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L-Functions for GSp(2) $\times$ GL(2): Archimedean Theory and Applications

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Abstract. Let Π be a generic cuspidal automorphic representation of GSp(2) defined over a totally real algebraic number field k whose archimedean type is either a (limit of) large discrete series representation or a certain principal series representation. Through explicit computation of archimedean local zeta integrals, we prove the functional equation of tensor product L-functions $L(s, \Pi \times \sigma)$ for an arbitrary cuspidal automorphic representation σ of GL(2). We also give an application to the spinor L-function of Π.

Introduction

Let G be a reductive algebraic group defined over a global field k, which is assumed to be split over k for simplicity. We fix an embedding $\iota\colon \widehat{\mathrm{G}} \hookrightarrow \mathrm{GL}(N, \mathbf{C})$ of the dual group $\widehat{\mathrm{G}}$ of G. Suppose that $\Pi = \bigotimes' \Pi_v$ (resp. $\sigma = \bigotimes' \sigma_v$) is a cuspidal automorphic representation of $\mathrm{G}_{\mathbf{A}_k}$ (resp. of $\mathrm{GL}(r)_{\mathbf{A}_k}$). Here $\mathbf{A}_k$ stands for the adele ring of k. Many mathematicians (*e.g.*, [G-PS-R]) have investigated the automorphic L-function $L(s, \Pi \times \sigma)$ arising from the tensor product representation

$$\iota \otimes \mathrm{id}\colon \widehat{\mathrm{G}} \times \mathrm{GL}(r, \mathbf{C}) \to \mathrm{GL}(Nr, \mathbf{C})$$

using the method of zeta integrals as well as the Langlands–Shahidi method. Recently the latter method has made marked progress and yielded functorial liftings of Π in several cases (see [Ki] for exposition). However, in order to know arithmetic properties of these L-functions, it seems necessary to do more detailed studies through zeta integrals, including careful analysis at the archimedean and ramified finite places. In this paper, we let $\mathrm{G} = \mathrm{GSp}(2)$ be the symplectic group with similitude of degree two, ι a natural inclusion to $\mathrm{GL}(4, \mathbf{C})$, and $r = 2$, and we investigate the L-function $L(s, \Pi \times \sigma)$ by means of the zeta integrals discovered by M. Novodvorsky in the mid-1970s [No-1, No-2]. The main goal of this paper is to prove the following.

Main Theorem (See Theorem 1.1 for the precise statement.) Suppose that k is a totally real algebraic number field. Moreover we assume the following:

- A.1 The cuspidal automorphic representation Π is globally generic, in the sense that Π has a non-vanishing Whittaker model.
- A.2 The local component Π_v at each real place v is equivalent to either a limit of large discrete series representation or an irreducible P_1-principal series representation.

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Then for an arbitrary cuspidal automorphic representation of $\mathrm{GL}(2)_{\mathbf{A}_k}$, the L-function $L(s, \Pi \times \sigma)$, originally defined for $\mathrm{Re}(s) \gg 0$, is continued to a meromorphic function on the whole s-plane and satisfies the expected functional equation. We also determine the possible poles of $L(s, \Pi \times \sigma)$.

We understand that our main results above together with the functorial lifting of Π to $\mathrm{GL}(4)$ can be obtained by the Langlands–Shahidi method (see [A-S-1, Section 5], [A-S-2]). Note that the description of the functorial lifting in [A-S-2] gives the location of possible poles mentioned above. We believe, however, that our concrete investigation through Novodvorsky's zeta integrals is of independent interest. To illustrate an interesting feature of the zeta integrals, we also consider the situation where a cusp form σ in the zeta integral is replaced by an Eisenstein series. Then, as in the case of the classical Rankin–Selberg convolution for $\mathrm{GL}(2) \times \mathrm{GL}(2)$ (cf. [Shm]), the global zeta integral is decomposed into a product of two spinor L-functions of Π. As an application, we give a new proof of the entireness of the spinor L-function in the "everything unramified" situation.

