14

Krull–Gabriel Dimension

Contents

The Krull–Gabriel dimension is an invariant which is defined for any essentially small abelian category. The definition is based on a filtration which can be used to describe injective objects of a locally finitely presented Grothendieck category. In fact, we use this technique to classify all pure-injective objects for some interesting examples, including sheaves on the projective line and representations of the Kronecker quiver.

14.1 The Krull–Gabriel Filtration

In this section we introduce the Krull–Gabriel filtration of an essentially small abelian category. This filtration is then used to classify all injective objects in a locally finitely presented Grothendieck category, provided its category of finitely presented objects is abelian and the Krull–Gabriel dimension is defined.

https://doi.org/10.1017/9781108979108.022 Published online by Cambridge University Press

This is done in two steps: first the indecomposable objects, and then the general case. There is an analogue of this filtration for modular lattices, which we apply to the lattice of subobjects of an object of an abelian category. From the classification of injective objects we deduce the classification of pure-injective objects of a locally finitely presented category. In particular, the last layer of the Krull–Gabriel filtration produces endofinite objects.

Filtrations of Abelian Categories

We begin with a brief discussion of ordinals. For historical reasons let -1 denote the ordinal possessing no element, and $n-1$ denotes the ordinal possessing a finite number of n elements. The sum of ordinals α and β (first α and then β) is denoted by $\alpha \perp \beta$. For example, we have $\alpha \perp \beta = \alpha + \beta + 1$ when α and β are finite.

Let $\mathcal{C}$ be an essentially small abelian category. We set $\mathcal{A} = \operatorname{Lex}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$ and recall that $\mathcal{A}$ is a locally finitely presented Grothendieck category with $\mathcal{C} \xrightarrow{\sim} \operatorname{fp} \mathcal{A}$.

The *Krull–Gabriel filtration* of $\mathcal{C}$ is given by a sequence of Serre subcategories which is indexed by the ordinals $\alpha \geq -1$.

- $\mathcal{C}_{-1}$ is the full subcategory containing only the zero objects.
- $\mathcal{C}_\alpha$ is the full subcategory of objects that become of finite length in $\mathcal{C}/\mathcal{C}_\beta$, if $\alpha = \beta + 1$.
- $\mathcal{C}_\alpha = \bigcup_{\gamma<\alpha} \mathcal{C}_\gamma$, if α is a limit ordinal.

We set $\mathcal{C}_\infty = \bigcup_\alpha \mathcal{C}_\alpha$. If $\mathcal{C}_\infty = \mathcal{C}$, then the smallest ordinal α such that $\mathcal{C} = \mathcal{C}_\alpha$ is called the *Krull–Gabriel dimension* and is denoted $\operatorname{KG.dim} \mathcal{C}$. We write $\operatorname{KG.dim} \mathcal{C} < \infty$ in this case, and we say that the Krull–Gabriel dimension is defined.

We collect some elementary properties of this dimension.

Lemma 14.1.1. *Let $\mathcal{C}' \subseteq \mathcal{C}$ be a Serre subcategory and set $\mathcal{C}'' = \mathcal{C}/\mathcal{C}'$. Then*

$$\sup(\operatorname{KG.dim} \mathcal{C}', \operatorname{KG.dim} \mathcal{C}'') \leq \operatorname{KG.dim} \mathcal{C} \leq \operatorname{KG.dim} \mathcal{C}' \perp \operatorname{KG.dim} \mathcal{C}''.$$

Proof First observe that $\mathcal{C}'_\alpha = \mathcal{C}_\alpha \cap \mathcal{C}'$ for every ordinal α; see Lemma 14.1.10 for a detailed proof. Also, the canonical functor $\mathcal{C} \twoheadrightarrow \mathcal{C}''$ maps $\mathcal{C}_\alpha$ into $\mathcal{C}''_\alpha$. Thus $\mathcal{C}_\alpha = \mathcal{C}$ implies $\mathcal{C}'_\alpha = \mathcal{C}'$ and $\mathcal{C}''_\alpha = \mathcal{C}''$. This yields the first inequality. For the second one suppose that $\mathcal{C}' \subseteq \mathcal{C}_\alpha$ for some ordinal α. Then $\mathcal{C} \twoheadrightarrow \mathcal{C}/\mathcal{C}_\alpha$ induces a functor $\mathcal{C}/\mathcal{C}' \twoheadrightarrow \mathcal{C}/\mathcal{C}_\alpha$ which maps $(\mathcal{C}/\mathcal{C}')_\beta$ into $(\mathcal{C}/\mathcal{C}_\alpha)_\beta$ for every ordinal β. The functor $\mathcal{C} \twoheadrightarrow \mathcal{C}/\mathcal{C}_\alpha$ identifies $\mathcal{C}_{\alpha\perp\beta}$ with $(\mathcal{C}/\mathcal{C}_\alpha)_\beta$. Thus $(\mathcal{C}/\mathcal{C}')_\beta = \mathcal{C}/\mathcal{C}'$ implies $\mathcal{C}_{\alpha\perp\beta} = \mathcal{C}$. □

Lemma 14.1.2. *Let $\mathcal{C}$ be finitely generated as an abelian category and suppose* $\operatorname{KG.dim} \mathcal{C} < \infty$. *Then* $\operatorname{KG.dim} \mathcal{C} = \beta + 1$ *for some ordinal* β.

Proof Suppose that X generates $\mathcal{C}$ as an abelian category, i.e. there are no proper Serre subcategories containing X. Let α be a limit ordinal. If $\mathcal{C} = \bigcup_{\gamma<\alpha} \mathcal{C}_\gamma$, then $X \in \mathcal{C}_\gamma$ for some $\gamma < \alpha$ and therefore $\operatorname{KG.dim} \mathcal{A} < \alpha$. □

Next we consider indecomposable injective objects in $\mathcal{A}$. Recall that $\operatorname{Sp} \mathcal{A}$ denotes a representative set of the isomorphism classes of indecomposable injective objects. For each ordinal α, let $\operatorname{Sp}_\alpha \mathcal{A}$ denote the set of functors $F \in \operatorname{Sp} \mathcal{A}$ such that $F(\mathcal{C}_\alpha) = 0$ and $F(X) \neq 0$ for some object X which is simple in $\mathcal{C}/\mathcal{C}_\alpha$.

Lemma 14.1.3. *The relation* $F(X) \neq 0$ *yields a bijection between the isomorphism classes of simple objects* $X \in \mathcal{C}/\mathcal{C}_\alpha$ *and the elements* $F \in \operatorname{Sp}_\alpha \mathcal{A}$.

Proof An object $F \in \operatorname{Sp} \mathcal{A}$ is an exact functor, and $F(\mathcal{C}_\alpha) = 0$ implies that F can be viewed as an object in $\operatorname{Lex}((\mathcal{C}/\mathcal{C}_\alpha)^{\mathrm{op}}, \mathrm{Ab})$. If $X \in \mathcal{C}/\mathcal{C}_\alpha$ is simple, then the corresponding representable functor is simple in $\operatorname{Lex}((\mathcal{C}/\mathcal{C}_\alpha)^{\mathrm{op}}, \mathrm{Ab})$, and $F(X) \neq 0$ implies that F is an injective envelope; see Proposition 11.1.31 and the subsequent remark. It remains to observe that non-isomorphic simples have non-isomorphic injective envelopes. □

Proposition 14.1.4. *Suppose that* $\operatorname{KG.dim} \mathcal{C} = \kappa$. *Then* $\operatorname{Sp} \mathcal{A}$ *equals the disjoint union* $\bigsqcup_{\alpha<\kappa} \operatorname{Sp}_\alpha \mathcal{A}$. *Moreover, for each ordinal* $\alpha < \kappa$ *there is a bijection between* $\operatorname{Sp}_\alpha \mathcal{A}$ *and the set of isomorphism classes of simple objects in* $\mathcal{C}/\mathcal{C}_\alpha$.

Proof Let $F \in \operatorname{Sp} \mathcal{A}$ and choose β minimal such that $F(\mathcal{C}_\beta) \neq 0$. Then β is not a limit ordinal, so of the form $\beta = \alpha + 1$, because otherwise any $0 \neq X \in \mathcal{C}_\beta$ belongs to $\mathcal{C}_\gamma$ for some $\gamma < \beta$. Thus there is some object X which is simple in $\mathcal{C}/\mathcal{C}_\alpha$ such that $F(X) \neq 0$. Therefore $F \in \operatorname{Sp}_\alpha \mathcal{A}$. The second assertion follows from the preceding lemma. □

If the Krull–Gabriel dimension of $\mathcal{C} = \operatorname{fp} \mathcal{A}$ is defined, then one obtains a classification of all injective objects in $\mathcal{A}$.

Proposition 14.1.5. *Suppose that* $\operatorname{KG.dim} \operatorname{fp} \mathcal{A} = \kappa$. *Then every injective object* X *in* $\mathcal{A}$ *is the injective envelope of a coproduct of indecomposable injectives* $\coprod_{i\in I} X_i$. *The isomorphism classes of the* X_i *and their multiplicities are uniquely determined by* X.

Proof First observe that every non-zero injective object X admits an indecomposable summand. To this end let α be an ordinal that is minimal such that X admits a non-zero morphism $\phi\colon C \to X$ with $C \in (\operatorname{fp} \mathcal{A})_\alpha$. Clearly,

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$\alpha = \beta + 1$. Then C has finite length in $\operatorname{fp}\mathcal{A}/(\operatorname{fp}\mathcal{A})_\beta$. Choose a composition series $0 = C_0 \subseteq C_1 \subseteq \cdots \subseteq C_n = C$ and let r be minimal such that $\phi(C_r) \neq 0$ but $\phi(C_{r-1}) = 0$. Set $C' = C_r/C_{r-1}$ and $D = \operatorname{Im}\phi$. Because C' is simple in $\operatorname{fp}\mathcal{A}/(\operatorname{fp}\mathcal{A})_\beta$, it follows that D is *uniform*, so $D \neq 0$ and any pair of non-zero subobjects has a non-zero intersection. Thus the injective envelope of D yields an indecomposable direct summand of X.

Given a non-zero injective object X, we use Zorn's lemma and obtain a maximal family of indecomposable injective subobjects $(X_i)_{i\in I}$ such that the sum $X' = \sum_{i\in I} X_i$ is direct. This yields a decomposition $X = E(X') \oplus X''$, and $X'' = 0$ by the first part of the proof.

