ON RELLICH'S THEOREM CONCERNING INFINITELY NARROW TUBES

Colin Clark[1)]

(received September 9, 1963)

Let G be a region in Euclidean n-space E_n and consider the eigenvalue problem $\Delta^2 u = \lambda u$ on G, with boundary conditions $u = 0$ on Γ, the boundary of G. (To be precise, we are considering the eigenvalue problem for the self-adjoint realization L associated with the Laplacian $-\Delta^2$ and zero boundary condition, acting in $L_2(G)$, cf Browder [2]). If G is bounded, the spectrum of this problem is discrete, but Rellich showed in 1952 [6] that the spectrum could also be discrete for certain unbounded regions which he introduced and called "infinitely narrow tubes".

<u>Definition.</u> G is an infinitely narrow tube (with X_1 as centre-axis) if G is not bounded, but lies in some half-space $x_1 \geq M$ and if

$$\lim_{\substack{x \in G \\ x_1 \to +\infty}} (x_2^2 + \dots + x_n^2) = 0 .$$

1) Research supported by the United States Air Force Office of Scientific Research, contract AF-AFOSR 61-89.

Canad. Math. Bull. vol. 7, no. 3, July 1964.

In this paper we give a simple proof of Rellich's theorem, based entirely on variational considerations (Rellich's proof utilized his own "selection principle").

The statement of the variational principle we will apply is as follows. Let ${}_0C_1(G)$ consist of all square integrable complex-valued functions which are piecewise continuously differentiable on G, continuous on $\bar{G}$ and zero on the boundary of G. Define

$$\lambda_1 = \inf_{u \in {}_0C_1(G)} \frac{\int_G |\nabla u|^2 dx}{\int_G |u|^2 dx} = \inf_{u \in {}_0C_1(G)} W(u) .$$

Then λ_1 is the smallest number in the spectrum of L. If the infimum is achieved for some u_1, then $Lu_1 = \lambda_1 u_1$. In this case define

$$\lambda_2 = \inf_{u \in {}_0C_1(G),\ u \perp u_1} W(u) .$$

Then $\lambda_2 \geq \lambda_1$, and if $\lambda_2 > \lambda_1$ the interval (λ_1, λ_2) is disjoint from the spectrum of L. Again if the infimum is achieved for some u_2, then $Lu_2 = \lambda_2 u_2$ and we may continue defining λ_3, etc.

This principle is commonly used (e.g. in [4]), but no general proof seems to have been published. A proof can be derived from construction of the Green's function associated with the operator $\Delta^2 + \lambda$ on G; this has been carried out by the author under additional restrictions (relating to the boundary of G). This work will be published elsewhere.

Let G be an infinitely narrow tube with X_1 as centre axis; let $G_1 = G \cap \{x \mid x_1 < R\}$ and $G_2 = G \cap \{x \mid x_1 > R\}$, where R is some real number (large enough to make $G_1 \neq \emptyset$). Let L_1 and L_2 denote the self-adjoint realizations of the operator $\Lambda^{-\nabla^2}$ with zero boundary conditions on G_1 and G_2 respectively.

To show that the spectrum of L is discrete, we show that it is discrete below M for every positive number M. This follows directly from the following:

PROPOSITION. If the spectrum of L_2 is empty below a given positive number M, then the spectrum of L is discrete below M.

Proof. G_1 being bounded, the spectrum of L_1 is discrete. Let $\omega_1, \ldots, \omega_k$ be all eigenfunctions of L_1 which correspond to eigenvalues $\leq M$; assume the ω_i to be orthonormal. By the variational principle applied to G_1 we know that

$$u \varepsilon \, {}_0C_1 (G_1) \text{ and } \int_{G_1} u(x)\, \omega_i(x) dx = 0 \quad (i = 1, 2, \ldots, k) .$$

implies

$$(1) \qquad \int_{G_1} |\nabla u|^2 dx \geq M \int_{G_1} |u|^2 dx .$$

Applying the variational principle to G_2 and using the hypothesis of the theorem, we have

$$u \varepsilon \, {}_0C_1 (G_2)$$

implies

$$(2) \qquad \int_{G_2} |\nabla u|^2 dx \geq M \int_{G_2} |u|^2 dx .$$

Now let

https://doi.org/10.4153/CMB-1964-043-6 Published online by Cambridge University Press

$$u_i(x) = \begin{cases} \omega_i(x) & x \in G_1 \\ 0 & x \in G_2 \end{cases},$$

so that $u_i \in {}_0C_1(G)$. We wish to show that

$$u \in {}_0C_1(G) \text{ and } \int_G u(x)u_i(x)dx = 0 \quad (i = 1, 2, \ldots, k)$$

implies

$$(3) \qquad \int_G |\nabla u|^2 dx \geq M \int_G |u|^2 dx \ .$$

Thus consider such a function u, and let $\varepsilon > 0$. It is easy to construct a function $\tilde{u} \in {}_0C_1(G)$ such that $\tilde{u}|G_i \in {}_0C_1(G_i)$ and

$$||u - \tilde{u}||_1^2 = \int_G \{ |\nabla(u - \tilde{u})|^2 + |u - \tilde{u}|^2 \} dx < \varepsilon^2$$

