a_{n+m+1} q_{n+m}(a_1, \ldots, a_n, a_{n+1}, \ldots, a_{n+m}) \\
&\qquad + q_{n+m-1}(a_1, \ldots, a_n, a_{n+1}, \ldots, a_{n+m-1}) \\
&\quad \ge a_{n+m+1} q_n(a_1, \ldots, a_n) q_m(a_{n+1}, \ldots, a_{n+m}) \\
&\qquad + q_n(a_1, \ldots, a_n) q_{m-1}(a_{n+1}, \ldots, a_{n+m-1}) \\
&\quad = q_n(a_1, \ldots, a_n) q_{m+1}(a_{n+1}, \ldots, a_{n+m+1}),
\end{aligned}$$

and

$$\begin{aligned}
&q_{n+m+1}(a_1, \ldots, a_n, a_{n+1}, \ldots, a_{n+m+1}) \\
&\quad = a_{n+m+1} q_{n+m}(a_1, \ldots, a_n, a_{n+1}, \ldots, a_{n+m}) \\
&\qquad + q_{n+m-1}(a_1, \ldots, a_n, a_{n+1}, \ldots, a_{n+m-1}) \\
&\quad \le 2a_{n+m+1} q_n(a_1, \ldots, a_n) q_m(a_{n+1}, \ldots, a_{n+m}) \\
&\qquad + 2q_n(a_1, \ldots, a_n) q_{m-1}(a_{n+1}, \ldots, a_{n+m-1}) \\
&\quad = 2q_n(a_1, \ldots, a_n) q_{m+1}(a_{n+1}, \ldots, a_{n+m+1}).
\end{aligned}$$

(3) By the recursive relations in equation (2.1), we deduce that

$$\begin{aligned}
\begin{pmatrix} p_{n+1} & p_n \\ q_{n+1} & q_n \end{pmatrix} &= \begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix} \begin{pmatrix} a_{n+1} & 1 \\ 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} p_0 & p_{-1} \\ q_0 & q_{-1} \end{pmatrix} \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_{n+1} & 1 \\ 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_{n+1} & 1 \\ 1 & 0 \end{pmatrix}.
\end{aligned}$$

Taking $a_1 = \cdots = a_n = a_{n+1} = i$ yields that

$$\begin{pmatrix} p_{n+1} & p_n \\ q_{n+1} & q_n \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} i & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} i & 1 \\ 1 & 0 \end{pmatrix}.$$

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The symmetric matrix $A = \left(\begin{smallmatrix} i & 1 \\ 1 & 0 \end{smallmatrix}\right)$ is diagonalizable:

$$P^{-1}AP = \begin{pmatrix} \tau(i) & 0 \\ 0 & \zeta(i) \end{pmatrix}$$

with $P = \left(\begin{smallmatrix} \tau(i) & \zeta(i) \\ 1 & 1 \end{smallmatrix}\right)$.

A direct calculation yields that

$$q_n(i, \ldots, i) = \frac{(\tau(i))^{n+1} - (\zeta(i))^{n+1}}{\tau(i) - \zeta(i)}.$$

Also,

$$\frac{(\tau(i))^{n+1} - (\zeta(i))^{n+1}}{\tau(i) - \zeta(i)} \leq \frac{2(\tau(i))^{n+1}}{\tau(i)} = 2(\tau(i))^n,$$

and, if n is even,

$$\frac{(\tau(i))^{n+1} - (\zeta(i))^{n+1}}{\tau(i) - \zeta(i)} \geq \frac{(\tau(i))^{n+1}}{2\tau(i)} = \frac{(\tau(i))^n}{2};$$

if n is odd (since $\zeta(i) \cdot \tau(i) = -1$),

$$\frac{(\tau(i))^{n+1} - (\zeta(i))^{n+1}}{\tau(i) - \zeta(i)} = \frac{(\tau(i))^{2(n+1)} - 1}{(\tau(i))^{n+2} + (\tau(i))^n} \geq \frac{(\tau(i))^n}{2}.$$

This completes the proof. □

For $n \geq 1$ and $(a_1, \ldots, a_n) \in \mathbb{N}^n$, we write

$$I_n(a_1, \ldots, a_n) = \{x \in [0, 1) \colon a_k(x) = a_k, 1 \leq k \leq n\},$$

and call it a basic interval of order n. The basic interval of order n which contains x will be denoted by $I_n(x)$, that is, $I_n(x) = I_n(a_1(x), \ldots, a_n(x))$.

LEMMA 2.2. [14] *For $n \geq 1$ and $(a_1, \ldots, a_n) \in \mathbb{N}^n$, we have*

$$\frac{1}{2q_n^2} \leq |I_n(a_1, \ldots, a_n)| = \frac{1}{q_n(q_n + q_{n+1})} \leq \frac{1}{q_n^2}. \tag{2.2}$$

Here and hereafter, $|\cdot|$ denotes the length of an interval.

The next lemma describes the distribution of basic intervals I_{n+1} of order $n+1$ inside an nth basic interval I_n.

LEMMA 2.3. [14] *Let $I_n(a_1, \ldots, a_n)$ be a basic interval of order n, which is partitioned into sub-intervals $I_{n+1}(a_1, \ldots, a_n, a_{n+1})$ with $a_{n+1} \in \mathbb{N}$. When n is odd, these sub-intervals are positioned from left to right, as a_{n+1} increases; when n is even, they are positioned from right to left.*

The following lemma displays the relationship between the ball $B(x, |I_n(x)|)$ and the basic interval $I_n(x)$.

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LEMMA 2.4. [3] *Let $x = [a_1, a_2, \ldots]$. We have:*

(1) *if $a_n \neq 1$, then $B(x, |I_n(x)|) \subset \bigcup_{j=-1}^{3} I_n(a_1, \ldots, a_n + j)$;*

(2) *if $a_n = 1$ and $a_{n-1} \neq 1$, then $B(x, |I_n(x)|) \subset \bigcup_{j=-1}^{3} I_{n-1}(a_1, \ldots, a_{n-1} + j)$;*

(3) *if $a_n = 1$ and $a_{n-1} = 1$, then $B(x, |I_n(x)|) \subset I_{n-2}(a_1, \ldots, a_{n-2})$.*

2.2. *Hausdorff dimension.* The following two properties, namely, Hölder property and the mass distribution principle, are often used to estimate the Hausdorff dimension of a fractal set.

LEMMA 2.5. [8] *If $f : X \to Y$ is an α-Hölder mapping between metric spaces, that is, there exists $c > 0$ such that for all $x_1, x_2 \in X$,*

$$d(f(x_1), f(x_2)) \leq c d(x_1, x_2)^{\alpha},$$

then $\dim_H f(X) \leq (1/\alpha) \dim_H X$.

LEMMA 2.6. [8] *Let $E \subseteq [0, 1]$ be a Borel set and μ be a measure with $\mu(E) > 0$. If for every $x \in E$,*

$$\liminf_{r \to 0} \frac{\log \mu(B(x, r))}{\log r} \geq s,$$

then $\dim_H E \geq s$.

We conclude this subsection by quoting a dimensional result related to continued fractions, which will be used in the proof of Theorem 1.3.

Let $\mathbf{K} = \{k_n\}_{n=1}^{\infty}$ be a subsequence of $\mathbb{N}$ which is not cofinite. Let $x = [a_1, a_2, \ldots]$ be an irrational number in $[0, 1)$. Eliminating all the terms a_{k_n} from the sequence $a_1, a_2, \ldots$, we obtain an infinite subsequence $c_1, c_2, \ldots$, and put $\phi_{\mathbf{K}}(x) = y$ with $y = [c_1, c_2, \ldots]$. In this way, we define a mapping $\phi_{\mathbf{K}} : [0, 1) \cap \mathbb{Q}^c \to [0, 1) \cap \mathbb{Q}^c$.

Let $\{M_n\}_{n \geq 1}$ be a sequence with $M_n \in \mathbb{N}$, $n \geq 1$. Set

$$S(\{M_n\}) = \{x \in [0, 1) \cap \mathbb{Q}^c : 1 \leq a_n(x) \leq M_n \text{ for all } n \geq 1\}.$$

LEMMA 2.7. [6] *Suppose that $\{M_n\}_{n=1}^{\infty}$ is a bounded sequence. If the sequence $\mathbf{K} = \{k_n\}_{n=1}^{\infty}$ is of density zero in $\mathbb{N}$, then*

$$\dim_H S(\{M_n\}) = \dim_H \phi_{\mathbf{K}} S(\{M_n\}).$$

2.3. *Pressure function and pre-dimensional number.* We now introduce the notions of the pressure function and pre-dimensional number in the continued fraction dynamical system. For more details, we refer the reader to [11].

For $\mathcal{A}$ a finite or infinite subset of $\mathbb{N}$, we set

$$X_{\mathcal{A}} = \{x \in [0, 1) : a_n(x) \in \mathcal{A} \text{ for all } n \geq 1\}.$$

The pressure function restricted to the subsystem $(X_{\mathcal{A}}, T)$ with potential $\phi : [0, 1) \to \mathbb{R}$ is defined as

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$$P_{\mathcal{A}}(T,\phi)=\lim_{n\to\infty}\frac{\log\sum_{(a_1,\ldots,a_n)\in\mathcal{A}^n}\sup_{x\in X_{\mathcal{A}}}\exp S_n\phi([a_1,\ldots,a_n+x])}{n},\tag{2.3}$$

where $S_n\phi(x)=\phi(x)+\cdots+\phi(T^{n-1}(x))$ denotes the ergodic sum of ϕ. When $\mathcal{A}=\mathbb{N}$, we write $P(T,\phi)$ for $P_{\mathbb{N}}(T,\phi)$.

The nth variation $\mathrm{Var}_n(\phi)$ of ϕ is defined as

$$\mathrm{Var}_n(\phi)=\sup\{|\phi(x)-\phi(y)|:\ I_n(x)=I_n(y)\}.$$

The following lemma shows the existence of the limit in equation (2.3).

LEMMA 2.8. [29] *The limit defining $P_{\mathcal{A}}(T,\phi)$ in equation (2.3) exists. Moreover, if $\phi\colon[0,1)\to\mathbb{R}$ satisfies $\mathrm{Var}_1(\phi)<\infty$ and $\mathrm{Var}_n(\phi)\to 0$ as $n\to\infty$, the value of $P_{\mathcal{A}}(T,\phi)$ remains the same even without taking the supremum over $x\in X_{\mathcal{A}}$ in equation (2.3).*

For $0<\alpha<1$ and $i\in\mathbb{N}$, we define

$$\widehat{s}_n(\mathcal{A},\alpha,\tau(i))=\inf\bigg\{\rho\ge 0:\ \sum_{a_1,\ldots,a_n\in\mathcal{A}}\bigg(\frac{1}{(\tau(i))^{n\alpha/(1-\alpha)}q_n(a_1,\ldots,a_n)}\bigg)^{2\rho}\le 1\bigg\}.$$

Following [31], we call $\widehat{s}_n(\mathcal{A},\alpha,\tau(i))$ the nth pre-dimensional number with respect to $\mathcal{A}$ and α. The properties of pre-dimensional numbers are presented in the following lemmas; the original ideas for the proofs date back to Good [10] (see also [20]).

LEMMA 2.9. [31] *Let $\mathcal{A}$ be a finite or infinite subset of $\mathbb{N}$. For $0<\alpha<1$ and $i\in\mathbb{N}$, the limit $\lim_{n\to\infty}\widehat{s}_n(\mathcal{A},\alpha,\tau(i))$ exists, denoted by $s(\mathcal{A},\alpha,\tau(i))$.*

By equation (2.2) and the definition of $\widehat{s}_n(\mathcal{A},\alpha,\tau(i))$, we know $0\le\widehat{s}_n(\mathcal{A},\alpha,\tau(i))\le 1$. Furthermore, Lemma 2.9 implies that $0\le s(\mathcal{A},\alpha,\tau(i))\le 1$.

LEMMA 2.10. [31] *For any $B\in\mathbb{N}$, put $\mathcal{A}_B=\{1,\ldots,B\}$. The limit $\lim_{B\to\infty}s(\mathcal{A}_B,\alpha,\tau(i))$ exists, and is equal to $s(\mathbb{N},\alpha,\tau(i))$.*

Similarly to pre-dimensional numbers $\{\widehat{s}_n(\mathcal{A},\alpha,\tau(i))\}$, we define

$$s_n(\mathcal{A},\alpha,\tau(i))=\inf\bigg\{\rho\ge 0:\ \sum_{a_1,\ldots,a_{n-\lfloor n\alpha\rfloor}\in\mathcal{A}}\bigg(\frac{1}{q_n(a_1,\ldots,a_{n-\lfloor n\alpha\rfloor},i,\ldots,i)}\bigg)^{2\rho}\le 1\bigg\}.$$

Remark 2.11. We remark that

$$\sum_{a_1,\ldots,a_n\in\mathcal{A}}\bigg(\frac{1}{(\tau(i))^{n\alpha/(1-\alpha)}q_n(a_1,\ldots,a_n)}\bigg)^{2\widehat{s}_n(\mathcal{A},\alpha,\tau(i))}\le 1$$

and

$$\sum_{a_1,\ldots,a_{n-\lfloor n\alpha\rfloor}\in\mathcal{A}}\bigg(\frac{1}{q_n(a_1,\ldots,a_{n-\lfloor n\alpha\rfloor},i,\ldots,i)}\bigg)^{2s_n(\mathcal{A},\alpha,\tau(i))}\le 1,$$

with equalities holding when $\mathcal{A}$ is finite.

