DIFFERENTIABLE POINTS IN THE CONFORMAL PLANE

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1. Introduction. The purpose of this note is the classification of the differentiable points on curves in the conformal plane. We introduce tangent and osculating circles at such points and study the intersection and support properties of these circles.

Our paper is related to the case $n = 3$ of the classification of the differentiable points on curves in projective n-space given in [**4**]. The connecting link is a stereographic projection of a curve in the conformal plane on a spherical one. Naturally, this note is also connected with the many discussions of the curvature and the osculating circles of curves in the Euclidean plane [**1**; **2**; **3**; **5**].

2. Pencils of circles. In the following, P, Q, . . . denote points in the conformal plane; C, C', . . . denote oriented circles. Such a circle C decomposes the plane into two open regions, its *interior* $\underline{C}$ and its *exterior*[1] $\bar{C}$. The circle through three mutually distinct points, P, Q, and R will occasionally be denoted by $C(P, Q, R)$.

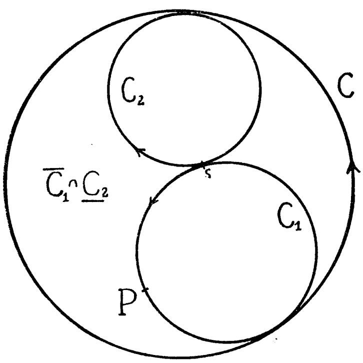

FIG. 1

The set of all circles that intersect two given circles at right angles form a linear pencil π of circles. A pencil π of the *first* kind possesses two fundamental points such that π is identical with the set of all circles through these points. A pencil π of the *second* kind has one fundamental point and is identical with the set of those circles that touch a given circle at that point. If π is of the *third* kind, then any two circles of π are disjoint. To any pencil π and to any

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[1] The meaning of these terms is not identical with that given in ordinary plane geometry. The interior of the oriented circle C lies at its left; cf. Fig. 1.

point Q which is not a fundamental point of π, there exists one and only one circle $C(\pi, Q)$ of π through Q. We consider the fundamental point of a pencil π of the second kind as a point-circle belonging to π.

The set of the circles perpendicular to a given pencil π form again a pencil ρ. If π is of the pth kind, then ρ is of the $(4 - p)$th kind ($p = 1, 2, 3$). The relation between π and ρ is involutory. If $C \subset \rho$, then every circle of π meets C, and C contains a fundamental point of π if and only if π is of the second kind.

3. Convergence. We call the sequence of points $P_1, P_2, \ldots$ *convergent* to P if there exists to every C with $P \subset \underline{C}$ a number $n = n(C)$ such that $P_\nu \subset \underline{C}$ if $\nu > n$.

In the same way, the convergence of circles to a point is defined.

We call the sequence $C_1, C_2, \ldots$ convergent to C if there exists to every pair $C' \subset \underline{C}$ and $C'' \subset \bar{C}$ a number $n = n(C', C'')$ such that $C' \subset \underline{C}_\nu$ and $C'' \subset \bar{C}_\nu$ for every $\nu > n$.

4. Support and intersection at a point of an arc. An *arc* A is the continuous image of a closed interval. Thus if a sequence of points of that *parameter interval* converges to a point s, then their image points converge to the image of s. We shall use the same letters $s, s', \ldots$ to denote both the *parameter* (i.e., the points of the parameter interval) and their images on A. The *end* (*interior*) points of A are the images of the end (interior) points of the parameter interval.

A *neighbourhood* of s on A is the image of a neighbourhood of the parameter s on the parameter interval. If s is an interior point of A, this neighbourhood is decomposed by s into two (open) *one-sided* neighbourhoods.

From our definition, different points of A, i.e., points with different parameters, may coincide with the same point of the conformal plane. However, we shall assume that each point s of A has a neighbourhood such that no other point of that neighbourhood coincides with s. (The notation $P \neq s$ will indicate that the points P and s do not coincide.)

Suppose s is an interior point of A. Then we call s a *point of support* (*intersection*) with respect to the circle C if a sufficiently small neighbourhood of s is decomposed by s into two one-sided neighbourhoods which lie in the same region (in different regions) bounded by C. C is then called a *supporting* (*intersecting*) circle of A at s. Thus C supports A at s if $s \not\subset C$. By definition, the point-circle s always supports A at s.