Now we explain the method of this paper in some detail. Although Novodvorsky's papers are rather sketchy, D. Bump [Bu] and D. Soudry [So-1] supply the missing details on the basic identity, the local functional equation, and the evaluation of the local zeta integrals at the unramified places (see also [G-PS-R, PS-S]). Hence our main task is to evaluate the local zeta integrals $Z^{(v)}(s, W, W', f)$ at each real place v, where W (resp. W') belongs to the local Whittaker model of Π_v (resp. of σ_v) and f is a section of a certain family of principal series representations on $\mathrm{GL}(2)_{k_v}$. To do the evaluation, we use the explicit formulae of local Whittaker functions on $G_{k_v} \cong \mathrm{GSp}(2, \mathbf{R})$ (and $\mathrm{GL}(2)_{k_v} \cong \mathrm{GL}(2, \mathbf{R})$) obtained in the previous papers [Mo-2] and [Mo-3] (cf. [O, Mi-O-1, Mi-O-2]). In fact, we did such computation for a special pair (Π_v, σ_v) in [Mo-2, Section 3]. In the case of more general pairs (Π_v, σ_v) (the "mixed weights case"), we have to choose (W, W', f) so as to make the integral $Z^{(v)}(s, W, W', f)$ computable and non-vanishing. An appropriate triplet (W, W', f) will be constructed by using certain elements in the universal enveloping algebra $U(\mathfrak{g}_v)$ of $\mathfrak{g}_v = \mathrm{Lie}(G_{k_v})$.

There are some related works to be mentioned here.

(i) Niwa [Ni] carried out the evaluation of the local zeta integral at the real place when both Π_v and σ_v are spherical principal series representations.

(ii) D. Soudry [So-2] (cf. [Wa, Section 7, p.361–362]) proposed another way of handling the archimedean local zeta integrals, based on the asymptotic expansion of local Whittaker functions.

(iii) Novodvorsky's zeta integral for $L(s, \Pi \times \sigma)$ works only when Π is globally generic. In particular, it cannot be applied to holomorphic cusp forms on $G_{\mathbf{A}_k}$. M. Furusawa [Fu] and B. Heim [He] discovered other kinds of zeta integrals for $L(s, \Pi \times \sigma)$ that are applicable to such cases. To deal with Heim's zeta integrals in the mixed weights case, S. Böcherer and B. Heim [Bo-He] use certain differential operators and get non-vanishing zeta integrals.

The organization of this paper is as follows. In Section 1, we introduce some basic ingredients of this paper and formulate our main results more precisely. In Sections 2 and 3, we recall some facts about Eisenstein series on $\mathrm{GL}(2)$ and Novodvorsky's zeta integrals for GSp(2) × GL(2), respectively. In Section 4, we write down the archimedean L- and ϵ-factors for the pair (Π_v, σ_v) determined by the Langlands parameters of Π_v and σ_v. Then, in Section 5, we compute the local zeta integrals at the real places to prove the main results. In the final section, we discuss the case where the cusp form on GL(2) is replaced by an Eisenstein series.

Notation

(i) Let k be a totally real algebraic number field. For a place v of k, we denote by k_v the completion of k at v. The module of an element $x \in \mathrm{k}_v$ is denoted by $|x|_v$ or simply by $|x|$. If v is a finite place, $\mathfrak{O}_v$ and $\mathfrak{P}_v = (\varpi_v)$ stand for the ring of integers in k_v and its prime ideal, respectively. We set $q_v := \sharp(\mathfrak{O}_v/\mathfrak{P}_v)$. We denote the adele ring of k and $\mathbf{Q}$ by $\mathbf{A}_\mathrm{k}$ and $\mathbf{A}$, respectively. The module of an element $x \in \mathbf{A}_\mathrm{k}$ is denoted by $|x|_{\mathbf{A}_\mathrm{k}}$. We fix a non-trivial character $\psi\colon \mathbf{A}_\mathrm{k}/\mathrm{k} \to \mathbf{C}^{(1)}$ by

$$\psi(x) := \mathbf{e}_\mathbf{A}((\mathrm{id} \otimes \mathrm{tr})(x)), \qquad x \in \mathbf{A}_\mathrm{k} \cong \mathbf{A} \otimes_\mathbf{Q} \mathrm{k}.$$