We sketch the argument for the uniqueness statement. One passes to the *spectral category* of $\mathcal{A}$, which is by definition the localisation $\mathcal{A}[\mathrm{Ess}^{-1}]$ with respect to the class Ess of essential monomorphisms (cf. Proposition 2.5.9). This is a split exact Grothendieck category, so each object decomposes into simple objects, and the canonical functor $\mathcal{A} \to \mathcal{A}[\mathrm{Ess}^{-1}]$ identifies each object in $\mathcal{A}$ with its injective envelope. Now one applies the Krull–Remak–Schmidt–Azumaya theorem (Theorem 2.5.8). □

Let us discuss a variation of the Kull-Gabriel filtration which yields a possible refinement. Fix again an essentially small abelian category $\mathcal{C}$ and consider an ascending chain of Serre subcategories $\mathcal{C}^\alpha \subseteq \mathcal{C}$ which is indexed by the ordinals $\alpha \leq \kappa$:

$$0 = \mathcal{C}^{-1} \subseteq \mathcal{C}^0 \subseteq \mathcal{C}^1 \subseteq \cdots \subseteq \mathcal{C}^\kappa = \mathcal{C}.$$

Suppose that $\mathcal{C}^{\alpha+1}/\mathcal{C}^\alpha$ is a length category for all $\alpha < \kappa$, and $\mathcal{C}^\alpha = \bigcup_{\beta<\alpha} \mathcal{C}^\beta$ for any limit ordinal $\alpha \leq \kappa$. As before, define $\operatorname{Sp}^\alpha \mathcal{A}$ to be the set of objects $F \in \operatorname{Sp}\mathcal{A}$ such that $F(\mathcal{C}^\alpha) = 0$ and $F(X) \neq 0$ for some object X which is simple in $\mathcal{C}^{\alpha+1}/\mathcal{C}^\alpha$.

Proposition 14.1.6. *We have* $\operatorname{KG.dim}\mathcal{C} \leq \kappa$, *and* $\operatorname{Sp}\mathcal{A}$ *equals the disjoint union* $\bigsqcup_{\alpha<\kappa} \operatorname{Sp}^\alpha \mathcal{A}$. *Moreover, for each* $\alpha < \kappa$ *there is a bijection between* $\operatorname{Sp}^\alpha \mathcal{A}$ *and the set of isomorphism classes of simple objects in* $\mathcal{C}^{\alpha+1}/\mathcal{C}^\alpha$.

Proof It follows by induction from Lemma 14.1.1 that $\operatorname{KG.dim}\mathcal{C} \leq \kappa$. The description of $\operatorname{Sp}\mathcal{A}$ is analogous to Proposition 14.1.4, and the proof is essentially the same. □

The Lattice of Subobjects

Let $\mathcal{C}$ be an essentially small abelian category. For an object $X \in \mathcal{C}$ we denote by $\mathbf{L}(X)$ its lattice of subobjects and note that this lattice is modular. The Krull–

Gabriel filtration of $\mathcal{C}$ is reflected by a cofiltration of $\mathbf{L}(X)$ for each object $X \in \mathcal{C}$. The description of this cofiltration requires the following definition.

Let L be a modular lattice. Recall that a lattice has *finite length* if there is a finite chain $0 = x_0 \le x_1 \le \cdots \le x_n = 1$ which cannot be refined. An equivalent condition is that L satisfies both chain conditions. Given elements $x, y \in L$ we write $x \sim y$ if the interval $[x \wedge y, x \vee y]$ has finite length. Then $\sim$ defines a *congruence relation* on L. This means the set of equivalence classes $L/\sim$ carries again the structure of a modular lattice and the canonical map $L \to L/\sim$ is a lattice homomorphism.

Let us define a cofiltration of L which is indexed by the ordinals $\alpha \ge -1$:

- $L_{-1} = L$,
- $L_\alpha = L_\beta/\sim$ when $\alpha = \beta + 1$,
- $L_\alpha = \operatorname{colim}_{\gamma<\alpha} L_\gamma$ when α is a limit ordinal.

We denote by L_∞ the colimit of this cofiltration. If $L_\infty = 0$ then the smallest ordinal α such that $L_\alpha = 0$ is called the *minimal dimension* (or simply *m-dimension*) of L and is denoted $\operatorname{m.dim} L$. We write $\operatorname{m.dim} L < \infty$ in this case, and $\operatorname{m.dim} L = \infty$ when $L_\infty \neq 0$.

Remark 14.1.7. If $\operatorname{m.dim} L < \infty$, then $\operatorname{m.dim} L = \beta + 1$ for some ordinal β. To see this consider the set I_α of elements $x \in L$ that $L \to L_\alpha$ maps to 0. Note that $I_\alpha = \bigcup_{\gamma<\alpha} I_\gamma$ when α is a limit ordinal. Moreover, $L_\alpha = 0$ if and only if $1 \in I_\alpha$.

We collect some basic properties of the m-dimension and begin with an elementary observation.

Lemma 14.1.8. *Let $F : \mathcal{C} \to \mathcal{D}$ be an exact functor between abelian categories and $X \in \mathcal{C}$ an object. Then F induces a lattice homomorphism $\mathbf{L}(X) \to \mathbf{L}(FX)$. This map is surjective when F is a quotient functor.*

Proof Given subobjects $U, V \subseteq X$, then $F(U \cap V) = F(U) \cap F(V)$ and $F(U + V) = F(U) + F(V)$ since F is exact. Thus $\mathbf{L}(X) \to \mathbf{L}(FX)$ is a homomorphism. When F is a quotient functor then we can apply the lemma below. □

Lemma 14.1.9. *Let $\mathcal{C}$ be an abelian category and $Q : \mathcal{C} \to \mathcal{C}/\mathcal{B}$ the quotient functor given by a Serre subcategory $\mathcal{B} \subseteq \mathcal{C}$. Then for any exact sequence $0 \to X \to Y \to Z \to 0$ in $\mathcal{C}/\mathcal{B}$ there is an exact sequence $0 \to X' \xrightarrow{\phi} Y \xrightarrow{\psi} Z' \to 0$*

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in $\mathcal{C}$ *inducing the following commutative diagram.*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & 0 \\
 & & \downarrow\wr & & \| & & \downarrow\wr & & \\
0 & \longrightarrow & X' & \xrightarrow{Q(\phi)} & Y & \xrightarrow{Q(\psi)} & Z' & \longrightarrow & 0
\end{array}$$

Proof We consider the morphism $X \to Y$ in $\mathcal{C}/\mathcal{B}$. Then there are subobjects $X_1 \subseteq X$ and $Y_1 \subseteq Y$ in $\mathcal{C}$ such that X/X_1 and Y_1 belong to $\mathcal{B}$, plus a morphism $\phi_1 \colon X_1 \to Y/Y_1$ in $\mathcal{C}$ inducing the following commutative square (Lemma 2.2.4).

$$\begin{array}{ccc}
X & \longrightarrow & Y \\
\uparrow & & \downarrow \\
X_1 & \xrightarrow{Q(\phi_1)} & Y/Y_1
\end{array}$$

We form in $\mathcal{C}$ the following pullback.

$$\begin{array}{ccc}
X_2 & \xrightarrow{\phi_2} & Y \\
\downarrow & & \downarrow \\
X_1 & \xrightarrow{\phi_1} & Y/Y_1
\end{array}$$

Now choose for $\phi \colon X' \to Y$ the inclusion $\operatorname{Im}\phi_2 \to Y$ and for $\psi \colon Y \to Z'$ the cokernel of ϕ. □

The following lemma describes the lattice of subobjects corresponding to the Krull–Gabriel filtration of an abelian category.

Lemma 14.1.10. *Let* $\mathcal{C}$ *be an essentially small abelian category and* $q_\alpha \colon \mathcal{C} \to \mathcal{C}/\mathcal{C}_\alpha$ *the quotient functor for an ordinal* α. *Given an object* $X \in \mathcal{C}$, *the canonical map* $\mathbf{L}(X) \to \mathbf{L}(q_\alpha X)$ *induces an isomorphism* $\mathbf{L}(X)_\alpha \xrightarrow{\sim} \mathbf{L}(q_\alpha X)$. *Therefore* $X \in \mathcal{C}_\alpha$ *if and only if* $\mathbf{L}(X)_\alpha = 0$, *and* $X \in \mathcal{C}_\infty$ *if and only if* $\mathbf{L}(X)_\infty = 0$.

Proof It follows from Lemma 14.1.8 that the map $\mathbf{L}(X) \to \mathbf{L}(q_\alpha X)$ is surjective. We compare this map with the canonical map $\mathbf{L}(X) \to \mathbf{L}(X)_\alpha$, and it suffices to show that both maps identify the same elements in $\mathbf{L}(X)$. This is done by induction. For the step $\beta \mapsto \beta + 1$ suppose $\mathbf{L}(X)_\beta \xrightarrow{\sim} \mathbf{L}(q_\beta X)$ and consider subobjects $U \subseteq V \subseteq X$. Then the map $\mathbf{L}(q_\beta X) \to \mathbf{L}(q_{\beta+1} X)$ identifies U and V if and only if V/U has finite length in $\mathcal{C}/\mathcal{C}_\beta$ if and only if $\mathbf{L}(X)_\beta \to \mathbf{L}(X)_{\beta+1}$ identifies U and V. This yields the isomorphism $\mathbf{L}(X)_\alpha \xrightarrow{\sim} \mathbf{L}(q_\alpha X)$. In particular, $X \in \mathcal{C}_\alpha$ if and only if $\mathbf{L}(X)_\alpha = 0$. □

A *dense chain* in a lattice is a sublattice $C \neq 0$ having the property that for

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every pair $x < y$ in C there is $z \in C$ with $x < z < y$. Note that having a dense chain is equivalent to having a sublattice isomorphic to

$$D = \{p \cdot 2^{-n} \in [0,1] \cap \mathbb{Q} \mid p, n \in \mathbb{N}\}.$$

Lemma 14.1.11. *There is a dense chain in L if and only if* $\operatorname{m.dim} L = \infty$.