(such a construction is explicitly carried out in [1], p. 38). Finally, let

$$u^* = \tilde{u} - \Sigma_{i=1}^k (\tilde{u}, u_i)u_i$$

(the inner product in $L_2(G)$). Since obviously $u^*|G_1 \in {}_0C_1(G_1)$ and is $\perp \omega_i$, we have by (1)

$$\int_{G_1} |\nabla u^*|^2 dx \geq M \int_{G_1} |u^*|^2 dx \ .$$

Hence, using (2) similarly,

$$\int_G |\nabla u^*|^2 dx = \left(\int_{G_1} + \int_{G_2} \right) |\nabla u^*|^2 dx$$

$$\geq M \int_{G_1} |u^*|^2 dx + M \int_{G_2} |u^*|^2 dx$$

$$= M \int_G |u^*|^2 dx \ .$$

https://doi.org/10.4153/CMB-1964-043-6 Published online by Cambridge University Press

Since

$$||u^* - u||_1 \leqq ||u^* - \tilde{u}||_1 + ||\tilde{u} - u||_1$$

$$\leqq \Sigma_i |(\tilde{u}, u_i)| \; ||u_i||_1 + \varepsilon$$

$$= \Sigma_i |(\tilde{u} - u, u_i)| \; ||u_i||_1 + \varepsilon$$

$$\leqq \varepsilon (\Sigma_i ||u_i||_1 + 1)$$

$$\leqq k\varepsilon \qquad (k = \text{constant}) ,$$

it follows that

$$\{ \int_G |\nabla u|^2 dx\}^{\frac{1}{2}} \geq \{ \int_G |\nabla u^*|^2 dx\}^{\frac{1}{2}} - \{ \int_G |\nabla u^* - \nabla u|^2 dx\}^{\frac{1}{2}}$$

$$\geq \sqrt{M} \{ \int_G |u^*|^2 dx\}^{\frac{1}{2}} - k\varepsilon$$

$$\geq \sqrt{M} [\{ \int_G |u|^2 dx\}^{\frac{1}{2}} - \{ \int_G |u - u^*|^2 dx\}^{\frac{1}{2}}] - k\varepsilon$$

$$\geq \sqrt{M} [\{ \int_G |u|^2 dx\}^{\frac{1}{2}} - k\varepsilon] - k\varepsilon .$$

Since ε is arbitrary, (3) follows, and the proof is completed by an application of the minimax form ([3], p. 406) of the variational principle to G itself.

COROLLARY. The spectrum of the operator L for an infinitely narrow tube is discrete.

Proof. Given $M > 0$, we can choose R so that G_2 is contained in the strip

$$-\infty < x_1 < \infty; \; 0 < x_i < \Pi \sqrt{\frac{n-1}{M}} \quad , \quad i = 2, 3, \ldots, n .$$

https://doi.org/10.4153/CMB-1964-043-6 Published online by Cambridge University Press

The spectrum for this strip is the interval $(M, +\infty)$, and therefore $\inf \sigma(L_2) \geq M$ (as follows by applying the variational principle to G and to the strip).

To conclude, we remark that A. M. Molcanov in a mysterious paper [5] has given necessary and sufficient conditions on G for discreteness of the spectrum of L over an arbitrary region G.

REFERENCES

1. F.H. Brownell and C. W. Clark, Asymptotic distribution of the eigenvalues for the lower part of the Schrődinger operator spectrum, Jour. Math. Mech. 10 (1961), pp. 31-70.

2. F. E. Browder, On the spectral theory of elliptic differential operators I, Math. An. 142 (1961), pp. 22-130.

3. Courant and Hilbert, Methods of Mathematical Physics, Vol. I; Interscience, N. Y. (1953).

4. D. S. Jones, The eigenvalues of $\Delta^2 u + \lambda u = 0$ when the boundary conditions are given on semi-infinite domains, Proc. Cambridge Philos. Soc. 49 (1953), pp. 668-684.

5. A. M. Molcanov, On conditions for discreteness of the spectrum of self-adjoint differential equations of the second order (Russian), Mosk. Mat. Obschestva (Trudy) 2 (1953), pp. 169-199.

6. F. Rellich, Das Eigenwertproblem von $\Delta u + \lambda u = 0$ in Halbrőhren, in Essays Presented to R. Courant, N. Y. (1948), pp. 329-344.

University of British Columbia

https://doi.org/10.4153/CMB-1964-043-6 Published online by Cambridge University Press