Here $\mathbf{e}_\mathbf{A}\colon \mathbf{A}/\mathbf{Q} \to \mathbf{C}^{(1)}$ is the non-trivial additive character of $\mathbf{A}/\mathbf{Q}$ characterized by $\mathbf{e}_\mathbf{A}(t_\infty) = \exp(2\pi\sqrt{-1}t_\infty)$ $(t_\infty \in \mathbf{R})$. For each place v of k, we denote by ψ_v the restriction of ψ to k_v. Let μ_v be the Haar measure on k_v which is self-dual with respect to ψ_v. If $k_v \cong \mathbf{R}$, then μ_v is the usual Lebesgue measure. For a finite place v, we have $\mu_v(\mathfrak{O}_v) = q_v^{-d_v/2}$, where d_v is the greatest integer such that ψ_v is trivial on $\mathfrak{P}_v^{-d_v}$. On the other hand, we normalize the Haar measure on the multiplicative group $\mathrm{k}_v^\times$ by $d^\times t := dt/|t|_v$ (resp. $d^\times t := (1 - q_v^{-1})^{-1} dt/|t|_v$) if v is a real place (resp. a finite place). As usual, we set $\Gamma_\mathbf{R}(s) := \pi^{-s/2}\Gamma(s/2)$ and $\Gamma_\mathbf{C}(s) := 2(2\pi)^{-s}\Gamma(s)$ $(s \in \mathbf{C})$, where $\Gamma(s)$ is the Gamma function. For a character $\chi\colon \mathrm{k}_v^\times \to \mathbf{C}^\times$, we denote Tate's L- and ϵ-factors by $L(s,\chi)$ and $\epsilon(s,\chi,\psi_v)$, respectively. Moreover we set

$$\gamma(s,\chi,\psi_v) := \epsilon(s,\chi,\psi_v) \times \frac{L(1-s,\chi^{-1})}{L(s,\chi)}.$$

(ii) Let G be the symplectic group with similitude of degree 2 defined over $\mathbf{Q}$:

$$\mathrm{G} = \mathrm{GSp}(2) := \{g \in \mathrm{GL}(4) \mid {}^t g J_4 g = \nu(g) J_4 \text{ for some } \nu(g) \in \mathbf{G}_m\},$$
$$J_4 = \begin{pmatrix} 0_2 & 1_2 \\ -1_2 & 0_2 \end{pmatrix}.$$

The center of G is given by $\mathrm{Z} := \{z1_4 \in \mathrm{G} \mid z \in \mathbf{G}_m\}$. Set $\mathrm{H} := \{h = (h_1,h_2) \in \mathrm{GL}(2) \times \mathrm{GL}(2) \mid \det(h_1) = \det(h_2)\}$ and regard it as an algebraic subgroup of G through the embedding

$$\mathrm{H} \ni h = (h_1,h_2) \mapsto \left(\begin{array}{cc|cc} a_1 & & b_1 & \\ & a_2 & & b_2 \\ \hline c_1 & & d_1 & \\ & c_2 & & d_2 \end{array}\right) \in \mathrm{G}, \qquad h_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}.$$

Fix a maximal unipotent subgroup N (resp. N^{H}) of G (resp. of H) by

$$\mathsf{N} := \left\{ n(x_0, x_1, x_2, x_3) := \left(\begin{array}{cc|cc} 1 & & x_1 & x_2 \\ & 1 & x_2 & x_3 \\ \hline & & 1 & \\ & & & 1 \end{array} \right) \left(\begin{array}{cc|cc} 1 & x_0 & & \\ & 1 & & \\ \hline & & 1 & \\ & & -x_0 & 1 \end{array} \right) \in \mathsf{G} \right\},$$
$$\mathsf{N}^{\mathsf{H}} := \mathsf{N} \cap \mathsf{H}.$$