Proof Let $\pi : L \to L_\infty$ be the canonical map. If $L_\infty \neq 0$, then L_∞ is a dense chain in L_∞. Thus we can construct inductively a dense chain isomorphic to D in L, since for any pair $x < y$ in L with $\pi(X) < \pi(y)$ there is some $z \in L$ with $\pi(X) < \pi(z) < \pi(y)$, and therefore $x < z' < y$ for $z' = (x \vee z) \wedge y$. Now suppose there is a dense chain in L, say between x and y. Using induction one shows that $\pi(x) \neq \pi(y)$, and therefore $\operatorname{m.dim} L = \infty$. □

We record a useful consequence.

Proposition 14.1.12. *Let $\mathcal{C}$ be an essentially small abelian category. Then $\mathcal{C}_\infty$ contains all noetherian and all artinian objects.*

Proof Let X be noetherian or artinian. Then $\mathbf{L}(X)$ does not contain a dense chain, and therefore $\operatorname{m.dim} \mathbf{L}(X) < \infty$ by Lemma 14.1.11. Thus $X \in \mathcal{C}_\infty$ by Lemma 14.1.10. □

Corollary 14.1.13. $\operatorname{KG.dim} \mathcal{C} < \infty$ *when all objects in $\mathcal{C}$ are noetherian.*

Example: Commutative Noetherian Rings

Let Λ be a commutative noetherian ring. We compute the Krull–Gabriel dimension of the abelian category $\operatorname{mod} \Lambda$. This justifies the terminology, because this dimension coincides with the Krull dimension of the ring Λ.

Let $X \in \mathcal{C}$ be an object of an abelian category $\mathcal{C}$ and suppose $X \in \mathcal{C}_\alpha$ for some ordinal α. Then the smallest ordinal α with this property is called the *Krull–Gabriel dimension* and is denoted $\operatorname{KG.dim} X$.

Proposition 14.1.14. *Let Λ be a commutative noetherian ring. For $\mathfrak{p} \in \operatorname{Spec} \Lambda$ and $n \in \mathbb{N}$ we have* $\operatorname{KG.dim} \Lambda/\mathfrak{p} \leq n$ *if and only if every proper chain*

$$\mathfrak{p} = \mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \mathfrak{p}_2 \subset \cdots \subset \mathfrak{p}_r$$

in $\operatorname{Spec} \Lambda$ *has length $r \leq n$. Therefore* $\operatorname{KG.dim}(\operatorname{mod} \Lambda) \leq n$ *if and only if every proper chain of prime ideals has length at most n.*

Proof We use the correspondence between Serre subcategories $\mathcal{C} \subseteq \operatorname{mod} \Lambda$ and specialisation closed subsets of $\operatorname{Spec} \Lambda$, which takes $\mathcal{C}$ to $\operatorname{Supp} \mathcal{C}$ (Proposition 2.4.8). Then the assertion follows easily by an induction on n from the lemma below. □

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Lemma 14.1.15. *Let* $\mathcal{C} \subseteq \operatorname{mod}\Lambda$ *be a Serre subcategory and* $\mathcal{V} = \operatorname{Supp}\mathcal{C}$. *Then the object* $\Lambda/\mathfrak{p}$ *is simple in* $(\operatorname{mod}\Lambda)/\mathcal{C}$ *if and only if* $\mathfrak{p} \notin \mathcal{V}$ *and* $\mathfrak{q} \in \mathcal{V}$ *for all* $\mathfrak{p} \subset \mathfrak{q}$.

Proof We may assume that $\Lambda/\mathfrak{p}$ is not in $\mathcal{C}$. Then it follows from Lemma 14.1.9 that $\Lambda/\mathfrak{p}$ is simple in $(\operatorname{mod}\Lambda)/\mathcal{C}$ if and only if for every proper epimorphism $\Lambda/\mathfrak{p} \to \Lambda/\mathfrak{a}$ in $\operatorname{mod}\Lambda$ we have $\Lambda/\mathfrak{a} \in \mathcal{C}$. Recall that $\operatorname{Supp}\Lambda/\mathfrak{a} = \mathcal{V}(\mathfrak{a})$ for every ideal $\mathfrak{a}$ (Lemma 2.4.1). Thus $\Lambda/\mathfrak{a} \in \mathcal{C}$ for every $\mathfrak{p} \subset \mathfrak{a}$ if and only if $\mathcal{V}(\mathfrak{a}) \subseteq \mathcal{V}$ for every $\mathfrak{p} \subset \mathfrak{a}$ if and only if $\mathfrak{q} \in \mathcal{V}$ for all $\mathfrak{p} \subset \mathfrak{q}$. □

Pure-Injective Objects

Let $\mathcal{A}$ be a locally finitely presented category. The Krull–Gabriel filtration of $\operatorname{Ab}(\mathcal{A})$ provides a method of classifying the pure-injective objects of $\mathcal{A}$. As before, this is done in two steps: first the indecomposable objects, and then the general case.

We use the embedding $\operatorname{ev}: \mathcal{A} \to \mathbf{P}(\mathcal{A})$ into the purity category, which identifies $\operatorname{Ab}(\mathcal{A}) = \operatorname{fp}\mathbf{P}(\mathcal{A})$ and $\operatorname{Ind}\mathcal{A} = \operatorname{Sp}\mathbf{P}(\mathcal{A})$; see Lemma 12.1.4 and Lemma 12.1.17. For each ordinal α set $\operatorname{Ind}_\alpha \mathcal{A} = \operatorname{Sp}_\alpha \mathbf{P}(\mathcal{A})$.

The following is a direct consequence of Proposition 14.1.4.

Corollary 14.1.16. *Suppose that* $\operatorname{KG.dim}\operatorname{Ab}(\mathcal{A}) = \kappa$. *Then* $\operatorname{Ind}\mathcal{A}$ *equals the disjoint union* $\bigsqcup_{\alpha<\kappa} \operatorname{Ind}_\alpha \mathcal{A}$. *Moreover, for each ordinal* $\alpha < \kappa$ *there is a bijection between* $\operatorname{Ind}_\alpha \mathcal{A}$ *and the set of isomorphism classes of simple objects in* $\operatorname{Ab}(\mathcal{A})/\operatorname{Ab}(\mathcal{A})_\alpha$. □

If the Krull–Gabriel dimension is defined, then one obtains a classification of all pure-injective objects. This follows from Proposition 14.1.5, since pure-injective objects in $\mathcal{A}$ identify with injective objects in $\mathbf{P}(\mathcal{A})$ by Lemma 12.1.8.

Corollary 14.1.17. *Suppose that* $\operatorname{KG.dim}\operatorname{Ab}(\mathcal{A}) = \kappa$. *Then every pure-injective object* X *in* $\mathcal{A}$ *is the pure-injective envelope of a coproduct of indecomposable pure-injectives* $\coprod_{i\in I} X_i$. *The isomorphism classes of the* X_i *and their multiplicities are uniquely determined by* X. □

The Krull–Gabriel filtration provides a useful method of classifying objects, even when $\operatorname{KG.dim}\operatorname{Ab}(\mathcal{A}) = \infty$. Given a class of objects $\mathcal{X} \subseteq \mathcal{A}$, we may consider the dimension $\operatorname{KG.dim}\operatorname{Ab}(\mathcal{X})$ of the corresponding abelian category $\operatorname{Ab}(\mathcal{X})$. For example, an object $X \in \mathcal{A}$ is endofinite if and only if $\operatorname{KG.dim}\operatorname{Ab}(X) \leq 0$, by Proposition 13.1.9. Also, $\operatorname{KG.dim}\operatorname{Ab}(X) < \infty$ when X is Σ-pure-injective, by Theorem 12.3.4 and Corollary 14.1.13.

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Endofinite Objects

Let $\mathcal{A}$ be a locally finitely presented category. Then the Krull–Gabriel filtration produces endofinite objects, provided the dimension is defined and is not a limit ordinal.

Proposition 14.1.18. *Suppose that* $\operatorname{KG.dim}\operatorname{Ab}(\mathcal{A}) = \kappa + 1$. *Then the objects in* $\operatorname{Ind}_\kappa \mathcal{A}$ *are endofinite.*

Proof Let $X \in \operatorname{Ind}_\kappa \mathcal{A}$. Then the abelian category $\operatorname{Ab}(X)$ is by definition a quotient of $\operatorname{Ab}(\mathcal{A})/\operatorname{Ab}(\mathcal{A})_\kappa$, which is a length category by assumption. Thus $\operatorname{Ab}(X)$ is a length category, and therefore X is endofinite by Proposition 13.1.9. □

Proposition 14.1.19. *Suppose that* $\operatorname{Ind}\mathcal{A}$ *is quasi-compact and let* $X \neq 0$ *be a* Σ*-pure-injective object in* $\mathcal{A}$*. Then the definable subcategory generated by* X *contains an indecomposable endofinite object.*

Proof The objects in $\operatorname{Ab}(X)$ are noetherian since X is Σ-pure-injective; see Theorem 12.3.4. Thus $\operatorname{KG.dim}\operatorname{Ab}(X) < \infty$ by Proposition 14.1.12. Because $\operatorname{Ind}\mathcal{A}$ is quasi-compact, the abelian category $\operatorname{Ab}(X)$ is finitely generated by Proposition 12.3.11. Thus $\operatorname{KG.dim}\operatorname{Ab}(X) = \kappa + 1$ for some ordinal κ by Lemma 14.1.2. Consider the quotient functor

$$Q\colon \operatorname{Ab}(\mathcal{A}) \twoheadrightarrow \operatorname{Ab}(X) \twoheadrightarrow \operatorname{Ab}(X)/\operatorname{Ab}(X)_\kappa$$

and choose any object $Y \in \operatorname{Ind}\mathcal{A}$ such that $\bar{Y}$ factors through Q. Then $\operatorname{Ab}(Y)$ is a quotient of $\operatorname{Ab}(X)/\operatorname{Ab}(X)_\kappa$ and therefore a length category. Thus Y is endofinite by Proposition 13.1.9. Also, Y belongs to the definable subcategory generated by X since $\operatorname{Ab}(Y)$ is a quotient of $\operatorname{Ab}(X)$. □

Remark 14.1.20. When X is Σ-pure-injective, then any object in the definable closure of X is actually a direct summand of a product of copies of X, by Theorem 12.3.4.