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By Lemmas 2.9 and 2.10, we have the following result.

LEMMA 2.12. *Let $\mathcal{A}$ be a finite or infinite subset of* $\mathbb{N}$. *For* $0 < \alpha < 1$ *and* $i \in \mathbb{N}$, *we have*

$$\lim_{n\to\infty} s_n(\mathcal{A}, \alpha, \tau(i)) = s(\mathcal{A}, \alpha, \tau(i)).$$

In particular, if $\mathcal{A} = \mathbb{N}$, *then*

$$\lim_{n\to\infty} s_n(\mathbb{N}, \alpha, \tau(i)) = s(\mathbb{N}, \alpha, \tau(i)).$$

Proof. For $\varepsilon > 0$ and n large enough, we have

$$2^{((n-\lfloor n\alpha\rfloor)/2)\varepsilon} > 64, \tag{2.4}$$

$$\frac{3}{(1-\alpha)(n\alpha-1)} + \frac{\log 4}{n\alpha-1} < \varepsilon, \tag{2.5}$$

$$|\widehat{s}_n(\mathcal{A}, \alpha, \tau(i)) - s(\mathcal{A}, \alpha, \tau(i))| < \frac{\varepsilon}{2}. \tag{2.6}$$

On the one hand, by Remark 2.11, we deduce that

$$\begin{aligned}
1 &\geq \sum_{a_1,\ldots,a_{n-\lfloor n\alpha\rfloor}\in\mathcal{A}} \bigg(\frac{1}{q_n(a_1,\ldots,a_{n-\lfloor n\alpha\rfloor}, i,\ldots,i)}\bigg)^{2s_n(\mathcal{A},\alpha,\tau(i))} \\
&\geq \sum_{a_1,\ldots,a_{n-\lfloor n\alpha\rfloor}\in\mathcal{A}} \bigg(\frac{1}{2q_{n-\lfloor n\alpha\rfloor}(a_1,\ldots,a_{n-\lfloor n\alpha\rfloor})q_{\lfloor n\alpha\rfloor}(i,\ldots,i)}\bigg)^{2s_n(\mathcal{A},\alpha,\tau(i))} \\
&\geq \sum_{a_1,\ldots,a_{n-\lfloor n\alpha\rfloor}\in\mathcal{A}} \bigg(\frac{1}{4q_{n-\lfloor n\alpha\rfloor}(a_1,\ldots,a_{n-\lfloor n\alpha\rfloor})(\tau(i))^{(\alpha/(1-\alpha))(n-\lfloor n\alpha\rfloor)}}\bigg)^{2s_n(\mathcal{A},\alpha,\tau(i))} \\
&\geq \sum_{a_1,\ldots,a_{n-\lfloor n\alpha\rfloor}\in\mathcal{A}} \bigg(\frac{1}{q_{n-\lfloor n\alpha\rfloor}(a_1,\ldots,a_{n-\lfloor n\alpha\rfloor})(\tau(i))^{(\alpha/(1-\alpha))(n-\lfloor n\alpha\rfloor)}}\bigg)^{2s_n(\mathcal{A},\alpha,\tau(i))+\varepsilon},
\end{aligned}$$

where the second inequality holds by Lemma 2.1(2); the third inequality is right by Lemma 2.1(3) and the fact that $(\alpha/(1-\alpha))(n-\lfloor n\alpha\rfloor) \geq \lfloor n\alpha\rfloor$ for $n \in \mathbb{N}$; the last inequality is true by Lemma 2.1(1) and equation (2.4). This means that

$$s_n(\mathcal{A}, \alpha, \tau(i)) + \frac{\varepsilon}{2} \geq \widehat{s}_{n-\lfloor n\alpha\rfloor}(\mathcal{A}, \alpha, \tau(i)).$$

On the other hand, we have

$$\begin{aligned}
1 &\geq \sum_{a_1,\ldots,a_{n-\lfloor n\alpha\rfloor}\in\mathcal{A}} \bigg(\frac{1}{q_{n-\lfloor n\alpha\rfloor}(a_1,\ldots,a_{n-\lfloor n\alpha\rfloor})(\tau(i))^{(\alpha/(1-\alpha))(n-\lfloor n\alpha\rfloor)}}\bigg)^{2\widehat{s}_{n-\lfloor n\alpha\rfloor}(\mathcal{A},\alpha,\tau(i))} \\
&\geq \sum_{a_1,\ldots,a_{n-\lfloor n\alpha\rfloor}\in\mathcal{A}} \bigg(\frac{1}{q_{n-\lfloor n\alpha\rfloor}(a_1,\ldots,a_{n-\lfloor n\alpha\rfloor})(\tau(i))^{\lfloor n\alpha\rfloor+1/(1-\alpha)}}\bigg)^{2\widehat{s}_{n-\lfloor n\alpha\rfloor}(\mathcal{A},\alpha,\tau(i))} \\
&\geq \sum_{a_1,\ldots,a_{n-\lfloor n\alpha\rfloor}\in\mathcal{A}} \bigg(\frac{1}{q_n(a_1,\ldots,a_{n-\lfloor n\alpha\rfloor}, i,\ldots,i)}\bigg)^{2\widehat{s}_{n-\lfloor n\alpha\rfloor}(\mathcal{A},\alpha,\tau(i))+\varepsilon},
\end{aligned}$$

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where the second inequality is obtained by $(\alpha/(1-\alpha))(n-\lfloor n\alpha\rfloor) \le \lfloor n\alpha\rfloor + 1/(1-\alpha)$ for $n \in \mathbb{N}$; the last inequality holds by Lemma 2.1(3) and equation (2.5). This implies that

$$s_n(\mathcal{A}, \alpha, \tau(i)) \le \widehat{s}_{n-\lfloor n\alpha\rfloor}(\mathcal{A}, \alpha, \tau(i)) + \frac{\varepsilon}{2}.$$

Thus, by equation (2.6), we obtain that

$$|s_n(\mathcal{A}, \alpha, \tau(i)) - s(\mathcal{A}, \alpha, \tau(i))| < \varepsilon$$

for n large enough. This completes the proof. □

For simplicity, write $s_n(\alpha, \tau(i))$ for $s_n(\mathbb{N}, \alpha, \tau(i))$, $s(\alpha, \tau(i))$ for $s(\mathbb{N}, \alpha, \tau(i))$.

LEMMA 2.13. [31] *For* $0 < \alpha < 1$ *and* $i \in \mathbb{N}$, *we have:*
(1) $s(\alpha, \tau(i)) > \frac{1}{2}$;
(2) $s(\alpha, \tau(i))$ *is non-increasing and continuous with respect to* α;
(3) $\lim_{\alpha\to 0} s(\alpha, \tau(i)) = 1$ *and* $\lim_{\alpha\to 1} s(\alpha, \tau(i)) = \frac{1}{2}$.

From a point of view of a dynamical system, $s(\alpha, \tau(i))$ can be regarded as the solution to the pressure function [32]

$$P\bigg(T, -s\bigg(\log|T'| + \frac{\alpha}{1-\alpha}\log\tau(i)\bigg)\bigg) = 0.$$

Furthermore, by Lemma 2.13, we may extend $s(\alpha, \tau(i))$ to $[0, 1]$ as follows:

$$s(\alpha, \tau(i)) = \begin{cases} 1, & \alpha = 0, \\ s(\alpha, \tau(i)), & 0 < \alpha < 1, \\ \frac{1}{2}, & \alpha = 1. \end{cases} \tag{2.7}$$

3. *Proof of Theorem 1.3: upper bound*

Recall that $y = [i, i, \ldots]$ with $i \in \mathbb{N}$. In this section, we devote to estimating the upper bound of $E(\hat{\nu}, \nu)$.

We first consider the case $\nu = 0$.

LEMMA 3.1. $\nu(x) = 0$ *for Lebesgue almost all* $x \in [0, 1)$.

Proof. Since $\sum_{n=1}^{\infty} |I_n(y)|^{1/m} < \infty$, we obtain by [21, Theorem 2B] that the set

$$\{x \in [0, 1) : |T^n(x) - y| < |I_n(y)|^{1/m} \text{ for infinitely many } n \in \mathbb{N}\}$$

is of measure zero. Now,

$$\begin{aligned} \{x \in [0, 1) : \nu(x) > 0\} &\subseteq \bigcup_{m=1}^{\infty} \bigg\{x \in [0, 1) : \nu(x) > \frac{1}{m}\bigg\} \\ &\subseteq \bigcup_{m=1}^{\infty} \{x \in [0, 1) : |T^n(x) - y| < |I_n(y)|^{1/m} \text{ for infinitely many } n \in \mathbb{N}\}. \end{aligned}$$

Hence, $\{x \in [0, 1) : \nu(x) > 0\}$ is a null set. This completes the proof. □

We now aim to determine the upper bound of $\dim_H E(\hat{\nu}, \nu)$ for $0 < \nu \le +\infty$.

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LEMMA 3.2. *Let $x \in E(\hat{\nu}, \nu)$, where $\upsilon > 0$. If the continued fraction expansion of x is not periodic, there exist two ascending sequences $\{n_k\}_{k=1}^{\infty}$ and $\{m_k\}_{k=1}^{\infty}$ depending on x such that:*

(1) $n_k < m_k < n_{k+1} < m_{k+1}$ *for* $k \geq 1$;

(2) $a_{n_k+1}(x) = \cdots = a_{m_k}(x) = i$ *for* $k \geq 1$;

(3) $\liminf_{k\to\infty}((m_k - n_k)/n_{k+1}) = \hat{\nu}$, $\limsup_{k\to\infty}((m_k - n_k)/n_k) = \nu$.

Proof. For $x = [a_1(x), a_2(x), \ldots] \in E(\hat{\nu}, \nu)$, we define two sequences $\{n'_k\}_{k\geq 1}$ and $\{m'_k\}_{k\geq 1}$ as follows:

$$m'_0 = 0,\ n'_k = \min\{n \geq m'_{k-1} : a_{n+1}(x) = i\},$$
$$m'_k = \max\{n \geq n'_k : a_{n'_k+1}(x) = \cdots = a_n(x) = i\}.$$

The fact that $\nu(x) > 0$ guarantees the existence of n'_k, and thus m'_k is well defined since the continued fraction expansion of x is not periodic. Further, for all $k \geq 1$, we have that $n'_k \leq m'_k < n'_{k+1}$ and

$$|T^{n'_k}(x) - y| < |I_{m'_k - n'_k}(y)|. \tag{3.1}$$

By Lemmas 2.1–2.3, we have

$$\begin{aligned}|T^{n'_k}(x) - y| &> |I_{m'_k - n'_k + 2}(\underbrace{i, \ldots, i}_{m_k - n_k + 1}, i+1)| \geq \frac{1}{2q^2_{m'_k - n'_k + 2}(i, \ldots, i, i+1)} \\ &> \frac{1}{8(i+2)^2 q^2_{m'_k - n'_k}(i, \ldots, i)} \geq \frac{1}{8(i+2)^2}|I_{m'_k - n'_k}(y)|. \end{aligned} \tag{3.2}$$

We also have $\limsup_{k\to\infty}(m'_k - n'_k) = +\infty$ since $\nu(x) > 0$. We then choose a subsequence of $\{(n'_k, m'_k)\}_{k\geq 1}$ as follows: put $(n_1, m_1) = (n'_1, m'_1)$; having chosen $(n_k, m_k) = (n'_{j_k}, m'_{j_k})$, we set $j_{k+1} = \min\{j > j_k : m'_j - n'_j > m_k - n_k\}$ and put $(n_{k+1}, m_{k+1}) = (n'_{j_{k+1}}, m'_{j_{k+1}})$. We claim that

$$\liminf_{k\to\infty} \frac{m_k - n_k}{n_{k+1}} = \hat{\nu}(x), \quad \limsup_{k\to\infty} \frac{m_k - n_k}{n_k} = \nu(x).$$

To prove the first assertion, we write $\liminf_{n\to\infty}((m_k - n_k)/n_{k+1}) = a$. For $\varepsilon > 0$, there is a subsequence $\{k_j\}_{j=1}^{\infty}$ such that

$$m_{k_j} - n_{k_j} \leq (a + \varepsilon) n_{k_j+1}.$$

Putting $N = n_{k_j} - 1$, we have for all $n \in [1, N]$ that

$$|T^n(x) - y| \geq \frac{1}{2(i+2)^2}|I_{m_{k_j} - n_{k_j}}(y)| > |I_{n_{k_j+1}}(y)|^{a+2\varepsilon} > |I_N(y)|^{a+3\varepsilon},$$

where the second inequality holds by the following fact: by Lemmas 2.2 and 2.1(3), we deduce

$$\lim_{n\to\infty} \frac{-\log|I_n(y)|}{2n} = \lim_{n\to\infty} \frac{\log q_n(y)}{n} = \log\tau(i).$$

We get that $\hat{\nu}(x) \leq a + 3\varepsilon$ by the definition of $\hat{\nu}(x)$.

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However, when $k \gg 1$, we have

$$m_k - n_k \geq (a - \varepsilon) n_{k+1}.$$

For $n_k \leq N < n_{k+1}$,

$$|T^{n_k}(x) - y| \leq |I_{m_k - n_k}(y)| < |I_{n_{k+1}}(y)|^{a-\varepsilon} < |I_N(y)|^{a-\varepsilon}.$$

From here, we deduce that $\hat{\nu}(x) \geq a - \varepsilon$.