We also define algebraic subgroups B' and N' by

$$\mathsf{B}' := \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \in \mathrm{GL}(2) \right\}, \quad \mathsf{N}' := \left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}(2) \right\}.$$

For a $\mathbf{Q}$-algebra R and an algebraic group L defined over $\mathbf{Q}$, we denote by L_R the group of R-valued points of L. We frequently write $\mathrm{GL}(2,R)$ and $\mathrm{GSp}(2,R)$ in place of $\mathrm{GL}(2)_R$ and $\mathrm{GSp}(2)_R$ if $R = \mathbf{R}, \mathbf{C}$. A maximal compact subgroup K (resp. K') of $\mathrm{GSp}(2)_{\mathbf{A}_\mathrm{k}}$ (resp. $\mathrm{GL}(2)_{\mathbf{A}_\mathrm{k}}$) is fixed as follows:

$$K := \prod_v K_v = \prod_{v:real} (O(4) \cap \mathrm{GSp}(2)_{\mathrm{k}_v}) \times \prod_{v<\infty} (\mathrm{GL}(4, \mathfrak{O}_v) \cap \mathrm{GSp}(2)_{\mathrm{k}_v});$$
$$K' := \prod_v K'_v = \prod_{v:real} O(2) \times \prod_{v<\infty} \mathrm{GL}(2, \mathfrak{O}_v).$$

1 Preliminaries and the Main Result

After recalling some representation theory of $\mathrm{GL}(2,\mathbf{R})$ and $\mathrm{GSp}(2,\mathbf{R})$, we shall state our main theorem.

1.1 Representations of $\mathrm{GL}(2,\mathbf{R})$

In this subsection, we introduce two kinds of (irreducible) admissible representations of $\mathrm{GL}(2,\mathbf{R})$.

(i) Let D_l ($l \in \mathbf{Z}$, $|l| \geq 1$) be the (limit of) discrete series representation of $\mathrm{SL}(2,\mathbf{R})$ with minimal $SO(2)$-type

$$SO(2) \ni r_\theta := \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \mapsto \exp(2\pi\sqrt{-1}l\theta) \in \mathbf{C}^{(1)}.$$

For each $c \in \mathbf{C}$ and $l \geq 1$, we denote by $D_l[c]$ the irreducible admissible representation σ of $\mathrm{GL}(2,\mathbf{R})$ characterized by $\sigma|_{\mathrm{SL}(2,\mathbf{R})} = D_l \oplus D_{-l}$, and $\sigma(z) = z^c$ ($z > 0$).

(ii) The principal series representation $I_{\mathsf{B}'_\mathbf{R}}^{\mathrm{GL}(2,\mathbf{R})}(\epsilon_1, \epsilon_2; c, \nu)$ of $\mathrm{GL}(2,\mathbf{R})$ is realized on the space of all C^∞-functions $f \colon \mathrm{GL}(2,\mathbf{R}) \to \mathbf{C}$ satisfying the relation

$$f(bh_2) = |b_1 b_2|^{c/2} \epsilon_1(b_1/|b_1|)\epsilon_2(b_2/|b_2|)|b_1/b_2|^{(\nu+1)/2} f(h),$$
$$\forall b = \begin{pmatrix} b_1 & * \\ 0 & b_2 \end{pmatrix} \in \mathsf{B}'_\mathbf{R}, \forall h_2 \in \mathrm{GL}(2,\mathbf{R}).$$

Here ϵ_i $(i = 1, 2)$ is a character of $\{\pm 1\}$ and $c, \nu \in \mathbf{C}$. The action of $\mathrm{GL}(2, \mathbf{R})$ on $I_{B'_{\mathbf{R}}}^{\mathrm{GL}(2)_{\mathbf{R}}}(\epsilon_1, \epsilon_2; c, \nu)$ is given by right translation. It is easily checked that $I_{B'_{\mathbf{R}}}^{\mathrm{GL}(2,\mathbf{R})}(\epsilon_1, \epsilon_2; c, \nu)$ is reducible if and only if $\nu \in \mathbf{Z}$, $\nu \neq 0$, and $(\epsilon_1 \cdot \epsilon_2)(-1) = (-1)^{\nu+1}$.