Interesting examples arise from Prüfer modules over Artin algebras, which are Σ-pure-injective by Proposition 12.3.9. With an additional assumption there are no finite length indecomposable direct summands.

Example 14.1.21. Let Λ be an Artin algebra and X a Prüfer module, given by an endomorphism $\phi\colon X \to X$ such that $X = \bigcup_{n\geq 0} \operatorname{Ker}\phi^n$ with $\operatorname{Ker}\phi$ of finite length. Suppose that each inclusion $\operatorname{Ker}\phi^n \to \operatorname{Ker}\phi^{n+1}$ is a radical morphism. Then there is a generic module in the definable subcategory generated by X.

Proof We can apply Proposition 14.1.19 since $\operatorname{Ind}\Lambda$ is quasi-compact. The

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assumption on the inclusions $\operatorname{Ker}\phi^n \to \operatorname{Ker}\phi^{n+1}$ implies that X has no indecomposable direct summand of finite length. It follows from Theorem 12.4.15 that there is no indecomposable module of finite length in the definable subcategory generated by X. □

14.2 Examples of Krull–Gabriel Filtrations

In this section we compute the Krull–Gabriel filtration for several examples. In each case we obtain as a consequence an explicit classification of all pure-injective objects.

Uniserial Categories

Let $\mathcal{A}$ be a locally finitely presented abelian category and set $\mathcal{C} = \operatorname{fp}\mathcal{A}$. Let us compute the Krull–Gabriel filtration of $\operatorname{Ab}(\mathcal{A}) = \operatorname{Fp}(\mathcal{C}, \operatorname{Ab})^{\mathrm{op}}$ when $\mathcal{C}$ is uniserial.

We write $\operatorname{Fp}_0(\mathcal{C}, \operatorname{Ab})$ for the Serre subcategory consisting of all finite length objects in $\operatorname{Fp}(\mathcal{C}, \operatorname{Ab})$, and $\operatorname{Eff}(\mathcal{C}, \operatorname{Ab})$ denotes the Serre subcategory of effaceable functors in $\operatorname{Fp}(\mathcal{C}, \operatorname{Ab})$ given by all functors F with presentation

$$0 \longrightarrow \operatorname{Hom}(Z, -) \longrightarrow \operatorname{Hom}(Y, -) \longrightarrow \operatorname{Hom}(X, -) \longrightarrow F \longrightarrow 0$$

coming from an exact sequence $0 \to X \to Y \to Z \to 0$ in $\mathcal{C}$.

Proposition 14.2.1. *Let* $\mathcal{C} = \operatorname{fp}\mathcal{A}$ *be uniserial. Then we have*

$$\operatorname{Fp}_0(\mathcal{C}, \operatorname{Ab})^{\perp} = \{X \in \operatorname{Ind}\mathcal{A} \mid X \notin \operatorname{fp}\mathcal{A}\} \quad \textit{and} \quad \operatorname{Eff}(\mathcal{C}, \operatorname{Ab})^{\perp} = \operatorname{Ind}\mathcal{A} \cap \operatorname{Inj}\mathcal{A}.$$

In particular, $\operatorname{Eff}(\mathcal{C}, \operatorname{Ab}) \subseteq \operatorname{Fp}_0(\mathcal{C}, \operatorname{Ab})$.

Proof Let $X \in \operatorname{Ind}\mathcal{A}$. Then X is the source of a left almost split morphism by Corollary 13.1.30, and therefore $\bar{X}$ is an injective envelope of a simple object S in $\mathbf{P}(\mathcal{A})$ by Theorem 12.3.13. Note that S is finitely presented when X is finitely presented. Thus $X \in \operatorname{fp}\mathcal{A}$ implies $X \notin \operatorname{Fp}_0(\mathcal{C}, \operatorname{Ab})^{\perp}$. On the other hand, when F in $\operatorname{Fp}(\mathcal{C}, \operatorname{Ab})$ is simple and the quotient of $\operatorname{Hom}(Y, -)$, then we may choose Y indecomposable, and it is pure-injective since Y is endofinite by Corollary 13.1.29. Thus $\bar{Y}$ is an injective envelope of F in $\mathbf{P}(\mathcal{A})$. It follows that $X \notin \operatorname{Fp}_0(\mathcal{C}, \operatorname{Ab})^{\perp}$ implies $X \in \operatorname{fp}\mathcal{A}$.

The identity $\operatorname{Eff}(\mathcal{C}, \operatorname{Ab})^{\perp} = \operatorname{Ind}\mathcal{A} \cap \operatorname{Inj}\mathcal{A}$ is Proposition 12.3.17.

The inclusion $\operatorname{Eff}(\mathcal{C}, \operatorname{Ab}) \subseteq \operatorname{Fp}_0(\mathcal{C}, \operatorname{Ab})$ is clear since $X \in \operatorname{Ind}\mathcal{A}$ and $X \notin \operatorname{fp}\mathcal{A}$ implies $X \in \operatorname{Inj}\mathcal{A}$, by Theorem 13.1.28. □

We recall from Proposition 2.3.3 the equivalence

$$\mathrm{Fp}(\mathcal{C}, \mathrm{Ab})/\mathrm{Eff}(\mathcal{C}, \mathrm{Ab}) \xrightarrow{\sim} \mathcal{C}^{\mathrm{op}}$$

and record some consequences.

Corollary 14.2.2. *We have* $\mathrm{KG.dim}\,\mathrm{Ab}(\mathcal{A}) \leq 1$. □

Corollary 14.2.3. *We have* $\mathrm{Fp}_0(\mathcal{C}, \mathrm{Ab}) = \mathrm{Eff}(\mathcal{C}, \mathrm{Ab})$ *if and only if all injective objects in* $\mathcal{C}$ *are zero.* □

Dedekind Domains

Let A be a *Dedekind domain*, that is, a commutative hereditary integral domain. For simplicity we assume that A is not a field. We write $\mathrm{mod}_0\, A$ for the category of finite length A-modules, $Q(A)$ for the quotient field, and $\mathrm{Max}\, A$ for the set of maximal ideals. Note that

$$\mathrm{mod}_0\, A = \coprod_{\mathfrak{p} \in \mathrm{Max}\, A} \mathcal{T}_{\mathfrak{p}} \qquad \text{and} \qquad (\mathrm{mod}\, A)/(\mathrm{mod}_0\, A) \xrightarrow{\sim} \mathrm{mod}\, Q(A),$$

where $\mathcal{T}_{\mathfrak{p}}$ denotes the uniserial category of finite length $\mathfrak{p}$-torsion modules. Recall that a module is $\mathfrak{p}$-*torsion* if each element is annihilated by some power of $\mathfrak{p}$.

Let us consider the functor $\pi\colon \mathrm{Fp}(\mathrm{mod}\, A, \mathrm{Ab}) \to (\mathrm{mod}\, A)^{\mathrm{op}}$ given by

$$\mathrm{Coker}\,\mathrm{Hom}(\phi, -) \longmapsto \mathrm{Ker}\,\phi \qquad (\phi \text{ a morphism in } \mathrm{mod}\, A).$$

We write $\mathrm{Eff}(\mathrm{mod}\, A, \mathrm{Ab})$ for the Serre subcategory of $\mathrm{Fp}(\mathrm{mod}\, A, \mathrm{Ab})$ given by all functors F with presentation

$$0 \longrightarrow \mathrm{Hom}(Z, -) \longrightarrow \mathrm{Hom}(Y, -) \longrightarrow \mathrm{Hom}(X, -) \longrightarrow F \longrightarrow 0$$

coming from an exact sequence $0 \to X \to Y \to Z \to 0$ in $\mathrm{mod}\, A$. Clearly, $\mathrm{Ker}\,\pi = \mathrm{Eff}(\mathrm{mod}\, A, \mathrm{Ab})$. Furthermore, let $\mathrm{Fp}(\mathrm{mod}\, A, \mathrm{Ab})'$ denote the kernel of the composite

$$\mathrm{Fp}(\mathrm{mod}\, A, \mathrm{Ab}) \xrightarrow{\ \pi\ } (\mathrm{mod}\, A)^{\mathrm{op}} \xrightarrow{-\otimes Q(A)} (\mathrm{mod}\, Q(A))^{\mathrm{op}}.$$

Lemma 14.2.4. *The functor* π *induces an equivalence*

$$\mathrm{Fp}(\mathrm{mod}\, A, \mathrm{Ab})/\mathrm{Eff}(\mathrm{mod}\, A, \mathrm{Ab}) \xrightarrow{\sim} (\mathrm{mod}\, A)^{\mathrm{op}}$$

which yields further equivalences

$$\mathrm{Fp}(\mathrm{mod}\, A, \mathrm{Ab})'/\mathrm{Eff}(\mathrm{mod}\, A, \mathrm{Ab}) \xrightarrow{\sim} (\mathrm{mod}_0\, A)^{\mathrm{op}}$$

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and

$$\operatorname{Fp}(\operatorname{mod} A, \mathrm{Ab})/\operatorname{Fp}(\operatorname{mod} A, \mathrm{Ab})' \xrightarrow{\sim} (\operatorname{mod} Q(A))^{\mathrm{op}}.$$

Proof The functor π is a left adjoint of $(\operatorname{mod} A)^{\mathrm{op}} \to \operatorname{Fp}(\operatorname{mod} A, \mathrm{Ab})$ given by $X \mapsto \operatorname{Hom}(X, -)$. Thus the first equivalence follows from Proposition 2.3.3. The other equivalences are then consequences which are derived from the equivalence $\frac{\operatorname{mod} A}{\operatorname{mod}_0 A} \xrightarrow{\sim} \operatorname{mod} Q(A)$; see Proposition 2.2.8. □

Next consider the functor

$$\tau\colon \operatorname{Eff}(\operatorname{mod} A, \mathrm{Ab}) \longrightarrow \operatorname{mod} A, \qquad F \mapsto F(A).$$

Lemma 14.2.5. *The functor τ induces an equivalence*

$$\operatorname{Eff}(\operatorname{mod} A, \mathrm{Ab})/(\operatorname{Ker} \tau) \xrightarrow{\sim} \operatorname{mod}_0 A,$$

and the inclusion $\operatorname{mod}_0 A \to \operatorname{mod} A$ *induces an equivalence*

$$\operatorname{Eff}(\operatorname{mod}_0 A, \mathrm{Ab}) \xrightarrow{\sim} \operatorname{Ker} \tau.$$

Proof A torsion module X with projective presentation $0 \to P_1 \to P_0 \to X \to 0$ induces an exact sequence

$$0 \to \operatorname{Hom}(X, -) \to \operatorname{Hom}(P_0, -) \to \operatorname{Hom}(P_1, -) \to \operatorname{Ext}^1(X, -) \to 0$$

in $\operatorname{Eff}(\operatorname{mod} A, \mathrm{Ab})$, and evaluating at A yields an isomorphism $\operatorname{Ext}^1(X, A) \cong \operatorname{Tr} X$. Thus the functor τ is a right adjoint of $\sigma\colon \operatorname{mod}_0 A \to \operatorname{Eff}(\operatorname{mod} A, \mathrm{Ab})$ given by $X \mapsto \operatorname{Ext}^1(\operatorname{Tr} X, -)$ and satisfying $\tau \circ \sigma \cong \mathrm{id}$. Now the first equivalence follows from Proposition 2.2.11.