It can happen that every neighbourhood of s has points $\neq s$ in common with C. Then C neither supports nor intersects A at s.

5. Tangent circles of s. Suppose the point $s \subset A$ satisfies the following

CONDITION I. To every point $P \neq s$ and to every sequence of points $s' \to s$ ($s' \neq s$; $s' \subset A$) there exists a circle C' such that $C(s', s, P) \to C'$.

Obviously, the *tangent circle* C' is independent of the choice of the sequence s'. We admit the point s itself as the tangent point-circle of A at s.

THEOREM 1. *The set $\tau = \tau(s)$ of all the tangent circles of A at s is a pencil of the second kind with the fundamental point s.*

Proof. Let P, Q, R be three mutually distinct points. If the point $R' \neq R$ converges to R, then the angle between the circles $C(R', R, P)$ and $C(R', R, Q)$ converges to zero.[2] We choose $R = s$ and $R' = s'$. Since the angle between two circles depends on them continuously, we conclude that any two tangent circles at s touch each other at that point. Thus two tangent circles that have another point in common are identical. In particular, there exists one and only one tangent circle at s through each point different from s.

Suppose the circle C touches the tangent circle C' at s. Let $P \subset C$, $P \neq s$. Then C also touches the tangent circle through P. Hence C is equal to that circle.

By Thereom 1, the tangent circle $C(\tau, P)$ through P depends continuously on P as long as $P \neq s$.

THEOREM 2. *Suppose the point $s \subset A$ satisfies Condition* I. *Let π be a pencil of the second kind with s as its fundamental point; $\pi \neq \tau$. If the points s' converge to s $(s' \neq s)$, then $C(\pi, s') \to s$.*

Proof. If our statement were false, there would exist a circle C such that $s \subset \underline{C}$ and a sequence of points $s' \to s$ $(s' \neq s)$ such that $C(\pi, s') \not\subset \underline{C}$ for each s'. Let C_1 and C_2 be the two circles of π that touch C. We may assume that π is oriented such that C lies in the closure of $\bar{C}_1 \cap \underline{C}_2$. Then this closed domain also contains the circles $C(\pi, s')$ and therefore the points s' (cf. Fig. 1).

Let P be any point of C_1; $P \neq s$. If a sequence of points Q converges to s through the above domain, then the circles $C(s, P, Q)$ converge to C_1. Choosing $Q = s'$, we obtain $C_1 = C(\tau, P)$, while $\tau \neq \pi$. This is a contradiction.

6. Non-tangent circles. Let s be an interior point of A. Suppose again that s satisfies Condition I (cf. §5).

THEOREM 3. *Every non-tangent circle either supports or intersects A at s.*

Proof. If the circle C neither supports nor intersects A at s, then $s \subset C$ and there exists a sequence of points $s' \to s$ such that $s' \subset A \cap C$ and $s' \neq s$. Let $P \subset C$, $P \neq s$. Then $C = C(s', s, P)$ for each s', and Condition I implies $C = C(\tau, P)$.

THEOREM 4. *Non-tangent circles through s all intersect or all support.*

Proof (cf. Fig. 2). Let C_1 and C_2 be two distinct non-tangent circles through s. We assume at first that they have another point $P \neq s$ in common. Suppose for example, that C_1 intersects and C_2 supports at s. Thus $A \cap \underline{C}_1$ and $A \cap \bar{C}_1$ are non-void. Without restriction of generality, we may assume that $A \subset \bar{C}_2$.

If $s' \subset A \cap \underline{C}_1$, then $C(s, s', P)$ lies in the closure of $(\underline{C}_1 \cap \bar{C}_2) \cup (\bar{C}_1 \cap \underline{C}_2)$.

[2] The circles themselves need not be convergent.