Letting $\varepsilon \to 0$, we complete the proof of the first assertion; the second one can be proved in a similar way. □

LEMMA 3.3. *If* $0 < \nu/(1+\nu) < \hat{\nu} \leq \infty$, $E(\hat{\nu}, \nu)$ *is at most countable and* $\dim_H E(\hat{\nu}, \nu) = 0$.

Proof. If $x \in E(\hat{\nu}, \nu)$ and its continued fraction expansion is not periodic, then by Lemma 3.2(2), there exist two sequences $\{n_k\}_{k=1}^{\infty}$ and $\{m_k\}_{k=1}^{\infty}$ depending on x such that

$$\liminf_{k\to\infty} \frac{m_k - n_k}{n_{k+1}} = \hat{\nu}, \quad \limsup_{k\to\infty} \frac{m_k - n_k}{m_k} = \frac{\nu}{1+\nu}.$$

This yields $\hat{\nu} \leq \nu/(1+\nu)$; the lemma follows. □

We start with constructing of a covering of $E(\hat{\nu}, \nu)$ in the case where $0 \leq \hat{\nu} \leq \nu/(1+\nu) < \infty$ and $0 < \nu \leq \infty$. Since $E(0, \nu)$ is a subset of $\{x \in [0, 1) : \nu(x) = \nu\}$, by Corollary 1.4, we have $\dim_H E(0, \nu) \leq s(\nu/(1+\nu), \tau(i))$, which is the desired upper bound estimate. Hence, we only need to deal with the case $0 < \hat{\nu} \leq \nu/(1+\nu) < \nu \leq \infty$. Whence, given any x in the set $E(\hat{\nu}, \nu)$ with non-periodic continued fraction expansion, we associate x with two sequences $\{n_k\}$, $\{m_k\}$ as in Lemma 3.2. The following properties hold.

(1) The sequence $\{m_k\}$ grows exponentially, more precisely, there exists $C > 0$, independent of x, such that when k is large enough,

$$k \leq C \log m_k. \tag{3.3}$$

Indeed, we have that $m_k - n_k \geq (\hat{\nu}/2) n_{k+1}$ for all large k, and thus,

$$m_k \geq \left(1 + \frac{\hat{\nu}}{2}\right) n_k \geq \left(1 + \frac{\hat{\nu}}{2}\right) m_{k-1}.$$

(2) Write $\xi = \nu^2/((1+\nu)(\nu - \hat{\nu}))$. For any $\varepsilon > 0$, there exist infinitely many k such that

$$\sum_{i=1}^{k} (m_i - n_i) \geq m_k(\xi - \varepsilon). \tag{3.4}$$

To prove this, we apply a general form of the Stolz–Cesàro theorem which states that: if b_n tends to infinity monotonically,

$$\liminf_n \frac{a_n - a_{n-1}}{b_n - b_{n-1}} \leq \liminf_n \frac{a_n}{b_n} \leq \limsup_n \frac{a_n}{b_n} \leq \limsup_n \frac{a_n - a_{n-1}}{b_n - b_{n-1}}.$$

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We deduce from Lemma 3.2 that

$$\limsup_{k\to\infty}\frac{m_k}{n_k}=1+\nu$$

and

$$\liminf_{k\to\infty}\frac{m_k}{n_{k+1}}\geq\liminf_{k\to\infty}\frac{m_k-n_k}{n_{k+1}}\cdot\liminf_{k\to\infty}\frac{m_k}{m_k-n_k}=\frac{\hat{\nu}(1+\nu)}{\nu}.$$

Hence,

$$\begin{aligned}\liminf_{k}\frac{\sum_{i=1}^{k}(m_i-n_i)}{m_{k+1}}&\geq\liminf_{k}\frac{m_k-n_k}{m_{k+1}-m_k}\\&\geq\liminf_{k}\frac{m_k-n_k}{n_{k+1}}\cdot\frac{1}{\limsup_k(m_{k+1}/n_{k+1})-\liminf_k(m_k/n_{k+1})}\\&\geq\frac{\hat{\nu}\nu}{(\nu-\hat{\nu})(1+\nu)},\end{aligned}\tag{3.5}$$

and thus

$$\sum_{i=1}^{k}(m_i-n_i)\geq\bigg(\frac{\hat{\nu}\nu}{(\nu-\hat{\nu})(1+\nu)}-\frac{\varepsilon}{2}\bigg)m_k+(m_k-n_k)$$

holds for k large enough. However, there exist infinitely many k such that

$$m_k-n_k\geq\bigg(\frac{\nu}{1+\nu}-\frac{\varepsilon}{2}\bigg)m_k.$$

We then readily check that equation (3.4) holds for such k.

We now construct a covering of $E(\hat{\nu},\nu)$. To this end, we collect all sequences $(\{n_k\},\{m_k\})$ associated with some $x\in E(\hat{\nu},\nu)$ as in Lemma 3.2 to form a set

$$\Omega=\{(\{n_k\},\{m_k\}):\ \text{conditions (1) and (3) in Lemma 3.2 are fulfilled}\}.$$

For $(\{n_k\},\{m_k\})\in\Omega$, write

$$\begin{aligned}H(\{n_k\},\{m_k\})&=\{x\in[0,1):\ \text{condition (2) in Lemma 3.2 is fulfilled}\},\\ \Lambda_{k,m_k}&=\{(n_1,m_1;\ldots;n_{k-1},m_{k-1};n_k): n_1<m_1<\cdots<m_{k-1}<n_k<m_k,\\&\qquad\text{equation (3.4) holds}\},\\ \mathcal{D}_{n_1,m_1;\ldots;n_k,m_k}&=\{(\sigma_1,\ldots,\sigma_{m_k})\in\mathbb{N}^{m_k}:\sigma_{n_j+1}=\cdots=\sigma_{m_j}=i\\&\qquad\text{for all } 1\leq j\leq k\}.\end{aligned}$$

Based on the previous analysis, we obtain a covering of $E(\hat{\nu},\nu)$, that is,

$$\begin{aligned}E(\hat{\nu},\nu)&\subseteq\bigcup_{(\{n_k\},\{m_k\})\in\Omega}H(\{n_k\},\{m_k\})\\&\subseteq\bigcap_{K=1}^{\infty}\bigcup_{k=K}^{\infty}\bigcup_{m_k\geq e^{k/C}}\ \bigcup_{(n_1,m_1,\ldots,m_{k-1},n_k)\in\Lambda_{k,m_k}}\ \bigcup_{(a_1,\ldots,a_{m_k})\in\mathcal{D}_{n_1,m_1;\ldots;n_k,m_k}}I_{m_k}(a_1,\ldots,a_{m_k}).\end{aligned}$$

For $\varepsilon > 0$, putting $t = s(\xi - \varepsilon, \tau(i)) + (\varepsilon/2)$, we have that $t > s(\xi, \tau(i))$ and $t > \frac{1}{2}$ by Lemma 2.13(2) and equation (2.7). We are now in a position to estimate $\mathcal{H}^{t+(\varepsilon/2)}(E(\hat{\nu}, \nu))$, the $(t + (\varepsilon/2))$-dimensional Hausdorff measure of $E(\hat{\nu}, \nu)$.

LEMMA 3.4. *For $\varepsilon > 0$, we have $\mathcal{H}^{t+(\varepsilon/2)}(E(\hat{\nu}, \nu)) < +\infty$.*

Proof. By Lemma 2.12, there exists $K \in \mathbb{N}$ such that for all $k \geq K$,

$$s_{m_k}(\xi - \varepsilon, \tau(i)) \leq t, \tag{3.6}$$

$$(4^{2t+\varepsilon} k)^{2C \log k} < 2^{((k-1)/4)\varepsilon}. \tag{3.7}$$

Writing $\psi(m_k) = m_k - \sum_{i=1}^{k}(m_i - n_i)$ when $m_k \geq K$, we have that

$$\begin{aligned}
&\sum_{(a_1,\ldots,a_{m_k}) \in \mathcal{D}_{n_1,m_1;\ldots;n_k,m_k}} |I_{m_k}(a_1, \ldots, a_{m_k})|^{t+(\varepsilon/2)} \\
&\leq \sum_{(a_1,\ldots,a_{m_k}) \in \mathcal{D}_{n_1,m_1;\ldots;n_k,m_k}} \left(\frac{1}{q_{m_k}(a_1, \ldots, a_{m_k})}\right)^{2t+\varepsilon} \\
&\leq \sum_{a_1,\ldots,a_{\psi(m_k)} \in \mathbb{N}} 2^{k(2t+\varepsilon)} \\
&\qquad \times \left(\frac{1}{q_{\psi(m_k)}(a_1, \ldots, a_{\psi(m_k)}) q_{m_1-n_1}(i, \ldots, i) \cdots q_{m_k-n_k}(i, \ldots, i)}\right)^{2t+\varepsilon} \\
&\leq \sum_{a_1,\ldots,a_{\psi(m_k)} \in \mathbb{N}} 4^{k(2t+\varepsilon)} \left(\frac{1}{q_{m_k}(a_1, \ldots, a_{\psi(m_k)}, i, \ldots, i)}\right)^{2t+\varepsilon} \\
&\leq \sum_{a_1,\ldots,a_{m_k - \lfloor m_k(\xi-\varepsilon)\rfloor} \in \mathbb{N}} 4^{k(2t+\varepsilon)} \left(\frac{1}{q_{m_k}(a_1, \ldots, a_{m_k - \lfloor m_k(\xi-\varepsilon)\rfloor}, i, \ldots, i)}\right)^{2s_{m_k}(\xi-\varepsilon,\tau(i))+\varepsilon} \\
&\leq 4^{k(2t+\varepsilon)} \left(\frac{1}{2}\right)^{((m_k-1)/2)\varepsilon},
\end{aligned}$$

where the first two inequalities hold by Lemmas 2.2 and 2.1; the penultimate one follows by equations (3.4) and (3.6); and the last one follows by Remark 2.11 and Lemma 2.1(1).

Therefore,

$$\begin{aligned}
&\mathcal{H}^{t+(\varepsilon/2)}(E(\hat{\nu}, \nu)) \\
&\quad \leq \liminf_{K\to\infty} \sum_{k=K}^{\infty} \sum_{m_k = e^{k/C}}^{\infty} \sum_{(n_1,m_1,\ldots,m_{k-1},n_k) \in \Lambda_{k,m_k}} \\
&\qquad \times \sum_{(a_1,\ldots,a_{m_k}) \in \mathcal{D}_{n_1,m_1;\ldots;n_k,m_k}} |I_{m_k}(a_1, \ldots, a_{m_k})|^{t+(\varepsilon/2)} \\
&\quad \leq \liminf_{K\to\infty} \sum_{k=K}^{\infty} \sum_{m_k = e^{k/C}}^{\infty} \sum_{n_k=1}^{m_k} \sum_{m_{k-1}=1}^{n_k} \cdots \sum_{m_1=1}^{n_2} \sum_{n_1=1}^{m_1} 4^{k(2t+\varepsilon)} \left(\frac{1}{2}\right)^{((m_k-1)/2)\varepsilon}
\end{aligned}$$

$$\le \liminf_{K\to\infty} \sum_{k=K}^{\infty} \sum_{m_k=e^{k/C}}^{\infty} (4^{2t+\varepsilon} m_k)^{2C\log m_k} \left(\frac{1}{2}\right)^{((m_k-1)/2)\varepsilon}$$

$$\le \liminf_{K\to\infty} \sum_{k=K}^{\infty} \sum_{m_k=e^{k/C}}^{\infty} \left(\frac{1}{2}\right)^{((m_k-1/4)\varepsilon} \le \frac{1}{1-(1/2)^{\varepsilon/4}} \sum_{k=1}^{\infty} \left(\frac{1}{2^{\varepsilon}}\right)^{(e^{k/C}-1)/4} < +\infty,$$

where the third and fourth inequalities follow from equations (3.3) and (3.7), respectively. □

By Lemma 3.4, we obtain the desired inequality $\dim_H E(\hat{\nu}, \nu) \le s(\xi, \tau(i))$ by letting $\varepsilon \to 0$.

Remark 3.5. In fact, the covering $\bigcup_{(\{n_k\},\{m_k\})\in\Omega} H(\{n_k\}, \{m_k\})$ of $E(\hat{\nu}, \nu)$ is also a covering of

$$E_*(\hat{\nu}, \nu) = \{x \in [0, 1] : \hat{\nu}(x) \ge \hat{\nu}, \nu(x) = \nu\},$$

because $\liminf_k((m_k - n_k)/n_{k+1}) \ge \hat{\nu}$ for $(\{n_k\}, \{m_k\}) \in \Omega$. It follows that $\dim_H E_*(\hat{\nu}, \nu) \le s(\xi, \tau(i))$.

4. *Proof of Theorem 1.3: lower bound*

In this section, we establish the lower bound of $\dim_H E(\hat{\nu}, \nu)$. Since $E(0, 0)$ is of full Lebesgue measure and $\dim_H E(\hat{\nu}, \nu) = 0$ for $\hat{\nu} > \nu/(1+\nu)$, we need only consider the cases $0 \le \hat{\nu} \le \nu/(1+\nu) < \nu < \infty$ or $\nu = \infty$.