The second equivalence is easily checked; it uses that $\operatorname{Hom}(X, A) = 0$ for all $X \in \operatorname{mod}_0 A$, and a quasi-inverse is given by $F \mapsto F|_{\operatorname{mod}_0 A}$. □

For an abelian category $\mathcal{A}$ let $\mathcal{A}_0$ denote the full subcategory consisting of all objects of finite length.

Lemma 14.2.6. *We have* $\operatorname{Eff}(\operatorname{mod}_0 A, \mathrm{Ab}) = \operatorname{Fp}_0(\operatorname{mod}_0 A, \mathrm{Ab})$. *In particular, the assignment*

$$A/\mathfrak{p}^n \longmapsto \operatorname{Hom}(A/\mathfrak{p}^n, -)/\operatorname{Rad}(A/\mathfrak{p}^n, -)$$

identifies the indecomposable objects in $\operatorname{mod}_0 A$ *with the simple objects of the category* $\operatorname{Eff}(\operatorname{mod}_0 A, \mathrm{Ab})$.

Proof The category $\operatorname{mod}_0 A$ is uniserial and has no non-zero injective objects. Thus finite length functors and effaceable functors in $\operatorname{Fp}(\operatorname{mod}_0 A, \mathrm{Ab})$ coincide. This follows from Corollary 14.2.3.

Each indecomposable object in $\operatorname{mod}_0 A$ is the source of an almost split

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morphism $A/\mathfrak{p}^n \to A/\mathfrak{p}^{n-1} \oplus A/\mathfrak{p}^{n+1}$, and this provides in $\mathrm{Fp}(\mathrm{mod}_0 A, \mathrm{Ab})$ the presentation of a simple functor

$$\mathrm{Hom}(A/\mathfrak{p}^{n-1}, -) \oplus \mathrm{Hom}(A/\mathfrak{p}^{n+1}, -) \longrightarrow \mathrm{Hom}(A/\mathfrak{p}^n, -) \longrightarrow S_{A/\mathfrak{p}^n} \longrightarrow 0.$$

Clearly, any simple functor S is of this form, because we have $S(X) \neq 0$ for some indecomposable object X. □

For $\mathfrak{p} \in \mathrm{Max}\, A$ set

$$A_{\mathfrak{p}^\infty} = \mathrm{colim}\, A/\mathfrak{p}^n \qquad \text{and} \qquad \hat{A}_{\mathfrak{p}} = \lim A/\mathfrak{p}^n.$$

Note that $A_{\mathfrak{p}^\infty}$ is an injective envelope of $A/\mathfrak{p}$, while

$$\hat{A}_{\mathfrak{p}} \cong \lim \mathrm{Hom}(A/\mathfrak{p}^n, A_{\mathfrak{p}^\infty}) \cong \mathrm{Hom}(\mathrm{colim}\, A/\mathfrak{p}^n, A_{\mathfrak{p}^\infty}) \cong \mathrm{Hom}(A_{\mathfrak{p}^\infty}, A_{\mathfrak{p}^\infty}).$$

In particular, both modules are indecomposable and pure-injective. The module $A_{\mathfrak{p}^\infty}$ is called a *Prüfer module*, while $\hat{A}_{\mathfrak{p}}$ is called *adic*.

Theorem 14.2.7. *The abelian category* $\mathrm{Ab}(A) = \mathrm{Fp}(\mathrm{mod}\, A, \mathrm{Ab})$ *admits a filtration*

$$\{0\} \subseteq \mathrm{Eff}(\mathrm{mod}_0 A, \mathrm{Ab}) \subseteq \mathrm{Eff}(\mathrm{mod}\, A, \mathrm{Ab}) \subseteq \mathrm{Fp}(\mathrm{mod}\, A, \mathrm{Ab})' \subseteq \mathrm{Ab}(A)$$

such that each quotient is a length category. This provides a complete list of indecomposable pure-injective A-modules, by taking the injective envelope of a functor that is simple in one of the quotient categories.

(1) *The simples in* $\mathrm{Eff}(\mathrm{mod}_0 A, \mathrm{Ab})$ *correspond to* $A/\mathfrak{p}^n$*,* $\mathfrak{p} \in \mathrm{Max}\, A$*,* $n \geq 1$.
(2) *The simples in* $\frac{\mathrm{Eff}(\mathrm{mod}\, A, \mathrm{Ab})}{\mathrm{Eff}(\mathrm{mod}_0 A, \mathrm{Ab})}$ *correspond to* $\hat{A}_{\mathfrak{p}}$*,* $\mathfrak{p} \in \mathrm{Max}\, A$.
(3) *The simples in* $\frac{\mathrm{Fp}(\mathrm{mod}\, A, \mathrm{Ab})'}{\mathrm{Eff}(\mathrm{mod}\, A, \mathrm{Ab})}$ *correspond to* $A_{\mathfrak{p}^\infty}$*,* $\mathfrak{p} \in \mathrm{Max}\, A$.
(4) *The simple in* $\frac{\mathrm{Fp}(\mathrm{mod}\, A, \mathrm{Ab})}{\mathrm{Fp}(\mathrm{mod}\, A, \mathrm{Ab})'}$ *corresponds to* $Q(A)$.

The modules $A/\mathfrak{p}^n$ *and* $Q(A)$ *are endofinite. The modules* $A_{\mathfrak{p}^\infty}$ *are* Σ*-pure-injective.*

Proof The filtration of $\mathrm{Fp}(\mathrm{mod}\, A, \mathrm{Ab})$ follows from the series of the above lemmas. This yields a classification of all indecomposable pure-injective A-modules, using the analogue of Corollary 14.1.16 which is based on the Krull–Gabriel filtration; see also Proposition 14.1.6.

Lemma 14.2.6 takes care of the first layer of the filtration. Lemma 14.2.4 describes the last two layers of the filtration, which yield the indecomposable injective A-modules by Proposition 12.3.17. The remaining layer is given by Lemma 14.2.5 and yields the remaining indecomposable pure-injectives.

It is clear that $A/\mathfrak{p}^n$ has endolength n, while $Q(A)$ has endolength 1. The

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module $A_{\mathfrak{p}^\infty}$ is actually injective, and therefore Σ-injective since the ring A is noetherian; see Theorem 11.2.12. □

Corollary 14.2.8. *We have* KG.dim Ab$(A) = 2$ *and obtain the following filtration of* Ind A*:*

$$\begin{aligned} \operatorname{Ind}_{-1} A &= \{A/\mathfrak{p}^n \mid \mathfrak{p} \in \operatorname{Max} A,\ n \geq 1\} \\ \operatorname{Ind}_0 A &= \{A_{\mathfrak{p}^\infty} \mid \mathfrak{p} \in \operatorname{Max} A\} \cup \{\hat{A}_{\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{Max} A\} \\ \operatorname{Ind}_1 A &= \{Q(A)\}. \end{aligned}$$

Proof For the category of finite length objects in $\operatorname{Ab}(A) = \operatorname{Fp}(\operatorname{mod} A, \operatorname{Ab})$ we have

$$\operatorname{Eff}(\operatorname{mod}_0 A, \operatorname{Ab}) = \operatorname{Fp}_0(\operatorname{mod}_0 A, \operatorname{Ab}) = \operatorname{Ab}(A)_0$$

and obtain the following commutative diagram where all functors are exact.

$$\begin{array}{ccccc} \operatorname{Eff}(\operatorname{mod}_0 A, \operatorname{Ab}) & \rightarrowtail & \operatorname{Fp}(\operatorname{mod} A, \operatorname{Ab}) & \twoheadrightarrow & \operatorname{Ab}(A)/\operatorname{Ab}(A)_0 \\ \downarrow & & \| & & \downarrow \\ \operatorname{Eff}(\operatorname{mod} A, \operatorname{Ab}) & \rightarrowtail & \operatorname{Fp}(\operatorname{mod} A, \operatorname{Ab}) & \twoheadrightarrow & (\operatorname{mod} A)^{\mathrm{op}} \\ \downarrow & & \| & & \downarrow \\ \operatorname{Fp}(\operatorname{mod} A, \operatorname{Ab})' & \rightarrowtail & \operatorname{Fp}(\operatorname{mod} A, \operatorname{Ab}) & \twoheadrightarrow & (\operatorname{mod} Q(A))^{\mathrm{op}} \end{array}$$

The assignment

$$(X, Y) \longmapsto \operatorname{Ext}^1(\operatorname{Tr} X, -) \oplus \operatorname{Hom}(Y, -)$$

provides an equivalence

$$(\operatorname{mod}_0 A) \times (\operatorname{mod}_0 A)^{\mathrm{op}} \xrightarrow{\sim} \operatorname{Ab}(A)_1/\operatorname{Ab}(A)_0.$$

Thus $\operatorname{Ab}(A)_1 = \operatorname{Fp}(\operatorname{mod} A, \operatorname{Ab})'$ and we obtain the Krull–Gabriel filtration

$$\{0\} \subseteq \operatorname{Eff}(\operatorname{mod}_0 A, \operatorname{Ab}) \subseteq \operatorname{Fp}(\operatorname{mod} A, \operatorname{Ab})' \subseteq \operatorname{Ab}(A).$$

It follows that KG.dim Ab$(A) = 2$. □

A subset $\mathcal{U} \subseteq \operatorname{Ind} A$ is Ziegler closed if and only if the following conditions are satisfied.