1.2 Representations of $\mathrm{GSp}(2, \mathbf{R})$

In this subsection, we introduce two kinds of (irreducible) admissible representations of $\mathrm{GSp}(2, \mathbf{R})$.

(i) Let $D_{(\lambda_1,\lambda_2)}$ be the (limit of) of large discrete series representation of $\mathrm{Sp}(2, \mathbf{R}) := \{g \in \mathrm{GSp}(2, \mathbf{R}) \mid \nu(g) = 1\}$ with Blattner parameter $(\lambda_1, \lambda_2) \in \mathbf{Z}^{\oplus 2}$, where (λ_1, λ_2) satisfies $1 - \lambda_1 \leq \lambda_2 \leq 0$ or $1 + \lambda_2 \leq -\lambda_1 \leq 0$. Here we refer the reader to [Mo-3, Subsection (1.2)] for the precise definition. For each $(\lambda_1, \lambda_2) \in \mathbf{Z}^{\oplus 2}$ satisfying $1 - \lambda_1 \leq \lambda_2 \leq 0$ and $c \in \mathbf{C}$, there exists an irreducible admissible representation $\pi = D_{(\lambda_1,\lambda_2)}[c]$ of $\mathrm{GSp}(2, \mathbf{R})$ characterized by

$$\pi|_{\mathrm{Sp}(2,\mathbf{R})} = D_{(\lambda_1,\lambda_2)} \oplus D_{(-\lambda_2,-\lambda_1)}, \qquad \pi(z) = z^c, \quad (z > 0).$$

(ii) We define the Jacobi parabolic subgroup P_1 of $\mathrm{GSp}(2, \mathbf{R})$ to be the stabilizer of the line $\mathbf{R} \cdot {}^t(1, 0, 0, 0)$ in $\mathrm{GSp}(2, \mathbf{R})$. We fix the Langlands decomposition $P_1 = M_1 A_1 N_1$ of P_1 as follows:

$$M_1 := \left\{ \left(\begin{array}{cc|cc} \epsilon_0 & & & \\ & \epsilon_0 & & \\ \hline & & 1 & \\ & & & 1 \end{array} \right) \left(\begin{array}{cc|cc} \epsilon_1 & & & \\ & a & & b \\ \hline & & \epsilon_1 & \\ & c & & d \end{array} \right) \;\middle|\; \epsilon_0, \epsilon_1 = \pm 1, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbf{R}) \right\};$$

$$A_1 := \{z\mathrm{diag}(a_1, 1, a_1^{-1}, 1) \mid z, a_1 > 0\}; \quad N_1 := \{n(x_0, x_1, x_2, 0) \mid x_i \in \mathbf{R}\}.$$

Let $(\sigma_{n,\pm}, V_{n,\pm})$ $(n \geq 1)$ be the irreducible unitary representation of M_1 characterized by $\sigma_{n,\pm}|_{\mathrm{SL}(2,\mathbf{R})} = D_n \oplus D_{-n}$ and $\sigma|_{\mathrm{SL}(2,\mathbf{R})}(\mathrm{diag}(-1, 1, -1, 1)) = \pm 1$. For $(c, \nu_1) \in \mathbf{C}^2$, we define a quasi-character $\chi_{(c,\nu_1)} \colon A_1 \to \mathbf{C}^\times$ by

$$\chi_{(c,\nu_1)}(z\mathrm{diag}(a_1, 1, a_1^{-1}, 1)) = z^c a_1^{\nu_1}.$$

Then the P_1-principal series representation $I(P_1; \sigma_{n,\pm}, c, \nu_1)$ of $\mathrm{GSp}(2, \mathbf{R})$ is realized on the space of all C^∞-functions $f \colon \mathrm{GSp}(2, \mathbf{R}) \longrightarrow V_{n,\pm}$ satisfying

$$f(mang) = \sigma_{n,\pm}(m)\chi_{(c,\nu_1+2)}(a) f(g)$$
$$\forall (m, a, n, g) \in M_1 \times A_1 \times N_1 \times \mathrm{GSp}(2, \mathbf{R}),$$

on which $\mathrm{GSp}(2, \mathbf{R})$ acts by right translation.