Let us start by treating the case $0 \le \hat{\nu} \le \nu/(1+\nu) < \nu < \infty$. We claim that there exist two sequences of natural numbers $\{n_k\}_{k=1}^{\infty}$ and $\{m_k\}_{k=1}^{\infty}$ satisfying the following conditions:

(1) $n_k < m_k < n_{k+1}$ and $(m_k - n_k) \le (m_{k+1} - n_{k+1})$ for $k \ge 1$;
(2) $\lim_{k\to\infty}((m_k - n_k)/n_{k+1}) = \hat{\nu}$;
(3) $\lim_{k\to\infty}((m_k - n_k)/n_k) = \nu$.

Indeed, when $\hat{\nu} > 0$, we may take

$$n_1 = 2,\ n_{k+1} = \left\lfloor \frac{\nu}{\hat{\nu}}\left(n_k + \frac{1}{\nu}\right)\right\rfloor + 2, m_k = \lfloor (1+\nu)n_k \rfloor + 1;$$

when $\hat{\nu} = 0$, we may take

$$n_k = \lfloor (1+\nu)2^{2^{2k}} \rfloor + 2, m_k = \lfloor (1+\nu)n_k \rfloor + 1.$$

From now on, we fix two such sequences $\{n_k\}, \{m_k\}$; for any $B \ge i+1$, we define

$$E(B) = \{x \in [0, 1) : 1 \le a_n(x) \le B, a_{n_k+1}(x) = \cdots = a_{m_k}(x) = i, n \ge 1 \text{ and } k \ge 1\}.$$

The lower bound estimate of $\dim_H E(\hat{\nu}, \nu)$ will be established in the following way: we provide a lower bound of $\dim_H E(B)$; build an injective mapping f from $E(B)$ to $E(\hat{\nu}, \nu)$ and prove that f is dimension-preserving.

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4.1. *Lower bound of* $\dim_H E(B)$. Before proceeding, we cite an analogous definition of the pre-dimensional numbers. Let $l_k = m_k - m_{k-1}$ for $k \geq 1$ ($m_0 = 0$ by convention). Let

$$\widetilde{f}_k(s, \tau(i)) = \sum_{1 \leq a_{m_{k-1}+1}, \ldots, a_{n_k} \leq B} \left(\frac{1}{q_{l_k}(a_{m_{k-1}+1}, \ldots, a_{n_k}, i, \ldots, i)} \right)^{2s}.$$

Recall $\xi = \nu^2/((1+\nu)(\nu - \hat{\nu}))$. We define $\widetilde{s}_{l_k}(\mathcal{A}_B, \xi, \tau(i))$ to be the solution of the equation $\widetilde{f}_k(s, \tau(i)) = 1$.

LEMMA 4.1. *The limit* $\lim_{k\to\infty} \widetilde{s}_{l_k}(\mathcal{A}_B, \xi, \tau(i))$ *exists, and is equal to* $s(\mathcal{A}_B, \xi, \tau(i))$.

Proof. By Lemma 2.12 and the fact $l_k \to \infty$ as $k \to \infty$ (cf. condition (1)), we deduce that, for any $\varepsilon > 0$, when $k \gg 1$,

$$|s_{l_k}(\mathcal{A}_B, \xi + \varepsilon, \tau(i)) - s(\mathcal{A}_B, \xi + \varepsilon, \tau(i))| < \frac{\varepsilon}{2}. \tag{4.1}$$

Further, from conditions (2) and (3), we have that

$$\lim_{k\to\infty} \frac{m_k - n_k}{l_k} = \lim_{k\to\infty} \frac{((m_k - n_k)/m_k) \cdot (m_k/n_k)}{(m_k/n_k) - ((m_{k-1} - n_{k-1})/n_k) \cdot (m_{k-1})/(m_{k-1} - n_{k-1})} = \xi,$$

and thus for $k \gg 1$,

$$\lfloor l_k(\xi - \varepsilon) \rfloor \leq m_k - n_k \leq \lfloor l_k(\xi + \varepsilon) \rfloor. \tag{4.2}$$

Hence, by equations (4.1) and (4.2), we obtain that

$$\begin{aligned}
&\sum_{1 \leq a_1, \ldots, a_{n_k - m_{k-1}} \leq B} \left(\frac{1}{q_{l_k}(a_1, \ldots, a_{n_k - m_{k-1}}, i, \ldots, i)} \right)^{2(s(\mathcal{A}_B, \xi+\varepsilon, \tau(i)) - \varepsilon)} \\
&\geq \sum_{1 \leq a_1, \ldots, a_{n_k - m_{k-1}} \leq B} \left(\frac{1}{q_{l_k}(a_1, \ldots, a_{n_k - m_{k-1}}, i, \ldots, i)} \right)^{2s_{l_k}(\mathcal{A}_B, \xi+\varepsilon, \tau(i)) - \varepsilon} \\
&\geq \sum_{1 \leq a_1, \ldots, a_{l_k - \lfloor l_k(\xi+\epsilon) \rfloor} \leq B} \left(\frac{1}{q_{l_k}(a_1, \ldots, a_{l_k - \lfloor l_k(\xi+\varepsilon) \rfloor}, i, \ldots, i)} \right)^{2s_{l_k}(\mathcal{A}_B, \xi+\varepsilon, \tau(i)) - \varepsilon} \\
&\geq \left(\frac{1}{q_{l_k}(B, \ldots, B)} \right)^{-\varepsilon} \geq \tau(B)^{l_k \varepsilon} \geq 1,
\end{aligned}$$

where $\tau(B) = (B + \sqrt{B^2 + 4})/2$. Moreover, we get that

$$\begin{aligned}
&\sum_{1 \leq a_1, \ldots, a_{n_k - m_{k-1}} \leq B} \left(\frac{1}{q_{l_k}(a_1, \ldots, a_{n_k - m_{k-1}}, i, \ldots, i)} \right)^{2(s(\mathcal{A}_B, \xi-\varepsilon, \tau(i)) + \varepsilon)} \\
&\leq \sum_{1 \leq a_1, \ldots, a_{l_k - \lfloor l_k(\xi-\varepsilon) \rfloor} \leq B} \left(\frac{1}{q_{l_k}(a_1, \ldots, a_{l_k - \lfloor l_k(\xi-\varepsilon) \rfloor}, i, \ldots, i)} \right)^{2s_{l_k}(\mathcal{A}_B, \xi-\varepsilon, \tau(i)) + \varepsilon} \\
&\leq \left(\frac{1}{q_{l_k}(1, \ldots, 1)} \right)^{\varepsilon} < 1.
\end{aligned}$$

By the monotonicity of $\widetilde{f}_k(s, \tau(i))$ with respect to s, we have

$$s(\mathcal{A}_B, \xi + \varepsilon, \tau(i)) - \varepsilon \le \widetilde{s}_{l_k}(\mathcal{A}_B, \xi, \tau(i)) \le s(\mathcal{A}_B, \xi - \varepsilon, \tau(i)) + \varepsilon,$$

which completes the proof. □

4.1.1. *Supporting measure.* We define a probability measure μ on $E(B)$ by distributing mass among the basic intervals. We introduce the symbolic space to code these basic intervals: write $\mathcal{A}_B = \{1, \ldots, B\}$; for $n \ge 1$, set

$$\mathcal{B}_n = \{(\sigma_1, \ldots, \sigma_n) \in \mathcal{A}_B^n : \sigma_j = i \text{ for } n_k < j \le m_k \text{ with some } k \ge 1\}.$$

Step I: For $(a_1, \ldots, a_{m_1}) \in \mathcal{B}_{m_1}$, we define

$$\mu(I_{m_1}(a_1, \ldots, a_{m_1})) = \left(\frac{1}{q_{l_1}(a_1, \ldots, a_{m_1})}\right)^{2\widetilde{s}_{l_1}(\mathcal{A}_B, \xi, \tau(i))}$$

and for $1 \le n < m_1$, set

$$\mu(I_n(a_1, \ldots, a_n)) = \sum_{a_{n+1}, \ldots, a_{m_1}} \mu(I_{m_1}(a_1, \ldots, a_n, a_{n+1}, \ldots, a_{m_1})),$$

where the summation is taken over all $(a_{n+1}, \ldots, a_{m_1})$ with $(a_1, \ldots, a_{m_1}) \in \mathcal{B}_{m_1}$.

Step II: Assuming that $\mu(I_{m_k}(a_1, \ldots, a_{m_k}))$ is defined for some $k \ge 1$, we define

$$\begin{aligned}\mu(I_{m_{k+1}}&(a_1, \ldots, a_{m_{k+1}})) \\ &= \mu(I_{m_k}(a_1, \ldots, a_{m_k})) \cdot \left(\frac{1}{q_{l_{k+1}}(a_{m_k+1}, \ldots, a_{m_{k+1}})}\right)^{2\widetilde{s}_{l_{k+1}}(\mathcal{A}_B, \xi, \tau(i))}\end{aligned}$$

and for $m_k < n < m_{k+1}$, set

$$\mu(I_n(a_1, \ldots, a_n)) = \sum_{a_{n+1}, \ldots, a_{m_{k+1}}} \mu(I_{m_{k+1}}(a_1, \ldots, a_n, a_{n+1}, \ldots, a_{m_{k+1}})).$$

Likewise, the last summation is taken under the restriction that $(a_1, \ldots, a_{m_{k+1}}) \in \mathcal{B}_{m_{k+1}}$.

Step III: We have distributed the measure among basic intervals. By the definition of $\widetilde{s}_{l_k}(\mathcal{A}_B, \xi, \tau(i))$, we readily check the consistency: for $n \ge 1$ and $(a_1, \ldots, a_n) \in \mathcal{B}_n$,

$$\mu(I_n(a_1, \ldots, a_n)) = \sum_{a_{n+1}} \mu(I_{n+1}(a_1, \ldots, a_n, a_{n+1})).$$

We then extend the measure to all Borel sets by Kolmogorov extension theorem. The extension measure is also denoted by μ.

From the construction, we know that μ is supported on $E(B)$ and

$$\mu(I_{m_k}(a_1, \ldots, a_{m_k})) = \prod_{j=1}^{k} \left(\frac{1}{q_{l_j}(a_{m_{j-1}+1}, \ldots, a_{m_j})}\right)^{2\widetilde{s}_{l_j}(\mathcal{A}_B, \xi, \tau(i))},$$

$$\sum_{a_1 \in \mathcal{B}_1} \mu(I_1(a_1)) = 1.$$

4.1.2. *Hölder exponent of* $\boldsymbol{\mu}$. We shall start with the study of a basic interval.

For $0 < \varepsilon < s(\mathcal{A}_B, \xi, \tau(i))/4$, by Lemmas 2.12, 4.1, and the fact that m_k grows exponentially, we can find $K \in \mathbb{N}$ such that for any $k, j \geq K$,

$$|\widetilde{s}_{l_k}(\mathcal{A}_B, \xi, \tau(i)) - s(\mathcal{A}_B, \xi, \tau(i))| < \varepsilon, \tag{4.3}$$

$$|\widetilde{s}_{l_k}(\mathcal{A}_B, \xi, \tau(i)) - s_j(\mathcal{A}_B, \xi, \tau(i))| < \frac{\varepsilon \log 2}{2\log(B+1)} := \varepsilon', \tag{4.4}$$

and

$$\max\{(B+1)^K, 2^k\} \leq \frac{1}{4}(q_{m_k}(a_1, \ldots, a_{m_k}))^{\varepsilon}. \tag{4.5}$$

LEMMA 4.2. *Let $n \geq m_K$. For $(a_1, \ldots, a_n) \in \mathcal{B}_n$, we have*

$$\mu(I_n(a_1, \ldots, a_n)) \leq C_0 \cdot |I_n(a_1, \ldots, a_n)|^{s(\mathcal{A}_B, \xi, \tau(i)) - 3\varepsilon},$$

where $C_0 = (B+1)^{2(l_1 + \cdots + l_{K-1})}$.

Proof. To shorten notation, we will write $\widetilde{s}_{l_k}, s_k$ and s instead of $\widetilde{s}_{l_k}(\mathcal{A}_B, \xi, \tau(i))$, $s_k(\mathcal{A}_B, \xi, \tau(i))$ and $s(\mathcal{A}_B, \xi, \tau(i))$, respectively. Fixing $(a_1, \ldots, a_n) \in \mathcal{B}_n$, we also write I_n for $I_n(a_1, \ldots, a_n)$, q_n for $q_n(a_1, \ldots, a_n)$, and q_{l_j} for $q_{l_j}(a_{m_{j-1}+1}, \ldots, a_{m_j})$ when no confusion can arise. The proof falls naturally into three parts according to the range of n.