(1) If $\mathfrak{p} \in \operatorname{Max} A$ and $\{n \geq 1 \mid A/\mathfrak{p}^n \in \mathcal{U}\}$ is infinite, then $A_{\mathfrak{p}^\infty}, \hat{A}_{\mathfrak{p}} \in \mathcal{U}$.
(2) If $\mathcal{U}$ contains infinitely many finite length modules or a module that is not of finite length, then $Q(A) \in \mathcal{U}$.

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The Projective Line

Let k be a field and $\mathbb{P}^1_k$ the projective line over k. We view $\mathbb{P}^1_k$ as a scheme and consider the category $\operatorname{Qcoh}\mathbb{P}^1_k$ of quasi-coherent sheaves on $\mathbb{P}^1_k$. This is a locally finitely presented category and the subcategory of finitely presented objects identifies with $\operatorname{coh}\mathbb{P}^1_k$. We denote by $\operatorname{Ind}\mathbb{P}^1_k$ the set $\operatorname{Ind}(\operatorname{Qcoh}\mathbb{P}^1_k)$ of indecomposable pure-injectives.

Recall that we have the following pullback of abelian categories

$$\begin{array}{ccc} \operatorname{coh}\mathbb{P}^1_k & \longrightarrow & \operatorname{mod} k[y] \\ \downarrow & & \downarrow \\ \operatorname{mod} k[y^{-1}] & \longrightarrow & \operatorname{mod} k[y, y^{-1}] \end{array}$$

which extends to a pullback of Grothendieck categories.

$$\begin{array}{ccc} \operatorname{Qcoh}\mathbb{P}^1_k & \longrightarrow & \operatorname{Mod} k[y] \\ \downarrow & & \downarrow \\ \operatorname{Mod} k[y^{-1}] & \longrightarrow & \operatorname{Mod} k[y, y^{-1}] \end{array}$$

This reflects the covering $\mathbb{P}^1_k = U' \cup U''$ where we identify

$$U' = \operatorname{Spec} k[y] \qquad U'' = \operatorname{Spec} k[y^{-1}] \qquad U' \cap U'' = \operatorname{Spec} k[y, y^{-1}].$$

We use this covering to describe $\operatorname{Ind}\mathbb{P}^1_k$, though it does not extend to a full covering of $\operatorname{Ind}\mathbb{P}^1_k$. To be more precise, the functors $\operatorname{Qcoh}\mathbb{P}^1_k \to \operatorname{Mod} k[y]$ and $\operatorname{Qcoh}\mathbb{P}^1_k \to \operatorname{Mod} k[y^{-1}]$ admit fully faithful right adjoints, which identify $\operatorname{Mod} k[y]$ and $\operatorname{Mod} k[y^{-1}]$ with definable subcategories of $\operatorname{Qcoh}\mathbb{P}^1_k$; this follows from Theorem 12.2.9. In particular, we have embeddings $\operatorname{Ind} k[y] \subseteq \operatorname{Ind}\mathbb{P}^1_k$ and $\operatorname{Ind} k[y^{-1}] \subseteq \operatorname{Ind}\mathbb{P}^1_k$ with

$$\operatorname{Ind} k[y] \cap \operatorname{Ind} k[y^{-1}] = \operatorname{Ind} k[y, y^{-1}].$$

However, $\operatorname{Ind}\mathbb{P}^1_k \neq \operatorname{Ind} k[y] \cup \operatorname{Ind} k[y^{-1}]$.

The classification of indecomposable pure-injective modules over the Dedekind domain $A = k[y]$ from Theorem 14.2.7 provides a description of most objects in $\operatorname{Ind}\mathbb{P}^1_k$. For instance, the inclusion $\operatorname{Ind} A \to \operatorname{Ind}\mathbb{P}^1_k$ extends the inclusion $\operatorname{Spec} A \to \mathbb{P}^1_k$ and is given by

$$A/\mathfrak{p}^n \mapsto \mathscr{O}_{\mathfrak{p}^n} \qquad A_{\mathfrak{p}^\infty} \mapsto \mathscr{O}_{\mathfrak{p}^\infty} := \operatorname{colim} \mathscr{O}_{\mathfrak{p}^n} \qquad \hat{A}_{\mathfrak{p}} \mapsto \hat{\mathscr{O}}_{\mathfrak{p}} := \lim \mathscr{O}_{\mathfrak{p}^n}.$$

Also, $Q(A) \mapsto \mathscr{Q}$, with $\mathscr{Q}$ the sheaf of rational functions.

Theorem 14.2.9. *The following is, up to isomorphism, a complete list of indecomposable pure-injective quasi-coherent sheaves on* $\mathbb{P}^1_k$.

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(1) *For each* $n \in \mathbb{Z}$*, the sheaf* $\mathscr{O}(n)$*.*
(2) *For each closed point* $\mathfrak{p} \in \mathbb{P}^1_k$ *and* $r \geq 1$*, the sheaf* $\mathscr{O}_{\mathfrak{p}^r}$*.*
(3) *For each closed point* $\mathfrak{p} \in \mathbb{P}^1_k$*, the sheaves* $\mathscr{O}_{\mathfrak{p}^\infty}$ *and* $\hat{\mathscr{O}}_{\mathfrak{p}}$*.*
(4) *The sheaf of rational functions* $\mathscr{Q}$*.*

The coherent sheaves and $\mathscr{Q}$ *are endofinite objects. The sheaves* $\mathscr{O}_{\mathfrak{p}^\infty}$ *are* Σ*-pure-injective objects.*

Note that $\mathscr{O}_{\mathfrak{p}^\infty}$ is the injective envelope of $\mathscr{O}_{\mathfrak{p}}$. Also, the sheaf of rational functions $\mathscr{Q}$ is an injective object. This follows from the analogous fact in $\operatorname{Mod} A$ for $A = k[y]$, using that the functor $\operatorname{Mod} A \to \operatorname{Qcoh} \mathbb{P}^1_k$ preserves injectivity.

The proof of the above theorem amounts to an analysis of the abelian category

$$\operatorname{Ab}(\mathbb{P}^1_k) := \operatorname{Ab}(\operatorname{Qcoh} \mathbb{P}^1_k) = \operatorname{Fp}(\mathcal{C}, \operatorname{Ab})^{\mathrm{op}} \quad \text{with} \quad \mathcal{C} = \operatorname{coh} \mathbb{P}^1_k,$$

and we begin with some preparations.

Let $\mathcal{C}_0 := \operatorname{coh}_0 \mathbb{P}^1_k$ denote the category of torsion sheaves, which equals the category of finite length objects in $\mathcal{C}$, and set

$$\mathcal{C}_+ := \{Y \in \mathcal{C} \mid \operatorname{Hom}(X, Y) = 0 \text{ for all } X \in \mathcal{C}_0\}.$$

Lemma 14.2.10. $(\mathcal{C}_0, \mathcal{C}_+)$ *is a split torsion pair for* $\mathcal{C}$*.*

Proof Fix an object $X \in \mathcal{C}$. Since X is noetherian, there exists a maximal subobject X_0 of finite length. Then every finite length subobject is necessarily contained in X_0, so it is unique. Next, if $\phi\colon S \to X/X_0$ is non-zero for some simple S, then ϕ is injective and we can form the pullback to obtain a larger finite length subobject of X, a contradiction. Thus $X/X_0 \in \mathcal{C}_+$, and therefore $(\mathcal{C}_0, \mathcal{C}_+)$ is a torsion pair.

We know from the classification of objects in $\mathcal{C}$ that $\mathcal{C} = \mathcal{C}_0 \vee \mathcal{C}_+$. Thus the torsion pair $(\mathcal{C}_0, \mathcal{C}_+)$ is split, so X_0 is a direct summand of X. Alternatively, one uses Serre duality (Example 6.5.4). □

Next we consider the subcategories $\vec{\mathcal{C}}_0$ and $\vec{\mathcal{C}}_+$ of $\operatorname{Qcoh} \mathbb{P}^1_k$ which are obtained by closing under filtered colimits. Observe that $\vec{\mathcal{C}}_0$ and $\vec{\mathcal{C}}_+$ are locally finitely presented categories, with $\operatorname{fp} \vec{\mathcal{C}}_0 = \mathcal{C}_0$ and $\operatorname{fp} \vec{\mathcal{C}}_+ = \mathcal{C}_+$.

Proposition 14.2.11. *We have*

$$\operatorname{Ind} \mathbb{P}^1_k = \operatorname{Ind} \vec{\mathcal{C}}_0 \sqcup \operatorname{Ind} \vec{\mathcal{C}}_+.$$

Proof The subcategory $\mathcal{C}_+ \subseteq \mathcal{C}$ is covariantly finite because the inclusion admits a left adjoint. From this it follows that $\vec{\mathcal{C}}_+$ is a definable subcategory; see Example 12.2.6. Therefore $\operatorname{Ind} \mathbb{P}^1_k \cap \vec{\mathcal{C}}_+ = \operatorname{Ind} \vec{\mathcal{C}}_+$.

The category $\mathcal{C}_0$ is uniserial, and therefore all indecomposable objects in $\vec{\mathcal{C}}_0$ are pure-injective, by Theorem 13.1.28. Thus $\operatorname{Ind}\mathbb{P}^1_k \cap \vec{\mathcal{C}}_0 = \operatorname{Ind}\vec{\mathcal{C}}_0$.