1.3 Main Results

Let $\Pi = \bigotimes' \Pi_v$ (resp. $\sigma = \bigotimes' \sigma_v$) be a cuspidal automorphic representation of $\mathrm{G}_{\mathbf{A}_k}$ (resp. of $\mathrm{GL}(2)_{\mathbf{A}_k}$) with *unitary* central character $\omega_\Pi \colon \mathbf{A}_k^\times \to \mathbf{C}^{(1)}$ (resp. $\omega_\sigma \colon \mathbf{A}_k^\times \to \mathbf{C}^{(1)}$). Throughout this paper, we make two fundamental assumptions on Π. In order

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to state our first assumption, we define the global Whittaker function $\mathcal{W}_F(g)$ attached to a cusp form $F \in \Pi$ by

$$\mathcal{W}_F(g) := \int_{\mathrm{N}_{\mathrm{k}}\backslash \mathrm{N}_{\mathbf{A}_{\mathrm{k}}}} F(ng)\psi_{\mathrm{N}}(n^{-1})dn, \quad g \in \mathrm{G}_{\mathbf{A}_{\mathrm{k}}}.$$

Here $\psi_{\mathrm{N}} \colon \mathrm{N}_{\mathbf{A}_{\mathrm{k}}} \to \mathbf{C}^{(1)}$ is a non-degenerate unitary character defined by

$$\psi_{\mathrm{N}}(n(x_0, x_1, x_2, x_3)) = \psi(-x_0 - x_3) \in \mathbf{C}^{(1)}.$$

Our first assumption is:

A.1 *For some cusp form F in Π, the global Whittaker function $\mathcal{W}_F$ attached to F does not vanish.*

Note that this assumption implies that the representation Π_v of $\mathrm{G}_{\mathrm{k}_v}$ for each place v can be realized as a subspace of

$$\{W \colon \mathrm{G}_{\mathrm{k}_v} \to \mathbf{C} \mid \text{smooth}, W(ng) = \psi_{\mathrm{N}}(n)W(g), \quad \forall (n, g) \in \mathrm{N}_{\mathrm{k}_v} \times \mathrm{G}_{\mathrm{k}_v}\}.$$

We denote this subspace by $\mathrm{Wh}(\Pi_v, \psi_v)$ and call it the local Whittaker model of Π_v (cf. [Mo-3, Section 3]). For a real place v, this means that the representation Π_v must be large in the sense of Vogan.

A.2 *For each real place v, the representation Π_v of $\mathrm{GSp}(2)_{\mathrm{k}_v} \cong \mathrm{GSp}(2, \mathbf{R})$ is equivalent to either $D_{(\lambda_1,\lambda_2)}[c]$ or an irreducible P_1-principal series representation $I(P_1; \sigma_{n,\pm}, c, \nu_1)$.*

On the other hand, we make no restrictions on $\sigma = \bigotimes' \sigma_v$. For a cusp form $\varphi \in \sigma$ on $\mathrm{GL}(2)_{\mathbf{A}_{\mathrm{k}}}$, the global Whittaker function $\mathcal{W}_\varphi(h_2)$ attached to φ is defined by

$$\mathcal{W}_\varphi(h_2) := \int_{\mathrm{k}\backslash \mathbf{A}_{\mathrm{k}}} \varphi\left(\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) h_2\right) \psi(-x)dx, \quad h_2 \in \mathrm{GL}(2)_{\mathbf{A}_{\mathrm{k}}}.$$