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By having s' converge to s, we conclude that $C(\tau, P)$ lies in the same closed domain. By having s' converge to s through $A \cap \bar{C}_1$, we obtain symmetrically

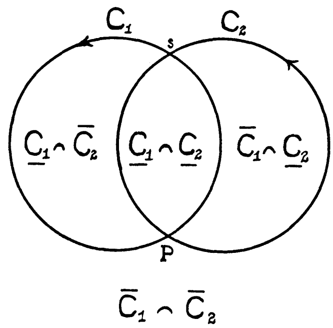

FIG. 2

that $C(\tau, P)$ also lies in the closure of $(\underline{C}_1 \cap \underline{C}_2) \cup (\bar{C}_1 \cup \bar{C}_2)$. Hence $C(\tau, P)$ lies in the intersection $C_1 \cup C_2$ of these two domains, i.e., $C(\tau, P)$ is either C_1 or C_2, contrary to our assumptions. Thus C_1 and C_2 either both support or both intersect.

If C_1 and C_2 meet only at s, then they touch at that point. Choose any non-tangent circle C_3 through s that does not belong to the pencil through C_1 and C_2. From the above, C_1 and C_3, and also C_3 and C_2, either both support or both intersect. Hence our statement remains valid for C_1 and C_2 also in this case. This completes our proof.

7. Differentiable points. We call the point s *differentiable* if it satisfies not only Condition I but also the following one:

CONDITION II. If s' converges to s ($s' \neq s$), then $C(\tau, s')$ converges to some circle $C(s)$ (cf. §5).

If s is differentiable, then the *osculating circle* $C(s)$ is obviously independent of the choice of the points s'. Furthermore, τ being closed, we certainly have $C(s) \subset \tau$.

THEOREM 5. *The point s is differentiable if and only if the limit circle*

$$\lim_{s' \to s} C(\pi, s') \qquad s' \neq s$$

exists for every π.

Proof. This follows at once when the continuity of A at s and the Conditions I and II are combined with §5.

As a consequence of Theorem 5, we may define

$$C(\pi, s) = \lim_{s' \to s} C(\pi, s') \qquad s' \neq s$$

if s is a fundamental point of π. In this way $C(\pi, s')$ will become continuous at s for every π.

The following example shows that Condition II does not follow from Condition I. We introduce retangular cartesian coordinates x, y. Then the arc A defined by

$$x = s, y = \begin{cases} (1 - \sqrt{1 - s^2}) \sin s^{-1}, & 0 < |s| \leqslant 1 \\ 0, & s = 0 \end{cases}$$

lies between the two circles $x^2 + (y \pm 1)^2 = 1$. In particular A satisfies Condition I at $s = 0$, and τ consists of all the circles that touch the x-axis at the origin. Since every neighbourhood of $s = 0$ on A has points in common with both of the above circles, II does not hold.

8. Non-osculating tangent circles of s. Let s be a differentiable interior point of A.

THEOREM 6. *Every non-osculating circle either supports or intersects at s.*

Proof. If the circle C neither supports nor intersects at s, then $C \subset \tau$ (cf. §6) and there exists a sequence of points $s' \to s$, $s' \neq s$ on C. Thus $C = C(\tau, s')$ for each s'. From Condition II,

$$C = \lim_{s' \to s} C(\tau, s') = C(s).$$

THEOREM 7. *If $C(s) \neq s$, then every non-osculating tangent circle supports.*

Proof. Suppose that $C \subset \tau$. $C \neq C(s)$. If a sequence of points s' exists such that $s' \subset A \cap \bar{C}$, $s' \neq s$, $s' \to s$, then each $C(\tau, s')$ lies in the closure of $\bar{C}$. Hence $C(s)$ will lie in the same domain and therefore even in $s \cup \bar{C}$. Similarly the existence of a sequence $s' \subset \underline{C}$, $s' \neq s$, $s' \to s$ implies $C(s) \subset s \cup \underline{C}$. Since $(s \cup \bar{C}) \cap (s \cup \underline{C}) = s$, $C(s) = s$.

It remains to consider the case that $C(s)$ is the point-circle s. In this case we prove

THEOREM 8. *Either the other tangent circles all support or they all intersect.*

Proof. Let C_1 and C_2 be two distinct non-osculating tangent circles. We may assume that τ is oriented such that $C_2 \subset (s \cup \bar{C}_1)$; thus $C_1 \subset (s \cup C_2)$ (compare

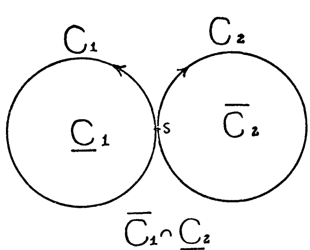

FIG. 3

Fig. 3). From the above, there exists a neighbourhood N of s on A which has no other points in common with $C_1 \cup C_2$.