Any object $X \in \operatorname{Ind}\mathbb{P}^1_k$ fits into a pure-exact sequence $0 \to X_0 \to X \to X_+ \to 0$ with $X_0 \in \vec{\mathcal{C}}_0$ and $X_+ \in \vec{\mathcal{C}}_+$; see Example 12.1.10. If $X_0 \neq 0$, then X_0 has an indecomposable direct summand which is pure-injective, again by Theorem 13.1.28. Thus $X_0 = X$. □

Proposition 14.2.12. *We have* KG.dim(grmod $k[x, y]$) = 2*, with filtration*

$$0 \subseteq \operatorname{grmod}_0 k[x,y] \subseteq \operatorname{grmod}_1 k[x,y] \subseteq \operatorname{grmod} k[x,y]$$

and subquotients $\operatorname{grmod}_0 k[x,y] \xrightarrow{\sim} \operatorname{grmod} k$*,*

$$\frac{\operatorname{grmod}_1 k[x,y]}{\operatorname{grmod}_0 k[x,y]} \xrightarrow{\sim} \operatorname{coh}_0 \mathbb{P}^1_k \quad \textit{and} \quad \frac{\operatorname{grmod} k[x,y]}{\operatorname{grmod}_1 k[x,y]} \xrightarrow{\sim} \operatorname{mod} k(t).$$

Proof Proposition 5.1.6 yields equivalences

$$\frac{\operatorname{grmod} k[x,y]}{\operatorname{grmod}_0 k[x,y]} \xrightarrow{\sim} \operatorname{coh}\mathbb{P}^1_k \quad \text{and} \quad \frac{\operatorname{coh}\mathbb{P}^1_k}{\operatorname{coh}_0 \mathbb{P}^1_k} \xrightarrow{\sim} \operatorname{mod} k(t).$$

From this the assertion follows. □

Proof of Theorem 14.2.9 First observe that all objects from the list are pure-injective. In fact, each object $X \in \mathcal{C}$ is endofinite since $\operatorname{Hom}(C, X)$ has finite length over k and therefore also over $\operatorname{End}(X)$ for all $C \in \mathcal{C}$. The inclusion $\operatorname{Ind} k[y] \to \operatorname{Ind}\mathbb{P}^1_k$ preserves endofiniteness and Σ-pure-injectivity, since these properties are preserved by any inclusion of a definable subcategory. Thus $\mathcal{Q}$ is endofinite and each $\mathcal{O}_{\mathfrak{p}^\infty}$ is Σ-pure-injective, by Theorem 14.2.7.

It remains to show that the list is complete. We argue via a filtration of $\operatorname{Ab}(\mathbb{P}^1_k)$. The split torsion pair $(\mathcal{C}_0, \mathcal{C}_+)$ yields a sequence of additive functors

$$\mathcal{C}_+ \xrightarrow{i} \mathcal{C} \xrightarrow{p} \mathcal{C}/\mathcal{C}_+ \xrightarrow{\sim} \mathcal{C}_0$$

which induces a diagram of exact functors

$$\operatorname{Fp}(\mathcal{C}_0, \operatorname{Ab}) \xrightarrow{p^*} \operatorname{Fp}(\mathcal{C}, \operatorname{Ab}) \xrightarrow{i^*} \operatorname{Fp}(\mathcal{C}_+, \operatorname{Ab})$$

by Proposition 2.2.20 and Example 2.2.21. Now observe that the canonical functor $\operatorname{grmod} k[x,y] \twoheadrightarrow \operatorname{coh}\mathbb{P}^1_k$ induces an equivalence $\operatorname{grproj} k[x,y] \xrightarrow{\sim} \mathcal{C}_+$ when restricted to the subcategory of projective modules. Thus we have filtrations

$$0 \subseteq \operatorname{Eff}(\mathcal{C}_0, \operatorname{Ab}) \subseteq \operatorname{Fp}(\mathcal{C}_0, \operatorname{Ab})$$

and

$$0 \subseteq \operatorname{grmod}_0 k[x,y] \subseteq \operatorname{grmod}_1 k[x,y] \subseteq \operatorname{grmod} k[x,y] \cong \operatorname{Fp}(\mathcal{C}_+, \operatorname{Ab})$$

such that each subquotient is a length category, by Corollary 14.2.3 and Proposition 14.2.12.

From these filtrations we obtain a classification of all indecomposable pure-injective objects, using the analogue of Corollary 14.1.16 which is based on the Krull–Gabriel filtration; see also Proposition 14.1.6. To be more precise, the filtration of $\operatorname{Fp}(\mathcal{C}_0, \operatorname{Ab})$ yields the objects in $\operatorname{Ind} \vec{\mathcal{C}}_0$, while the filtration of $\operatorname{Fp}(\mathcal{C}_+, \operatorname{Ab})$ yields the objects in $\operatorname{Ind} \vec{\mathcal{C}}_+$, keeping in mind the decomposition

$$\operatorname{Ind} \mathbb{P}^1_k = \operatorname{Ind} \vec{\mathcal{C}}_0 \sqcup \operatorname{Ind} \vec{\mathcal{C}}_+$$

from Proposition 14.2.11. We have $\mathscr{O}_{\mathfrak{p}^r}$ and $\mathscr{O}_{\mathfrak{p}^\infty}$ in $\vec{\mathcal{C}}_0$, with the sheaves $\mathscr{O}_{\mathfrak{p}^r}$ corresponding to the simple objects in $\operatorname{Fp}(\mathcal{C}_0, \operatorname{Ab})$. On the other hand, the sheaves $\mathscr{O}(n)$ correspond to the simple objects in $\operatorname{Fp}(\mathcal{C}_+, \operatorname{Ab})$, the sheaves $\hat{\mathscr{O}}_{\mathfrak{p}}$ correspond to the simple objects in the next layer of $\operatorname{Fp}(\mathcal{C}_+, \operatorname{Ab})$, while the sheaf of rational functions $\mathscr{Q}$ arises from the last layer of $\operatorname{Fp}(\mathcal{C}_+, \operatorname{Ab})$. □

Remark 14.2.13. We have $\operatorname{KG.dim} \operatorname{Ab}(\mathbb{P}^1_k) = 2$ and obtain the following filtration of $\operatorname{Ind} \mathbb{P}^1_k$:

$$\begin{aligned}
\operatorname{Ind}_{-1} \mathbb{P}^1_k &= \{\mathscr{O}(n) \mid n \in \mathbb{Z}\} \cup \{\mathscr{O}_{\mathfrak{p}^r} \mid \mathfrak{p} \in \mathbb{P}^1_k,\ r \geq 1\} \\
\operatorname{Ind}_0 \mathbb{P}^1_k &= \{\mathscr{O}_{\mathfrak{p}^\infty} \mid \mathfrak{p} \in \mathbb{P}^1_k\} \cup \{\hat{\mathscr{O}}_{\mathfrak{p}} \mid \mathfrak{p} \in \mathbb{P}^1_k\} \\
\operatorname{Ind}_1 \mathbb{P}^1_k &= \{\mathscr{Q}\}.
\end{aligned}$$

This is the analogue of Corollary 14.2.8 with a similar proof.

The Kronecker Quiver

We consider the following Kronecker quiver

$$\circ \rightrightarrows \circ$$

and fix a field k. A representation (V, W, ϕ, ψ) consists of a pair of vector spaces together with a pair of linear maps between them

$$V \overset{\phi}{\underset{\psi}{\rightrightarrows}} W.$$

The representations of the Kronecker quiver identify with modules over its path algebra, which is the Kronecker algebra.

The sheaf $\mathscr{T} = \mathscr{O} \oplus \mathscr{O}(1)$ is a tilting object of $\operatorname{coh} \mathbb{P}^1_k$ and its endomorphism algebra $\Lambda = \operatorname{End}(\mathscr{T})$ identifies with the Kronecker algebra, because of (5.1.7) and (5.1.8). Thus the functor $\operatorname{Hom}(\mathscr{T}, -)$ induces a triangle equivalence

$$\mathbf{D}^b(\operatorname{coh} \mathbb{P}^1_k) \xrightarrow{\sim} \mathbf{D}^b(\operatorname{mod} \Lambda).$$

This follows from Theorem 5.1.2. In fact, the tilting object $\mathscr{T}$ induces a split torsion pair $(\mathcal{T}, \mathcal{F})$ for $\operatorname{coh} \mathbb{P}^1_k$, and we have

$$\mathcal{T} = (\operatorname{coh}_0 \mathbb{P}^1_k) \vee (\operatorname{add}\{\mathscr{O}(n) \mid n \geq 0\}) \qquad \text{and} \qquad \mathcal{F} = \operatorname{add}\{\mathscr{O}(n) \mid n < 0\}.$$

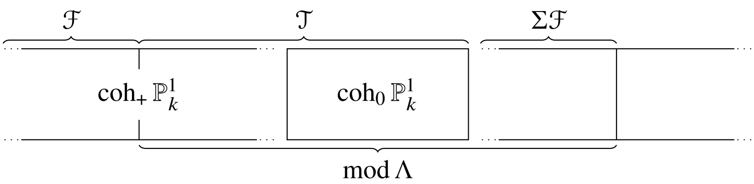
On the other hand, there is a split torsion pair $(\mathcal{U}, \mathcal{V})$ for $\operatorname{mod} \Lambda$ with equivalences

$$\operatorname{Hom}(\mathscr{T}, -) : \mathcal{T} \xrightarrow{\sim} \mathcal{V} \quad \text{and} \quad \operatorname{Ext}^1(\mathscr{T}, -) : \mathcal{F} \xrightarrow{\sim} \mathcal{U}.$$

An explicit description is given in Proposition 5.1.17. In particular, we have

$$\mathcal{V} = (\operatorname{reg} \Lambda) \vee (\operatorname{add}\{P_n \mid n \geq 0\}) \qquad \text{and} \qquad \mathcal{U} = \operatorname{add}\{I_n \mid n \geq 0\}.$$

The following diagram illustrates the tilting from $\operatorname{coh} \mathbb{P}^1_k$ to $\operatorname{mod} \Lambda$.



Next we extend the torsion pairs by taking for each subcategory the closure under filtered colimits; see Example 12.1.10. Observe that $\vec{\mathcal{F}}$ and $\vec{\mathcal{U}}$ are locally finitely presented categories such that each object decomposes into a coproduct of finitely presented objects, and such that each object is pure-injective and pure-projective; see Theorem 13.1.20 and Example 13.1.21.

The following is the analogue of Proposition 14.2.11.