Case 1: $n = m_k$ for $k \geq K$. By Lemmas 2.2 and 2.3, equations (4.3), (4.5), and the fact $q_{l_j} \leq (B+1)^{l_j}$, we obtain

$$\begin{aligned}\mu(I_{m_k}) &= \prod_{j=1}^{k} q_{l_j}^{-2\widetilde{s}_{l_j}} = \prod_{j=1}^{K} q_{l_j}^{-2\widetilde{s}_{l_j}} \cdot \prod_{j=K+1}^{k} q_{l_j}^{-2\widetilde{s}_{l_j}} \leq C_0 \prod_{j=1}^{K} q_{l_j}^{-2(s-\varepsilon)} \cdot \prod_{j=K+1}^{k} q_{l_j}^{-2(s-\varepsilon)} \\ &\leq C_0 2^{2(k-1)}(q_{m_k})^{-2(s-\varepsilon)} \leq \frac{C_0}{4}(q_{m_k})^{-2(s-2\varepsilon)} \leq C_0 |I_{m_k}|^{s-2\varepsilon}.\end{aligned}$$

Case 2: $m_k < n < n_{k+1}$ for $k \geq K$. In this case, we have

$$\begin{aligned}\mu(I_n) &= \sum_{a_{n+1},\ldots,a_{m_{k+1}}} \mu(I_{m_{k+1}}) = \sum_{a_{n+1},\ldots,a_{m_{k+1}}} \prod_{j=1}^{k+1} (q_{l_j})^{-2\widetilde{s}_{l_j}} \\ &= \prod_{j=1}^{k} (q_{l_j})^{-2\widetilde{s}_{l_j}} \cdot \sum_{a_{n+1},\ldots,a_{m_{k+1}}} (q_{l_{k+1}})^{-2\widetilde{s}_{l_{k+1}}}.\end{aligned}$$

We have already seen in Case 1 that $\prod_{j=1}^{k}(q_{l_j})^{-2\widetilde{s}_{l_j}} \leq C_0 2^{2(k-1)}(q_{m_k})^{-2(s-\varepsilon)}$. Additionally,

$$\begin{aligned}\sum_{a_{n+1},\ldots,a_{m_{k+1}}} (q_{l_{k+1}})^{-2\widetilde{s}_{l_{k+1}}} &\leq (q_{n-m_k}(a_{m_k+1}, \ldots, a_n))^{-2(s-\varepsilon)} \\ &\quad \cdot \sum_{a_{n+1},\ldots,a_{n_{k+1}}} (q_{m_{k+1}-n}(a_{n+1}, \ldots, a_{n_{k+1}}, i, \ldots, i))^{-2\widetilde{s}_{l_{k+1}}}.\end{aligned}$$

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We then obtain that

$$\mu(I_n) \le C_0 2^{2k} (q_n)^{-2(s-\varepsilon)} \cdot \sum_{a_{n+1},\ldots,a_{m_{k+1}}} (q_{m_{k+1}-n}(a_{n+1}, \ldots, a_{n_{k+1}}, i, \ldots, i))^{-2\widetilde{s}_{l_{k+1}}}.$$

Now we need an upper estimate of the last sum. By the definition of $\widetilde{s}_{l_{k+1}}$, we have that

$$\sum_{1\le a'_{m_k+1},\ldots,a'_n,a_{n+1},\ldots,a_{n_{k+1}}\le B} (q_{l_{k+1}}(a'_{m_k+1}, \ldots, a'_n, a_{n+1}, \ldots, a_{n_{k+1}}, i, \ldots, i))^{-2\widetilde{s}_{l_{k+1}}} = 1.$$

This yields that

$$\sum_{a'_{m_k+1},\ldots,a'_n} (q_{n-m_k}(a'_{m_k+1}, \ldots, a'_n))^{-2\widetilde{s}_{l_{k+1}}} \times \sum_{a_{n+1},\ldots,a_{n_{k+1}}} (q_{m_{k+1}-n}(a_{n+1}, \ldots, a_{n_{k+1}}, i, \ldots, i))^{-2\widetilde{s}_{l_{k+1}}} \le 4.$$

We will bound the first sum from below to reach the desired upper estimate of the second sum. We consider two cases.

(1) If $n - m_k < K$,

$$\sum_{a'_{m_k+1},\ldots,a'_n} (q_{n-m_k}(a'_{m_k+1}, \ldots, a'_n))^{-2\widetilde{s}_{l_{k+1}}} \ge (q_{n-m_k}(B, \ldots, B))^{-2} \ge (B+1)^{-2K}.$$

And thus, by equation (4.5), we reach that

$$\mu(I_n) \le C_0 2^{2k+2}(B+1)^{2K}(q_n)^{-2(s-\varepsilon)} \le \frac{C_0}{4}(q_{m_k})^{-2(s-3\varepsilon)} \le C_0|I_n|^{s-3\varepsilon}.$$

(2) If $n - m_k \ge K$, then, by equations (4.4), (4.5), and Remark 2.11, we have

$$\begin{aligned}\sum_{a'_{m_k+1},\ldots,a'_n} (q_{n-m_k}(a'_{m_k+1}, \ldots, a'_n))^{-2\widetilde{s}_{l_{k+1}}} &\ge \sum_{a'_{m_k+1},\ldots,a'_n} (q_{n-m_k}(a'_{m_k+1}, \ldots, a'_n))^{-2s_{n-m_k}-\varepsilon'}\\ &\ge \sum_{a'_{m_k+1},\ldots,a'_{n-\lfloor (n-m_k)\xi\rfloor}} (q_{n-m_k}(a'_{m_k+1}, \ldots, a'_{n-\lfloor (n-m_k)\xi\rfloor}, i, \ldots, i))^{-2s_{n-m_k}-\varepsilon'}\\ &\ge (q_{n-m_k}(B, \ldots, B))^{-\varepsilon'} \ge (B+1)^{-(n-m_k)\varepsilon'} \ge (B+1)^{-n\varepsilon'} \ge 2^{-n\varepsilon/2}.\end{aligned}$$

Therefore,

$$\mu(I_n) \le C_0 2^{2k+2} 2^{n\varepsilon/2}(q_n)^{-2(s-2\varepsilon)} \le \frac{C_0}{2}(q_n)^{-2(s-3\varepsilon)} \le C_0|I_n|^{s-3\varepsilon}.$$

Case 3: $n_{k+1} \le n < m_{k+1}$ for $k \ge K$. In this case, since $(a_{n+1}, \ldots, a_{m_{k+1}}) = (i, \ldots, i)$, we have

$$\mu(I_n(a_1, \ldots, a_n)) = \mu(I_{m_{k+1}}(a_1, \ldots, a_{m_{k+1}})),$$

then

$$\mu(I_n) \le C_0 |I_{m_{k+1}}|^{s-2\varepsilon} \le C_0 |I_n|^{s-2\varepsilon}.$$

These conclude the verification of the lemma. $\square$

Now we study the Hölder exponent for the measure of a general ball $B(x, r)$.

LEMMA 4.3. *For $x \in E(B)$ and $r > 0$ small enough, we have*

$$\mu(B(x, r)) \le C_0 \cdot r^{s(\mathcal{A}_B,\xi,\tau(i))-4\varepsilon}.$$

Proof. Let $x = [a_1, a_2, \ldots]$ be its continued fraction expansion. Let $n \ge K + 2$ be the integer such that

$$|I_{n+1}(a_1, \ldots, a_{n+1})| \le r < |I_n(a_1, \ldots, a_n)|.$$

Therefore, it follows from Lemmas 2.4 and 4.2 that

$$\begin{aligned}\mu(B(x, r)) &\le \mu(I_{n-2}(a_1, \ldots, a_{n-2})) \le C_0 \cdot |I_{n-2}(a_1, \ldots, a_{n-2})|^{s(\mathcal{A}_B,\xi,\tau(i))-3\varepsilon} \\ &\le C_0(B + 1)^6 \cdot |I_{n+1}(a_1, \ldots, a_{n+1})|^{s(\mathcal{A}_B,\xi,\tau(i))-3\varepsilon} \\ &\le C_0 \cdot |I_{n+1}(a_1, \ldots, a_{n+1})|^{s(\mathcal{A}_B,\xi,\tau(i))-4\varepsilon} \le C_0 \cdot r^{s(\mathcal{A}_B,\xi,\tau(i))-4\varepsilon}.\end{aligned}$$ $\square$

Applying mass distribution principle (see Lemma 2.6), letting $\varepsilon \to 0$, we conclude that

$$\dim_H E(B) \ge s(\mathcal{A}_B, \xi, \tau(i)).$$

4.2. *Lower bound of* $\dim_H E(\hat{\nu}, \nu)$. We build a mapping f from $E(B)$ to $E(\hat{\nu}, \nu)$ and prove that f is dimension-preserving.

Fix an integer $d > B$. For $x = [a_1, a_2, \ldots]$ in $E(B)$, we remark that the continued fraction of x is the concatenation of $\mathbb{B}_0 = [a_1, \ldots, a_{n_1}]$ and the blocks

$$\mathbb{B}_k = [\underbrace{i, \ldots, i}_{m_k - n_k}, a_{m_k+1}, \ldots, a_{n_{k+1}}] \quad (k = 1, 2, \ldots).$$

In the block $\mathbb{B}_k$, from the beginning, we insert a digit d after each $m_k - n_k$ digits to obtain a new block $\mathbb{B}'_k$, that is,

$$\mathbb{B}'_k = [d, i, \ldots, i, d, a_{m_k+1}, \ldots, a_{m_k+(m_k-n_k)}, d, \ldots, a_{n_{k+1}}].$$

Concatenating the blocks $\mathbb{B}_0, \mathbb{B}'_1, \mathbb{B}'_2, \ldots$, we get $[\mathbb{B}_0, \mathbb{B}'_1, \mathbb{B}'_2, \ldots]$, which is a continued fraction expansion of some $\bar{x}$. We then define $f(x) = \bar{x}$. Let $\mathbf{K} = \{k_n\} \subset \mathbb{N}$ be the collection of the occurrences of the digit d in the continued expansion of $\bar{x}$. It is trivially seen that $\mathbf{K}$ is independent of the choice of $x \in E(B)$, and, in the notation of Lemma 2.7, $\phi_{\mathbf{K}}(\bar{x}) = x$ for $x \in E(B)$.

Let h_k be the length of the block $\mathbb{B}'_k$. Noting that the number of the inserted digit d is at most $(n_{k+1} - m_k)/(m_k - n_k) + 1 = o(h_k)$ in the block $\mathbb{B}'_k$, we readily check that $\mathbf{K}$ is a subset of $\mathbb{N}$ of density zero. Hence, by Lemma 2.7, we have

$$\dim_H f(E(B)) = \dim_H E(B).$$

It remains to prove that $f(E(B))$ is a Cantor subset of $E(\hat{\nu}, \nu)$.

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LEMMA 4.4. $f(E(B)) \subset E(\hat{\nu}, \nu)$.

Proof. Fix $\bar{x} \in f(E(B))$.

For $\varepsilon > 0$ and n large enough, there exists some k such that $(\sum_{j=0}^{k-1} h_j) \leq n < (\sum_{j=0}^{k} h_j)$. From the construction, we deduce that if $n = (\sum_{j=0}^{k-1} h_j) + 1$, then

$$|T^n(\bar{x}) - y| < |I_{m_k - n_k}(y)| \leq |I_n(y)|^{\nu - \varepsilon},$$

where the last inequality holds by the fact $\lim_k (\sum_{j=0}^{k-1} h_j / n_k) = 1$; if $(\sum_{j=0}^{k-1} h_j) < n < (\sum_{j=0}^{k} h_j)$, then

$$|T^n(\bar{x}) - y| \geq \frac{1}{2(d+2)^2} |I_{m_k - n_k}(y)| \geq |I_n(y)|^{\nu + \varepsilon}.$$

We then prove that $\nu(\bar{x}) = \nu$ by the arbitrariness of ε.

However, for $(\sum_{j=0}^{k-1} h_j) \leq N < (\sum_{j=0}^{k} h_j)$ with k large enough, we pick $n = (\sum_{j=0}^{k-1} h_j) + 1$ to obtain that

$$|T^n(\bar{x}) - y| < |I_{m_k - n_k}(y)| \leq |I_{\sum_{j=0}^{k} h_j}(y)|^{\hat{\nu} - \varepsilon} < |I_N(y)|^{\hat{\nu} - \varepsilon}.$$

When $N = (\sum_{j=0}^{k} h_j)$, for all $n \in [1, N]$, we have that

$$|T^n(\bar{x}) - y| \geq \frac{1}{2(d+2)^2} |I_{m_k - n_k}(y)| \geq |I_N(y)|^{\hat{\nu} + \varepsilon}.$$

We prove that $\hat{\nu}(\bar{x}) = \hat{\nu}$.

Hence, $\bar{x} \in E(\hat{\nu}, \nu)$, as desired. $\square$

Consequently, for $0 \leq \hat{\nu} \leq \nu/(1+\nu) < \nu < \infty$, we have $\dim_H E(\hat{\nu}, \nu) \geq s(\mathcal{A}_B, \xi, \tau(i))$.

Letting $B \to \infty$ yields $\dim_H E(\hat{\nu}, \nu) \geq s(\xi, \tau(i))$, where $\xi = \nu^2/((1+\nu)(\nu - \hat{\nu}))$.

We conclude this section by determining the lower bound of $\dim_H E(\hat{\nu}, +\infty)$. We first study the case $0 < \hat{\nu} < 1$. Let

$$n_1 = 2, \quad n_{k+1} = n_k^k + 2n_k, \quad m_0 = 0, \quad m_k = \lfloor \hat{\nu} n_k^k \rfloor + n_k, \quad B_k = \lfloor m_k \log m_k \rfloor.$$

Thus,

$$\lim_{k\to\infty} \frac{m_k - n_k}{n_{k+1}} = \hat{\nu}, \quad \lim_{k\to\infty} \frac{m_k - n_k}{n_k} = \infty, \quad \lim_{k\to\infty} \frac{m_k - n_k}{m_k} = 1.$$

Define

$$E = \{x \in [0,1) : a_n(x) \leq B_k \text{ if } m_k < n \leq n_{k+1} \text{ for some } k;\ a_n(x) = i \text{ otherwise}\}.$$

As before, for any $x = [a_1, a_2, \ldots]$ in E, we construct an element $\bar{x} := f(x)$: insert a digit $B_k + 1$ after positions n_k and $m_k + i(m_k - n_k)$, $0 \leq i \leq t_k$ in the continued fraction expansion of x, where $t_k = \max\{t \in \mathbb{N} : m_k + t(m_k - n_k) < n_{k+1}\}$; the resulting sequence is the continued fraction of $\bar{x}$.