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<table>
<tr><th>Charac-teristic</th><th>Non-tangent circles through s</th><th>Tangent circles $\neq$ $C(s)$</th><th colspan="2">$C(s)$</th><th colspan="3">Examples</th></tr>
<tr><td>(1, 1, 1)</td><td>intersect</td><td rowspan="4">support</td><td rowspan="4">$C(s) \neq s$</td><td>intersects</td><td rowspan="4">$n < m$</td><td>$n \equiv 1$, $m \equiv 0$</td><td>regular point</td></tr>
<tr><td>(1, 1, 2)</td><td>intersect</td><td>supports</td><td>$n \equiv m \equiv 1$</td><td>vertex</td></tr>
<tr><td>(2, 2, 1)</td><td>support</td><td>intersects</td><td>$n \equiv 0$, $m \equiv 1$</td><td>cusp of the first kind[3]</td></tr>
<tr><td>(2, 2, 2)</td><td>support</td><td>supports</td><td>$n \equiv m \equiv 0$</td><td>cusp of the second kind[3]</td></tr>
<tr><td>$(1, 1, 2)_0$</td><td>intersect</td><td>support</td><td colspan="2" rowspan="4">point-circle</td><td rowspan="4">$n > m$</td><td>$n \equiv m \equiv 1$</td><td rowspan="4"></td></tr>
<tr><td>$(1, 2, 1)_0$</td><td>intersect</td><td>intersect</td><td>$n \equiv 1$, $m \equiv 0$</td></tr>
<tr><td>$(2, 1, 1)_0$</td><td>support</td><td>intersect</td><td>$n \equiv 0$, $m \equiv 1$</td></tr>
<tr><td>$(2, 2, 2)_0$</td><td>support</td><td>support</td><td>$n \equiv m \equiv 0$</td></tr>
<tr><td>$(1, 1, \infty)$</td><td>intersect</td><td rowspan="2">support</td><td colspan="2" rowspan="2">neither intersects nor supports</td><td rowspan="2">$n < m$</td><td>$n \equiv 1$</td><td rowspan="2"></td></tr>
<tr><td>$(2, 2, \infty)$</td><td>support</td><td>$n \equiv 0$</td></tr>
</table>

[3]These names apply if we choose the infinite point on $C(s)$ but different from s.

Suppose that C_1, for instance, supports, while C_2 intersects A at s. Then some points of N lie in $\bar{C}_2$ and therefore in $\bar{C}_1$. Hence $N \subset s \cup \bar{C}_1$. Furthermore, from our assumption, there is a sequence of points s' which converge to s through C_2 and therefore through $\bar{C}_1 \cap \underline{C}_2$. Consequently $C(\tau, s') \subset s \cup (\bar{C}_1 \cap \underline{C}_2)$. Hence $C(\tau, s')$ cannot converge to $C(s) = s$ if s' converges to s. This is a contradiction.

9. A classification of the differentiable points. Sections 6 and 8 yield a classification of the differentiable interior points of A, as on page 517. The first eight examples refer to the curves $x = s^n$, $y = s^{n+m}$; the last two refer to $x = s^n$, $y = s^{n+m} \sin s^{-1}$. In all these cases we consider the point $s = 0$. Congruences are mod 2.

The *characteristic* (a_0, a_1, a_2), where a_0, $a_1 = 1$ or 2, and $a_2 = 1, 2$, or ∞, has the following properties: a_0 is even or odd according as the non-tangent circles through s support or intersect; $a_0 + a_1$ is even or odd according as the non-osculating tangent circles support or intersect; $a_0 + a_1 + a_2$ is even if $C(s)$ supports, odd if $C(s)$ intersects, while $a_2 = \infty$ if $C(s)$ neither supports nor intersects. We shall use the notation $(a_0, a_1, a_2)_0$ whenever $C(s)$ is the point-circle s (cf. §4).

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