Proposition 14.2.14. *We have*

$$\operatorname{Ind} \mathbb{P}^1 = \operatorname{Ind} \vec{\mathcal{T}} \sqcup \operatorname{Ind} \vec{\mathcal{F}} \qquad \textit{and} \qquad \operatorname{Ind} \Lambda = \operatorname{Ind} \vec{\mathcal{U}} \sqcup \operatorname{Ind} \vec{\mathcal{V}}.$$

Proof The proof is essentially the same as that for Proposition 14.2.11. That each indecomposable pure-injective object over $\mathbb{P}^1_k$ or Λ belongs to one of the subcategories uses the fact that the objects in $\vec{\mathcal{F}}$ and $\vec{\mathcal{U}}$ are pure-injective and pure-projective so that both torsion pairs yield split exact sequences. □

The functor $\operatorname{Hom}(\mathscr{T}, -)$ preserves filtered colimits and extends therefore to an equivalence $\vec{\mathcal{T}} \xrightarrow{\sim} \vec{\mathcal{V}}$. Analogously, $\operatorname{Ext}^1(\mathscr{T}, -)$ yields an equivalence $\vec{\mathcal{F}} \xrightarrow{\sim} \vec{\mathcal{U}}$. Combining these equivalences with Proposition 5.1.17 and Proposition 14.2.14 gives the following bijection $\operatorname{Ind} \mathbb{P}^1 \xrightarrow{\sim} \operatorname{Ind} \Lambda$:

$$\mathscr{O}_{\mathfrak{p}^n} \mapsto R_{\mathfrak{p}^n} \qquad \mathscr{O}_{\mathfrak{p}^\infty} \mapsto R_{\mathfrak{p}^\infty} \qquad \hat{\mathscr{O}}_{\mathfrak{p}} \mapsto \hat{R}_{\mathfrak{p}} \qquad \mathscr{Q} \mapsto Q$$

and

$$\mathscr{O}(n) \mapsto P_n \quad (n \geq 0) \qquad \mathscr{O}(n) \mapsto I_{-n+1} \quad (n < 0).$$

In fact, we use that $\mathscr{O}_{\mathfrak{p}^\infty} = \operatorname{colim} \mathscr{O}_{\mathfrak{p}^n}$, so

$$R_{\mathfrak{p}^\infty} := \operatorname{colim} R_{\mathfrak{p}^n} \cong \operatorname{colim} \operatorname{Hom}(\mathscr{T}, \mathscr{O}_{\mathfrak{p}^n}) \cong \operatorname{Hom}(\mathscr{T}, \mathscr{O}_{\mathfrak{p}^\infty}).$$

Analogously, $\hat{\mathscr{O}}_{\mathfrak{p}} = \lim \mathscr{O}_{\mathfrak{p}^n}$, so

$$\hat{R}_{\mathfrak{p}} := \lim R_{\mathfrak{p}^n} \cong \lim \operatorname{Hom}(\mathscr{T}, \mathscr{O}_{\mathfrak{p}^n}) \cong \operatorname{Hom}(\mathscr{T}, \hat{\mathscr{O}}_{\mathfrak{p}}).$$

Finally, the distinguished sheaf $\mathscr{Q}$ (indecomposable endofinite but not finitely presented) corresponding to the generic point of $\mathbb{P}^1_k$ is mapped to the distinguished module Q, which is indecomposable endofinite but not finitely presented.

The following theorem summarises the description of $\operatorname{Ind} \Lambda$.

Theorem 14.2.15. *The following is, up to isomorphism, a complete list of indecomposable pure-injective Λ-modules.*

(1) *For each $n \geq 0$, the modules P_n and I_n.*
(2) *For each closed $\mathfrak{p} \in \mathbb{P}^1_k$ and $r \geq 1$, the module $R_{\mathfrak{p}^r}$.*
(3) *For each closed $\mathfrak{p} \in \mathbb{P}^1_k$, the Prüfer module $R_{\mathfrak{p}^\infty}$ and the adic module $\hat{R}_{\mathfrak{p}}$.*
(4) *The generic module Q.*

The finitely presented modules and Q are endofinite. The modules $R_{\mathfrak{p}^\infty}$ are Σ-pure-injective. □

Remark 14.2.16. We have $\operatorname{KG.dim} \operatorname{Ab}(\Lambda) = 2$ and obtain the following filtration of $\operatorname{Ind} \Lambda$:

$$\begin{aligned} \operatorname{Ind}_{-1} \Lambda &= \operatorname{Ind} \Lambda \cap \operatorname{mod} \Lambda \\ \operatorname{Ind}_0 \Lambda &= \{R_{\mathfrak{p}^\infty} \mid \mathfrak{p} \in \mathbb{P}^1_k\} \cup \{\hat{R}_{\mathfrak{p}} \mid \mathfrak{p} \in \mathbb{P}^1_k\} \\ \operatorname{Ind}_1 \Lambda &= \{Q\}. \end{aligned}$$

This is the analogue of Corollary 14.2.8 with a similar proof.

A subset $\mathcal{U} \subseteq \operatorname{Ind} \Lambda$ is Ziegler closed if and only if the following conditions are satisfied.

(1) $R_{\mathfrak{p}^\infty} \in \mathcal{U}$ provided $\operatorname{Hom}(R_{\mathfrak{p}}, X) \neq 0$ for infinitely many $X \in \mathcal{U} \cap \operatorname{mod} \Lambda$.
(2) $\hat{R}_{\mathfrak{p}} \in \mathcal{U}$ provided $\operatorname{Hom}(X, R_{\mathfrak{p}}) \neq 0$ for infinitely many $X \in \mathcal{U} \cap \operatorname{mod} \Lambda$.
(3) $Q \in \mathcal{U}$ provided $\mathcal{U}$ contains infinitely many finite length modules or a module that is not of finite length.

https://doi.org/10.1017/9781108979108.022 Published online by Cambridge University Press

Injective Cohomology Representations

Let G be a finite group and k a field of characteristic $p > 0$. We recall the functor $T\colon \operatorname{Inj} H^*(G,k) \to \operatorname{StMod} kG$; it identifies the torsion free injective $H^*(G,k)$-modules with a definable subcategory of $\operatorname{Mod} kG$ which we denote by $\mathcal{T}(G,k)$ (Corollary 12.4.24).

The *p-rank* of a finite group G is the largest integer n such that G has an elementary abelian subgroup of order p^n. We note that the Krull dimension of $H^*(G,k)$ equals the p-rank of G by a theorem of Quillen [29, Theorem 5.3.8].

Proposition 14.2.17. $\operatorname{KG.dim} \operatorname{Ab}(\mathcal{T}(G,k)) + 1$ *equals the p-rank of G.*

Proof Set $R = H^*(G,k)$. We consider the category of graded R-modules and the definable subcategory $\operatorname{Inj} R$ of injective R-modules. It follows from Proposition 14.1.14 that the Krull dimension of $H^*(G,k)$ equals

$$\operatorname{KG.dim} \operatorname{Ab}(\operatorname{Inj} R) = \operatorname{KG.dim} \operatorname{Ab}(\operatorname{grmod} R).$$

For the definable subcategory $\operatorname{Inj}_+ R$ of torsion free modules we have

$$\begin{aligned}\operatorname{KG.dim} \operatorname{Ab}(\operatorname{Inj}_+ R) &= \operatorname{KG.dim} \operatorname{Ab}((\operatorname{grmod} R)/(\operatorname{grmod}_0 R)) \\ &= \operatorname{KG.dim} \operatorname{Ab}(\operatorname{grmod} R) - 1.\end{aligned}$$

We claim that

$$\operatorname{KG.dim} \operatorname{Ab}(\operatorname{Inj}_+ R) = \operatorname{KG.dim} \operatorname{Ab}(\mathcal{T}(G,k)),$$

and then the proof is complete because of Quillen's result. The claim follows from an iterated application of the lemma below, because the functor T preserves products and coproducts, so it identifies isolated points. □

Let $\mathcal{A}$ be a locally finitely presented category and $\mathcal{B} \subseteq \mathcal{A}$ a definable subcategory. We consider the Krull–Gabriel filtration of $\operatorname{Ab}(\mathcal{B})$ and this yields subsets $\operatorname{Ind}_\alpha \mathcal{B} \subseteq \operatorname{Ind} \mathcal{B} := \mathcal{B} \cap \operatorname{Ind} \mathcal{A}$ for each ordinal $\alpha \geq -1$.

Lemma 14.2.18. *Suppose that* $\operatorname{Ab}(\mathcal{B})$ *is noetherian. Then for* $\alpha \geq -1$ *and* $X \in \mathcal{U} := \bigsqcup_{\beta \geq \alpha} \operatorname{Ind}_\beta \mathcal{B}$ *the following are equivalent.*

(1) *X is isolated, so $\{X\} \subseteq \mathcal{U}$ is open.*
(2) $X \in \operatorname{Ind}_\alpha \mathcal{B}$.
(3) *If X is isomorphic to a direct summand of a product $\prod_{i \in I} Y_i$ of indecomposable objects in $\mathcal{U}$, then $X \cong Y_i$ for some $i \in I$.*

Proof Apply Theorem 12.3.13. The assumption on $\operatorname{Ab}(\mathcal{B})$ to be noetherian is needed; it implies that each simple object in $\mathbf{P}(\mathcal{B})$ is finitely presented. □

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Example 14.2.19. Let $G = \mathbb{Z}/2\times\mathbb{Z}/2$ and k be a field of characteristic two. We denote by Λ the path algebra of the Kronecker quiver. Then (5.1.23) provides a functor $\operatorname{Mod}\Lambda \to \operatorname{Mod} kG$ which identifies the Prüfer modules $R_{\mathfrak{p}^\infty}$ ($\mathfrak{p} \in \mathbb{P}^1_k$) and the generic module Q with the indecomposable objects in $\mathcal{T}(G, k)$.

Notes

The method of classifying the pure-injective objects via the Krull–Gabriel filtration is taken from Jensen and Lenzing [118], which is modelled after [79]. Note that the corresponding dimensions may differ, depending on the use of *all* simple objects versus the use of the *finitely presented* simple objects in a locally finitely presented Grothendieck category.

The general decomposition theory of injective objects in Grothendieck categories is based on the spectral category in the sense of Gabriel and Oberst [82].

The classification of pure-injectives works well for modules over Dedekind domains, and in particular for abelian groups; for the classical approach see Kaplansky [119] and Fuchs [77].

The striking parallel between abelian groups and modules over tame hereditary algebras was pointed out by Ringel [171]. The Krull–Gabriel dimension for a tame hereditary algebra is equal to two by work of Geigle [87], and for the Ziegler topology we refer to [161, 174]. The computation of the Krull–Gabriel filtration for uniserial categories is taken from joint work with Vossieck [135].