The method establishing the lower bound of $\dim_H E(B)$ applies to show that $\dim_H E \geq 1/2$. Moreover, $f(E)$ is a subset of $E(\hat{\nu}, +\infty)$. It remains to prove that the

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Hausdorff dimension of $f(E)$ coincides with the one of E. To this end, we shall show that f^{-1} is a $(1-\varepsilon)$-Hölder mapping for any $\varepsilon > 0$. We remark that Lemma 2.7 may not apply directly here since $\{B_k\}$ is an unbounded sequence.

LEMMA 4.5. *For $\varepsilon > 0$, f^{-1} is a $(1-\varepsilon)$-Hölder mapping.*

Proof. We write

$$m'_k = m_k + \sum_{l=1}^{k-1}(t_l + 2), \quad n'_k = n_k + \sum_{l=1}^{k-1}(t_l + 2),$$

and define the marked set

$$\mathbf{K} = \{m'_k + i(m_k - n_k) + 1 \colon 0 \le i \le t_k, k \ge 1\} \cup \{n'_k \colon k \ge 1\}.$$

Let $\Delta_n = \sharp\{i \le n \colon i \in \mathbf{K}\}$, where $\sharp$ denotes the cardinality of a finite set. Let $k \in \mathbb{N}$ such that $m'_k \le n < m'_{k+1}$. We have

$$\begin{aligned}\frac{\Delta_n \log B_k}{n} &\le \frac{(\sum_{l=1}^{k-1}(t_l+2) + (n - m'_k)/(m_k - n_k) + 1) \log B_k}{n} \\ &\le \frac{(\sum_{l=1}^{k-1}(t_l+2) + 1) \log B_k}{m'_k} + \frac{\log B_k}{m_k - n_k} + \frac{\log B_k}{m_k} \to 0.\end{aligned}$$

So there exists $K \in \mathbb{N}$, such that for $k \ge K$ and $n \ge m'_K$,

$$(B_k + 2)^{2\Delta_n + 4} < 2^{(n-1)\varepsilon}. \tag{4.6}$$

For $\overline{x_1} = f(x_1)$ and $\overline{x_2} = f(x_2)$ in $f(E)$, we assume without loss of generality that

$$|\overline{x_1} - \overline{x_2}| < \frac{1}{2(B_K + 2)^2 q^2_{m'_K}(\overline{x_1})};$$

otherwise, $|f^{-1}(\overline{x_1}) - f^{-1}(\overline{x_2})| < C|\overline{x_1} - \overline{x_2}|^{1-\varepsilon}$ for some C, as desired. Let

$$n = \min\{j \ge 1 \colon a_{j+1}(\overline{x_1}) \ne a_{j+1}(\overline{x_2})\}.$$

By Lemma 2.2, we have $m'_k \le n < m'_{k+1}$ for some $k \ge K$ and $n + 1 < n'_{k+1}$. Assume that $\overline{x_1} > \overline{x_2}$ and n is even (the same conclusion can be drawn for the remaining cases). There exist $1 \le \tau_{n+1}(\overline{x_1}) < \sigma_{n+1}(\overline{x_2}) \le B_k + 1$ such that $\overline{x_1} \in I_{n+1}(a_1, \ldots, a_n, \tau_{n+1}(\overline{x_1}))$, $\overline{x_2} \in I_{n+1}(a_1, \ldots, a_n, \sigma_{n+1}(\overline{x_2}))$. Combining Lemma 2.3 and the construction yields that $\overline{x_1} - \overline{x_2}$ is greater than the length of basic interval $I_{n+2}(a_1, \ldots, a_n, \sigma_{n+1}(\overline{x_2}), B_k + 1)$. This implies that

$$\overline{x_1} - \overline{x_2} \ge |I_{n+2}(a_1, \ldots, a_n, \sigma_{n+1}(\overline{x_2}), B_k + 1)| \ge \frac{1}{2(B_k + 2)^4 q_n^2}.$$

Furthermore, noting that $f^{-1}(\overline{x_1}), f^{-1}(\overline{x_2}) \in I_{n-\Delta_n}(c_1, \ldots, c_{n-\Delta_n})$, where $(c_1, \ldots, c_{n-\Delta_n})$ is obtained by eliminating all the terms a_i with $i \in \mathbf{K}$ from $(a_1, \ldots, a_n)$, we conclude that

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$$|f^{-1}(\overline{x_1}) - f^{-1}(\overline{x_2})| \le |I_{n-\Delta_n}(a_1, \ldots, a_{n-\Delta_n})| \le \frac{1}{q_{n-\Delta_n}^2} \le (B_k+2)^{2\Delta_n}\frac{1}{q_n^2} \le 2|\overline{x_1} - \overline{x_2}|^{1-\varepsilon},$$

where the penultimate inequality follows by equation (4.6). This completes the proof. □

Now we deduce that $\dim_H E(\hat{\nu}, \infty) \ge (1-\varepsilon)/2$ by Lemma 2.5. Letting $\varepsilon \to 0$, we establish that $\dim_H E(\hat{\nu}, \infty) \ge \frac{1}{2}$ when $0 < \hat{\nu} < 1$. A slight change in the proof actually shows that the estimate $\dim_H E(\hat{\nu}, \infty) \ge \frac{1}{2}$ also works for $\hat{\nu} = 0$ or 1. Indeed, when $\hat{\nu} = 0$, we may take

$$n_k = 2^{2^{2k}}, \quad m_k = n_k^2, \quad B_k = 2^{n_k};$$

when $\hat{\nu} = 1$, we may take

$$m_k = (k+1)!\,, \quad n_1 = 1, \quad n_{k+1} = m_k + \frac{m_k}{\log m_k}, \quad B_k = \lfloor 2^{\sqrt{m_k}} \rfloor.$$

Remark 4.6. Combining Remark 3.5 and the fact

$$E(\hat{\nu}, \nu) \subseteq E_*(\hat{\nu}, \nu) = \{x \in [0,1] : \hat{\nu}(x) \ge \hat{\nu},\ \nu(x) = \nu\},$$

we have $\dim_H E_*(\hat{\nu}, \nu) = s(\xi, \tau(i))$.

Remark 4.7. By Lagrange's theorem, any quadratic irrational number y_1 is represented by a periodic continued fraction expansion, that is,

$$y_1 = [a_1(y_1), a_2(y_1), \ldots, a_{k_0}(y_1), \overline{a_{k_0+1}(y_1), \ldots, a_{k_0+h}(y_1)}]$$

for some positive integers k_0 and h. By Lemma 2.1(2), we readily check that the limit $\lim_n((\log q_n(y_1))/n)$ exists, denoted by $\log g(y_1)$. However, unlike in the special $y = [i, i, \ldots]$, we cannot obtain a closed-form expression for $\log g(y_1)$.

In Theorem 1.3, we replace $y = [i, i, \ldots]$ by a general quadratic irrational number y_1 and consider the Hausdorff dimension of the corresponding set $E(\hat{\nu}, \nu)$. We obtain that

$$\dim_H E(\hat{\nu}, \nu) = \begin{cases} 1 & \text{if } \nu = 0, \\ s\left(\dfrac{\nu^2}{(1+\nu)(\nu-\hat{\nu})}, g(y_1)\right) & \text{if } 0 \le \hat{\nu} \le \dfrac{\nu}{1+\nu} < \nu \le \infty, \\ 0 & \text{otherwise,} \end{cases}$$

where $s(\nu^2/((1+\nu)(\nu-\hat{\nu})), g(y_1))$ is the solution to

$$P\left(T, -s\left(\log|T'| + \frac{\alpha}{1-\alpha}\log g(y_1)\right)\right) = 0$$

with $\alpha = \nu^2/((1+\nu)(\nu-\hat{\nu}))$. The proof can be established following the same line as for the original theorem, with some crucial modifications as follows.

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(1) In the proof of Lemma 3.2: two sequences $\{n'_k\}_{k\geq 1}$ and $\{m'_k\}_{k\geq 1}$ are modified as

$$m'_0 = 0,\ n'_k = \min\{n \geq m'_{k-1} \colon a_{n+1}(x) = a_1(y_1)\},$$
$$m'_k = \max\{n \geq n'_k \colon (a_{n'_k+1}(x), \ldots, a_n(x)) = (a_1(y_1), \ldots, a_{n-n'_k}(y_1))\}.$$

The partial quotient $a_n(y_1)$ of y_1 is bounded uniformly in n, which guarantees that equations (3.1) and (3.2) hold.

The limit

$$\lim_{n\to\infty} \frac{-\log |I_n(y_1)|}{2n} = \lim_{n\to\infty} \frac{\log q_n(y_1)}{n} = \log g(y_1).$$

(2) For $B > \max\{a_1(y_1), \ldots, a_{k_0+h}(y_1)\}$, the cantor-like subset $E(B)$ is modified as

$$\{x \in [0, 1) \colon 1 \leq a_n(x) \leq B, (a_{n_k+1}(x), \ldots, a_{m_k}(x))$$
$$= (a_1(y_1), \ldots, a_{m_k-n_k}(y_1)), n \geq 1, k \geq 1\}.$$

5. *Proofs of Theorems 1.1 and 1.2*

In this section, we study the Hausdorff dimensions of the following sets:

$$E(\hat{\nu}) = \{x \in [0, 1) \colon \hat{\nu}(x) = \hat{\nu}\},$$

$$\mathcal{U}(\hat{\nu}) = \{x \in [0, 1) \colon \text{for all } N \gg 1, \text{ there exists } n \in [1, N],$$
$$\text{such that } |T^n(x) - y| < |I_N(y)|^{\hat{\nu}}\}.$$

A direct corollary of the definition is: if $\hat{\nu}_1 > \hat{\nu} \geq 0$,

$$E(\hat{\nu}_1) \subseteq \mathcal{U}(\hat{\nu}) \subseteq \{x \in [0, 1) \colon \hat{\nu}(x) \geq \hat{\nu}\}.$$

Hence, the proofs of Theorems 1.1 and 1.2 will be divided into two parts: the upper bound of $\dim_H\{x \in [0, 1) \colon \hat{\nu}(x) \geq \hat{\nu}\}$ and the lower bound of $\dim_H E(\hat{\nu})$.

Lemma 3.1 combined with the fact $E(0, 0) \subset E(0)$ implies that the sets $E(0), \mathcal{U}(0)$, and $\{x \in [0, 1) \colon \hat{\nu}(x) \geq 0\}$ are of full Lebesgue measure; we only need to deal with the case $\hat{\nu} > 0$.

We start with the upper bound of $\dim_H\{x \in [0, 1) \colon \hat{\nu}(x) \geq \hat{\nu}\}$.

LEMMA 5.1. *If* $0 < \hat{\nu} \leq 1$, *we have*

$$\dim_H\{x \in [0, 1) \colon \hat{\nu}(x) \geq \hat{\nu}\} \leq s\bigg(\frac{4\hat{\nu}}{(1 + \hat{\nu})^2}, \tau(i)\bigg).$$

If $\hat{\nu} > 1$, *then* $\dim_H\{x \in [0, 1) \colon \hat{\nu}(x) \geq \hat{\nu}\} = 0$.

Proof. For $\varepsilon > 0$ small enough, we define

$$E_\varepsilon(\hat{\nu}, \nu) = \bigg\{x \in [0, 1) \colon \hat{\nu}(x) \geq \hat{\nu}, \nu \leq \nu(x) \leq \frac{\nu + \varepsilon}{1 - \varepsilon}\bigg\}.$$

Since

$$\{x \in [0, 1) \colon \hat{\nu}(x) \geq \hat{\nu}\} \subseteq \bigcup_{\nu \in \mathbb{Q}^+} E_\varepsilon(\hat{\nu}, \nu),$$

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where $\mathbb{Q}^+$ denotes the set of positive rational numbers, we have

$$\dim_H\{x \in [0, 1)\colon \hat{\nu}(x) \ge \hat{\nu}\} \le \sup\,\{\dim_H E_\varepsilon(\hat{\nu}, \nu)\colon \nu \in \mathbb{Q}^+\}.$$

If $\hat{\nu} > 1$, the set $E_\varepsilon(\hat{\nu}, \nu)$ is at most countable by Lemmas 3.2 and 3.3, and $\dim_H\{x \in [0, 1)\colon \hat{\nu}(x) \ge \hat{\nu}\} = 0$.

If $0 < \hat{\nu} \le 1$, we obtain

$$\dim_H E_\varepsilon(\hat{\nu}, \nu) \le s\bigg(\frac{\nu^2}{(\nu - \hat{\nu} + \hat{\nu}\varepsilon + \varepsilon)(1 + \nu)}, \tau(i)\bigg)$$

in much the same way as the proof for the upper bound of $\dim_H E(\hat{\nu}, \nu)$; we sketch the main differences as follows.

In Lemma 3.3, $\limsup((m_k - n_k)/m_k)$ is estimated by

$$\frac{\nu}{1+\nu} \le \limsup_{k\to\infty} \frac{m_k - n_k}{m_k} \le \frac{\nu + \varepsilon}{1 + \nu}.$$

Equations (3.4)–(3.7) are replaced by

$$\sum_{i=1}^{k}(m_i - n_i) \ge m_k\bigg(\frac{(\nu + \varepsilon)^2}{(\nu - \hat{\nu} + \hat{\nu}\varepsilon + \varepsilon)(1 + \nu)} - \varepsilon\bigg),$$

$$1 + \nu \le \limsup_{k\to\infty} \frac{m_k}{n_k} \le \frac{1 + \nu}{1 - \varepsilon},$$

$$\liminf_{k\to\infty} \frac{m_k}{n_{k+1}} \ge \frac{\hat{\nu}(1 + \nu)}{\nu + \varepsilon},$$

$$\liminf_{k\to\infty} \frac{\sum_{i=1}^{k}(m_i - n_i)}{m_{k+1}} \ge \frac{\hat{\nu}(\nu + \varepsilon)(1 - \varepsilon)}{(\nu - \hat{\nu} + \hat{\nu}\varepsilon + \varepsilon)(1 + \nu)},$$

respectively. The set Ω is replaced by

$$\bigg\{(\{n_k\}, \{m_k\})\colon \liminf_{k\to\infty} \frac{m_k - n_k}{n_{k+1}} \ge \hat{\nu},\ \nu \le \limsup_{k\to\infty} \frac{m_k - n_k}{n_k} \le \frac{\nu + \varepsilon}{1 - \varepsilon},\ n_k < m_k < n_{k+1} \text{ for all } k \ge 1\bigg\}.$$

Finally, since the function $\nu^2/((\nu - \hat{\nu} + \hat{\nu}\varepsilon + \varepsilon)(1 + \nu))$ of ν attains its minimum at $\nu = (2\hat{\nu} - 2(\hat{\nu} + 1)\varepsilon)/(1 - \hat{\nu} + (\hat{\nu} + 1)\varepsilon)$, we have by Lemma 2.13(2) that

$$\begin{aligned}\dim_H\{x \in [0, 1)\colon \hat{\nu}(x) \ge \hat{\nu}\} &\le \sup\bigg\{s\bigg(\frac{\nu^2}{(\nu - \hat{\nu} + \hat{\nu}\varepsilon + \varepsilon)(1 + \nu)}, \tau(i)\bigg)\colon \nu \in \mathbb{Q}^+\bigg\} \\ &\le s\bigg(\frac{4(\hat{\nu} - (\hat{\nu} + 1)\varepsilon)}{(1 + \hat{\nu} - (\hat{\nu} + 1)\varepsilon)^2}, \tau(i)\bigg) \to s\bigg(\frac{4\hat{\nu}}{(1 + \hat{\nu})^2}, \tau(i)\bigg)\end{aligned}$$

as $\varepsilon \to 0$. □

We now deal with the lower bound of the $\dim_H E(\hat{\nu})$ for $0 < \hat{\nu} \le 1$.

LEMMA 5.2. *For* $0 < \hat{\nu} \le 1$, *we have* $\dim_H E(\hat{\nu}) \ge s(4\hat{\nu}/(1 + \hat{\nu})^2, \tau(i))$.

Proof. Noting that $E(\hat{\nu}, \nu)$ is a subset of $E(\hat{\nu})$ for $\nu \geq \hat{\nu}/(1-\hat{\nu})$ (or $\hat{\nu} \leq \nu/(1+\nu)$), we have

$$\dim_H E(\hat{\nu}) \geq \dim_H E(\hat{\nu}, \nu) = s\bigg(\frac{\nu^2}{(1+\nu)(\nu-\hat{\nu})}, \tau(i)\bigg).$$

Since the function $\nu^2/((1+\nu)(\nu-\hat{\nu}))$ is continuous for $\nu \in [\hat{\nu}/(1-\hat{\nu}), \infty]$, and attains its minimum at $\nu = 2\hat{\nu}/(1-\hat{\nu})$, so by Lemma 2.13(2), we have

$$\dim_H E(\hat{\nu}) \geq \dim_H E\bigg(\hat{\nu}, \frac{2\hat{\nu}}{1-\hat{\nu}}\bigg) = s\bigg(\frac{4\hat{\nu}}{(1+\hat{\nu})^2}, \tau(i)\bigg). \qquad \square$$

6. *Proof of Theorem 1.5*

In this section, we prove Theorem 1.5 by considering the upper and lower bounds of $\dim_H(F(\alpha) \cap G(\beta))$. Recall that

$$F(\alpha) = \bigg\{x \in [0,1) \colon \liminf_{n\to\infty} \frac{R_n(x)}{n} = \alpha\bigg\},$$
$$G(\beta) = \bigg\{x \in [0,1) \colon \limsup_{n\to\infty} \frac{R_n(x)}{n} = \beta\bigg\}.$$

The proof of Theorem 1.5 goes along the lines as that of Theorem 1.3 with some minor modifications.

Noting that $\{x \in [0,1) \colon \lim_{n\to\infty}(R_n(x)/\log_{(\sqrt{5}+1)/2} n) = \frac{1}{2}\} \subset F(0) \cap G(0)$, we have $F(0) \cap G(0)$ is of full Lebesgue measure. Furthermore, since $F(\alpha) \cap G(1) \subset G(1)$ and $F(0) \cap G(\beta) \subset G(\beta)$, we have $\dim_H F(\alpha) \cap G(1) \leq s(1, \tau(1))$ and $\dim_H(F(0) \cap G(\beta)) \leq s(\beta, \tau(1))$. We only need to consider the case $0 < \alpha \leq \beta < 1$.

6.1. *Upper bound of* $\dim_H(F(\alpha) \cap G(\beta))$.

For $x = [a_1(x), a_2(x), \ldots] \in F(\alpha) \cap G(\beta)$ with non-periodic continued fraction expansion, we associate x with two sequences $\{n_k\}$ and $\{m_k\}$ that satisfy the following properties:

(1) $n_k < m_k < n_{k+1} < m_{k+1}$ for $k \geq 1$;

(2) $a_{n_k}(x) = \cdots = a_{m_k}(x)$ for $k \geq 1$;

(3) $\liminf_{k\to\infty}((m_k - n_k)/n_{k+1}) = \alpha/(1-\alpha)$, $\limsup_{k\to\infty}((m_k - n_k)/m_k) = \beta$;

(4) the sequence $\{m_k\}$ grows exponentially;

(5) write $\xi = (\beta^2(1-\alpha))/(\beta-\alpha)$. For any $\varepsilon > 0$, there exist infinitely many k such that

$$\sum_{i=1}^{k}(m_i - n_i + 1) \geq m_k(\xi - \varepsilon).$$

To this end, we define two ascending sequences $\{n'_k\}$ and $\{m'_k\}$ as follows:

$$n'_1 = 1, \quad a_{n'_k}(x) = a_{n'_k+1}(x) = \cdots = a_{m'_k}(x) \neq a_{m'_k+1}(x), n'_{k+1} = m'_k + 1.$$

Since $\beta = \limsup R_n(x)/n > 0$, we have that $\limsup_{k\to\infty}(m'_k - n'_k) = +\infty$, which enables us to pick a non-decreasing subsequence of $\{(n'_k, m'_k)\}_{k\geq 1}\colon$, put $(n_1, m_1) = (n'_1, m'_1)$; having chosen $(n_k, m_k) = (n'_{j_k}, m'_{j_k})$ for $k \geq 1$, we set $j_{k+1} = \min\{j > j_k \colon m'_j - n'_j > m_k - n_k\}$, and put $(n_{k+1}, m_{k+1}) = (n'_{j_{k+1}}, m'_{j_{k+1}})$.

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We readily check the following properties.

(a) The sequence $\{m_k - n_k\}_{k\geq 1}$ is non-decreasing and $\lim_{k\to\infty}(m_k - n_k) = +\infty$.

(b) If $m_k \leq n \leq n_{k+1} + (m_k - n_k)$ for $k \geq 1$, then $R_n(x) = m_k - n_k + 1$ and

$$\frac{m_k - n_k + 1}{n_{k+1} + (m_k - n_k)} \leq \frac{R_n(x)}{n} \leq \frac{m_k - n_k + 1}{m_k}.$$

(c) If $n_{k+1} + (m_k - n_k) < n < m_{k+1}$ for $k \geq 1$, then $R_n(x) = n - n_{k+1} + 1$, and

$$\frac{m_k - n_k + 1}{n_{k+1} + (m_k - n_k)} \leq \frac{R_n(x)}{n} \leq \frac{m_{k+1} - n_{k+1} + 1}{m_{k+1}}.$$

Properties (a) and (b) imply

$$\alpha = \liminf_{n\to\infty} \frac{R_n(x)}{n} = \liminf_{k\to\infty} \frac{m_k - n_k + 1}{n_{k+1} + (m_k - n_k)} \tag{6.1}$$

and

$$\beta = \limsup_{n\to\infty} \frac{R_n(x)}{n} = \limsup_{k\to\infty} \frac{m_{k+1} - n_{k+1} + 1}{m_{k+1}}. \tag{6.2}$$

From equation (6.1), we obtain that

$$\liminf_{k\to\infty} \frac{R_{n_{k+1}}(x)}{n_{k+1}} = \liminf_{k\to\infty} \frac{m_k - n_k + 1}{n_{k+1}} = \frac{\alpha}{1-\alpha}, \tag{6.3}$$

which combined with equation (6.2) yields $\alpha/(1-\alpha) \leq \beta$, or equivalently $\alpha \leq \beta/(1+\beta)$. Thus, the set $F(\alpha) \cap G(\beta)$ is at most countable when $\alpha > \beta/(1+\beta)$. Moreover, equation (6.3) implies that $\{m_k\}_{k\geq 1}$ grows at least exponentially, namely, there exists $C > 0$, independent of x, such that $k \leq C \log m_k$ for k large enough. Further, by equations (6.2) and (6.3), we also have

$$\liminf_{k\to\infty} \frac{m_k}{n_{k+1}} \geq \liminf_{k\to\infty} \frac{m_k - n_k}{n_{k+1}} \cdot \liminf_{k\to\infty} \frac{m_k}{m_k - n_k} = \frac{\alpha}{\beta(1-\alpha)},$$

which combined with the Stolz–Cesàro theorem implies that

$$\liminf_{k} \frac{\sum_{i=1}^{k-1}(m_i - n_i + 1)}{m_k} \geq \frac{\alpha\beta(1-\beta)}{\beta - \alpha},$$

and thus, for $\varepsilon > 0$ and k large enough,

$$\sum_{i=1}^{k}(m_i - n_i + 1) \geq \bigg(\frac{\alpha\beta(1-\beta)}{\beta-\alpha} - \frac{\varepsilon}{2}\bigg)m_k + (m_k - n_k + 1).$$

Since there exist infinitely many k such that

$$m_k - n_k + 1 \geq m_k\bigg(\beta - \frac{\varepsilon}{2}\bigg), \tag{6.4}$$

property (5) holds for such k.

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6.2. *Covering of $F(\alpha)\cap G(\beta)$.* We collect all sequences $(\{n_k\},\{m_k\})$ associated with $x\in F(\alpha)\cap G(\beta)$ as above to form a set

$$\Omega=\{(\{n_k\},\{m_k\})\colon \text{Properties (1) and (3) are fulfilled}\}.$$

For $(\{n_k\},\{m_k\})\in\Omega$ and $\{b_k\}\subset\mathbb{N}$, write

$$\begin{aligned}H(\{n_k\},\{m_k\})&=\{x\in[0,1)\colon \text{Property (2) is fulfilled}\},\\ \Lambda_{k,m_k}&=\{(n_1,m_1;\ldots;n_{k-1},m_{k-1};n_k)\colon n_1<m_1<\cdots<m_{k-1}<n_k,\\ &\qquad \text{equation (6.4) holds}\},\\ \mathcal{D}_{n_1,m_1;\ldots;n_k,m_k}(\{b_k\})&=\{(\sigma_1,\ldots,\sigma_{m_k})\in\mathbb{N}^{m_k}\colon \sigma_{n_j}=\cdots=\sigma_{m_j}=b_j\\ &\qquad \text{for all } 1\le j\le k\}.\end{aligned}$$

We obtain a covering of $F(\alpha)\cap G(\beta)$:

$$\begin{aligned}F(\alpha)\cap G(\beta)&\subseteq\bigcup_{(\{n_k\},\{m_k\})\in\Omega}H(\{n_k\},\{m_k\})\\ &\subseteq\bigcap_{K=1}^{\infty}\bigcup_{k=K}^{\infty}\bigcup_{m_k=e^{k/C}}^{\infty}\bigcup_{(n_1,m_1,\ldots,m_{k-1},n_k)\in\Lambda_{k,m_k}}\bigcup_{(b_1,\ldots,b_k)\in\mathbb{N}^k}\\ &\quad\times\bigcup_{(a_1,\ldots,a_{m_k})\in\mathcal{D}_{n_1,m_1;\ldots;n_k,m_k}(\{b_k\})}I_{m_k}(a_1,\ldots,a_{m_k}).\end{aligned}$$

Writing $t=s(\xi-\varepsilon,\tau(1))+(\varepsilon/2)$, we estimate the $(t+(\varepsilon/2))$-dimensional Hausdorff measure of $F(\alpha)\cap G(\beta)$. Setting $\psi(m_k)=m_k-\sum_{i=1}^{k}(m_i-n_i+1)$, we have $M=512\sum_{i=1}^{\infty}(\tau(i))^{-2t}$. For sufficiently large k, we first have the following estimate:

$$\begin{aligned}&\sum_{b_k=1}^{\infty}\cdots\sum_{b_1=1}^{\infty}\sum_{(a_1,\ldots,a_{m_k})\in\mathcal{D}_{n_k,m_k}(\{b_k\})}|I_{m_k}(a_1,\ldots,a_{m_k})|^{t+(\varepsilon/2)}\\ &\quad\le\sum_{b_k=1}^{\infty}\cdots\sum_{b_1=1}^{\infty}\sum_{a_1,\ldots,a_{\psi(m_k)}\in\mathbb{N}}4^{k(t+(\varepsilon/2))}\bigg(\frac{1}{q_{\psi(m_k)}(a_1,\ldots,a_{\psi(m_k)})}\bigg)^{2t+\varepsilon}\\ &\qquad\times\prod_{j=1}^{k}\bigg(\frac{1}{q_{m_j-n_j+1}(b_j,\ldots,b_j)}\bigg)^{2t+\varepsilon}\\ &\quad\le 4^{k(t+(\varepsilon/2))}\sum_{a_1,\ldots,a_{\psi(m_k)}\in\mathbb{N}}\bigg(\frac{1}{q_{\psi(m_k)}(a_1,\ldots,a_{\psi(m_k)})}\bigg)^{2t+\varepsilon}\\ &\qquad\times\prod_{j=1}^{k}\bigg(\sum_{i=1}^{\infty}\Big(1/(q_{m_j-n_j+1})(i,\ldots,i)\Big)^{2t+\varepsilon}\bigg)\\ &\quad\le(4^{t+(\varepsilon/2)}M)^k\sum_{a_1,\ldots,a_{\psi(m_k)}\in\mathbb{N}}\bigg(\frac{1}{q_{\psi(m_k)}(a_1,\ldots,a_{\psi(m_k)})}\bigg)^{2t+\varepsilon}\\ &\qquad\times\prod_{j=1}^{k}\bigg(\frac{1}{q_{m_j-n_j+1}(1,\ldots,1)}\bigg)^{2t+\varepsilon}\end{aligned}$$

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$$\le (16^{t+(\varepsilon/2)}M)^k \sum_{a_1,\ldots,a_{\psi(m_k)}\in\mathbb{N}} \bigg(\frac{1}{q_{m_k}(a_1,\ldots,a_{\psi(m_k)},1,\ldots,1)}\bigg)^{2t+\varepsilon}$$

$$\le (16^{t+(\varepsilon/2)}M)^k$$

$$\times \sum_{a_1,\ldots,a_{m_k-\lfloor m_k(\xi-\varepsilon)\rfloor}\in\mathbb{N}} \bigg(\frac{1}{q_{m_k}(a_1,\ldots,a_{m_k-\lfloor m_k(\xi-\delta)\rfloor},1,\ldots,1)}\bigg)^{2s_{m_k}(\xi-\varepsilon,\tau(1))+\varepsilon}$$

$$\le (16^{t+(\varepsilon/2)}M)^k \bigg(\frac{1}{2}\bigg)^{((m_k-1)/2)\varepsilon}, \tag{6.5}$$

where the third inequality holds since

$$\sum_{i=1}^{\infty}\bigg(\frac{1}{q_n(i,\ldots,i)}\bigg)^{2t+\varepsilon} = \bigg(\frac{1}{q_n(1,\ldots,1)}\bigg)^{2t+\varepsilon}\sum_{i=1}^{\infty}\bigg(\frac{q_n(1,\ldots,1)}{q_n(i,\ldots,i)}\bigg)^{2t+\varepsilon}$$
$$\le \bigg(\frac{1}{q_n(1,\ldots,1)}\bigg)^{2t+\varepsilon}\sum_{i=1}^{\infty}\bigg(\frac{4\tau(1)}{\tau(i)}\bigg)^{2t+\varepsilon};$$

the penultimate one follows from equations (6.4) and (3.7), and the last one is by Remark 2.11 and Lemma 2.1(1).

Hence,

$$\mathcal{H}^{t+(\varepsilon/2)}(F(\alpha)\cap G(\beta)) \le \liminf_{K\to\infty}\sum_{k=K}^{\infty}\sum_{m_k=e^{k/C}}^{\infty}\sum_{(n_1,m_1,\ldots,m_{k-1},n_k)\in\Lambda_{k,m_k}}\sum_{b_k=1}^{\infty}\cdots\sum_{b_1=1}^{\infty}$$

$$\times \sum_{(a_1,\ldots,a_{m_k})\in\mathcal{D}_{n_k,m_k}(\{b_k\})} |I_{m_k}(a_1,\ldots,a_{m_k})|^{t+(\varepsilon/2)}$$

$$\overset{(6.5)}{\le} \liminf_{K\to\infty}\sum_{k=K}^{\infty}\sum_{m_k=e^{k/C}}^{\infty}\sum_{n_k=1}^{m_k}\sum_{m_{k-1}=1}^{n_k}\cdots\sum_{m_1=1}^{n_2}\sum_{n_1=1}^{m_1}(16^{t+(\varepsilon/2)}M)^k\bigg(\frac{1}{2}\bigg)^{((m_k-1)/2)\varepsilon}$$

$$\le \liminf_{K\to\infty}\sum_{k=K}^{\infty}\sum_{m_k=e^{k/C}}^{\infty}(16^{t+(\varepsilon/2)}Mm_k)^{2C\log m_k}\bigg(\frac{1}{2}\bigg)^{((m_k-1)/2)\varepsilon}$$

$$\le \liminf_{K\to\infty}\sum_{k=K}^{\infty}\sum_{m_k=e^{k/C}}^{\infty}\bigg(\frac{1}{2}\bigg)^{((m_k-1)/4)\varepsilon} \le \frac{1}{1-(\frac{1}{2})^{\varepsilon/4}}\sum_{k=1}^{\infty}\bigg(\frac{1}{2^{\varepsilon}}\bigg)^{(e^{k/C}-1)/4} < +\infty,$$

where the penultimate one holds since $(16^{t+(\varepsilon/2)}Mk)^{2C\log k} < 2^{((k-1)/4)\varepsilon}$ for k large enough.

6.3. *Lower bound of* $\dim_H(F(\alpha)\cap G(\beta))$. Note that $F(\alpha)\cap F(\beta)$ is at most countable for $\alpha > \beta/(1+\beta)$; we assume that $\alpha \le \beta/(1+\beta)$. Let $\{n_k\}$ and $\{m_k\}$ be two strictly increasing sequences satisfying the following conditions:

(1) $(m_k - n_k) \le (m_{k+1} - n_{k+1})$ and $n_k < m_k < n_{k+1}$ for $k \ge 1$;
(2) $\lim_{k\to\infty}((m_k - n_k)/n_{k+1}) = \alpha/(1-\alpha)$;
(3) $\lim_{k\to\infty}((m_k - n_k)/m_k) = \beta$.

With the help of these sequences, we construct a Cantor subset of $F(\alpha) \cap G(\beta)$ to provide a lower bound estimate of its dimension. The set $E(B)$ is defined in much the same way as in §4.2, the only difference being that the digit i is replaced by the digit 1; the mapping f is defined in exactly the same way. It remains to verify that the set $f(E(B))$ is a subset of $F(\alpha) \cap G(\beta)$.

LEMMA 6.1. *For any $B \geq 2$, $f(E(B)) \subset F(\alpha) \cap G(\beta)$.*

Proof. Recall the definitions of

$$t_k = \max\{t \in \mathbb{N} \colon m_k + t(m_k - n_k) < n_{k+1}\},$$

$$m'_k = m_k + \sum_{l=1}^{k-1}(t_l + 2), \quad n'_k = n_k + \sum_{l=1}^{k-1}(t_l + 2).$$

We know that

$$\lim_{k\to\infty} \frac{\sum_{l=1}^{k}(t_l + 2)}{n_{k+1}} = 0, \quad \lim_{k\to\infty} \frac{n_k}{n'_k} = \lim_{k\to\infty} \frac{m_k}{m'_k} = 1.$$

For $x \in f(E(B))$, and $m'_k \leq n < m'_{k+1}$ with $k \in \mathbb{N}$, we have that

$$R_n(x) = \begin{cases} m_k - n_k & \text{if } m'_k \leq n \leq n'_{k+1} + m_k - n_k, \\ n - n'_{k+1} & \text{if } n'_{k+1} + m_k - n_k < n < m'_{k+1}. \end{cases}$$

Observing that for $n'_{k+1} + m_k - n_k < n < m'_{k+1}$,

$$\frac{m_k - n_k}{n'_{k+1} + m_k - n_k} \leq \frac{R_n(x)}{n} = \frac{n - n'_{k+1}}{n} \leq \frac{m_{k+1} - n_{k+1}}{m'_{k+1}},$$

we deduce that

$$\liminf_{n\to\infty} \frac{R_n(x)}{n} = \lim_{k\to\infty} \frac{R_{n'_{k+1}+m_k-n_k}(x)}{n'_{k+1} + m_k - n_k} = \lim_{k\to\infty} \frac{m_k - n_k}{n_{k+1} + m_k - n_k} = \alpha$$

and

$$\limsup_{n\to\infty} \frac{R_n(x)}{n} = \lim_{k\to\infty} \frac{m_{k+1} - n_{k+1}}{m'_{k+1}} = \lim_{k\to\infty} \frac{m_{k+1} - n_{k+1}}{m_{k+1}} = \beta.$$

Hence, $x \in F(\alpha) \cap G(\beta)$. □

7. *Proof of Theorem 1.6*

The proof of Theorem 1.6 will be divided into two parts according as $\alpha = 0$ or $0 < \alpha \leq 1$. We first note that $F(\alpha) \cap G(\beta)$ is at most countable for $\alpha > \frac{1}{2} \geq \beta/(1 + \beta)$ by equations (6.2) and (6.3). Moreover, since $G(0) \subseteq F(0)$ and $G(0)$ is of full Lebesgue measure, we have $\dim_H F(0) = 1$. Hence, we only need to deal with the case $0 < \alpha \leq \frac{1}{2}$.

Lower bound of $F(\alpha)$. Since $F(\alpha) \cap G(\beta) \subseteq F(\alpha)$ for any $\beta \geq \alpha/(1 - \alpha)$, we have

$$\dim_H F(\alpha) \geq \dim_H F(\alpha) \cap G(\beta) = s\left(\frac{\beta^2(1 - \alpha)}{\beta - \alpha}, \tau(1)\right).$$

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The function $(\beta^2(1-\alpha))/(\beta-\alpha)$ is continuous for $\beta \in [\alpha/(1-\alpha), 1]$, and attains its minimum at the point $\beta = 2\alpha \geq \alpha/(1-\alpha)$, so by Lemma 2.13(2), we obtain

$$\dim_H F(\alpha) \geq \dim_H(F(\alpha) \cap G(2\alpha)) = s(4\alpha(1-\alpha), \tau(1)).$$

Upper bound of $F(\alpha)$. For $x \in F(\alpha)$, there exists $\beta_0 \in (0, 1]$ such that

$$\liminf_{n\to\infty} \frac{R_n(x)}{n} = \alpha, \quad \limsup_{n\to\infty} \frac{R_n(x)}{n} = \beta_0.$$

Then, $0 < \alpha \leq \beta_0/(1+\beta_0) < \beta_0 < 1$ or $\beta_0 = 1$. If $0 < \alpha \leq \beta_0/(1+\beta_0) < \beta_0 < 1$, then for $0 < \varepsilon < (\alpha(1-2\alpha))/(2(1-\alpha))$, there exists $\beta \in \mathbb{Q}^+$ such that $0 < \alpha \leq \beta_0/(1+\beta_0) \leq \beta \leq \beta_0 \leq \beta + \varepsilon < 1$.

Let

$$E_{\alpha,\beta,\epsilon} = \left\{ x \in [0,1) \colon \liminf_{n\to\infty} \frac{R_n(x)}{n} = \alpha, \beta \leq \limsup_{n\to\infty} \frac{R_n(x)}{n} \leq \beta + \varepsilon < 1 \right\},$$

we have

$$F(\alpha) \subseteq \bigg(\bigcup_{\beta\in\mathbb{Q}^+} E_{\alpha,\beta,\varepsilon} \bigg) \cup (F(\alpha) \cap G(1)).$$

So $\dim_H F(\alpha) \leq \max\{\frac{1}{2}, \sup\{\dim_H E_{\alpha,\beta,\varepsilon} \colon \beta \in \mathbb{Q}^+\}\}$.

It remains to estimate the upper bound of $\dim_H E_{\alpha,\beta,\varepsilon}$. Following the same line as the proof for the upper bound of $\dim_H(F(\alpha) \cap G(\beta))$, we obtain that

$$\dim_H E_{\alpha,\beta,\varepsilon} \leq s\bigg(\frac{\beta^2(1-\alpha)}{\beta-\alpha+\varepsilon}, \tau(1) \bigg).$$

Thus, since the function $(\beta^2(1-\alpha))/(\beta-\alpha+\epsilon)$ with respect to β attains its minimum at $\beta = 2(\alpha-\varepsilon)$, we have that

$$\begin{aligned} \dim_H F(\alpha) &\leq \sup \left\{ s\bigg(\frac{\beta^2(1-\alpha)}{\beta-\alpha+\varepsilon} \bigg) \colon \beta \in \mathbb{Q}^+ \right\} \leq s(4(\alpha-\varepsilon)(1-\alpha), \tau(1)) \\ &\to s(4\alpha(1-\alpha), \tau(1)) \quad \text{as } \varepsilon \to 0. \end{aligned}$$

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