It is well known that $\mathcal{W}_\varphi(h_2)$ does not vanish for $\varphi \neq 0$. Hence each local component σ_v of a cuspidal automorphic representation σ can be realized as a subspace of

$$\left\{ W' \colon \mathrm{GL}(2)_{\mathrm{k}_v} \to \mathbf{C} \;\middle|\; \begin{array}{r} W' \text{ is smooth and } W'\left(\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) g\right) = \psi_v(x)W'(g) \\ \forall (x, g) \in \mathrm{k}_v \times \mathrm{GL}(2)_{\mathrm{k}_v} \end{array} \right\}.$$

We denote this subspace by $\mathrm{Wh}(\sigma_v, \psi_v)$ and call it a local Whittaker model of σ_v. For a real place v, the existence of local Whittaker models implies that σ_v is equivalent to one of the representations introduced in Subsection 1.1. For each finite place v, we define the local L-factor $L(s, \Pi_v \times \sigma_v)$ and ϵ-factor $\epsilon(s, \Pi_v \times \sigma_v, \psi_v)$ as in Subsection 3.3 below. For almost all finite places v, $L(s, \Pi_v \times \sigma_v)$ are Euler factors of degree eight. On the other hand, for each real place v, we define $L(s, \Pi_v \times \sigma_v)$ and $\epsilon(s, \Pi_v \times \sigma_v, \psi_v)$ through the Langlands parameters of Π_v and σ_v, whose explicit forms are given in Section 4. Then we have the L-function $L(s, \Pi \times \sigma)$ for $\Pi \times \sigma$ and its ϵ-factor $\epsilon(s, \Pi \times \sigma)$ by

$$L(s, \Pi \times \sigma) := \prod_v L(s, \Pi_v \times \sigma_v), \quad \epsilon(s, \Pi \times \sigma) := \prod_v \epsilon(s, \Pi_v \times \sigma_v, \psi_v),$$

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where v runs through all the places of k. Next we set

$$\Pi^\vee := \{\widetilde{F} \mid F \in \Pi\}, \qquad \widetilde{F}(g) := \omega_\Pi(\nu(g))^{-1}F(g),$$
$$\sigma^\vee := \{\widetilde{\varphi} \mid \varphi \in \sigma\}, \qquad \widetilde{\varphi}(h) := \omega_\sigma(\det(h))^{-1}\varphi(h).$$

Then the cuspidal automorphic representations $\Pi^\vee = \bigotimes' \Pi_v^\vee$ and $\sigma^\vee = \bigotimes' \sigma_v^\vee$ are the contragredient representations of Π and σ, respectively. Since $\Pi^\vee$ also satisfies the assumptions A.1 and A.2, we can define the L-function for $\Pi^\vee \times \sigma^\vee$:

$$L(s, \Pi^\vee \times \sigma^\vee) := \prod_v L(s, \Pi_v^\vee \times \sigma_v^\vee).$$

Now we can state our main results.

Theorem 1.1 *Under the assumptions* A.1 *and* A.2*, we have the following assertions:*

(i) *The L-function $L(s, \Pi \times \sigma)$ is continued to a meromorphic function on the whole s-plane and satisfies the following functional equation*

$$L(s, \Pi \times \sigma) = \epsilon(s, \Pi \times \sigma) \times L(1 - s, \Pi^\vee \times \sigma^\vee).$$

(ii) *If $\omega_\Pi \cdot \omega_\sigma = |\cdot|_{\mathbf{A}_k}^\lambda$ with some $\lambda \in \sqrt{-1}\mathbf{R}$, then the possible poles of $L(s, \Pi \times \sigma)$ are located at $s = -\lambda/2, -\lambda/2 + 1$, and are at most simple. Otherwise, $L(s, \Pi \times \sigma)$ is an entire function.*

2 A Review of the Eisenstein Series on GL(2)

In this section, we review some of the basics about the Eisenstein series on GL(2), which will be used to define the global zeta integral in the next section. The main reference is [J-2].

2.1 Induced Representation (Local)

In this subsection, we fix a finite or an infinite place v of k. Let $\chi_i \colon k_v^\times \to \mathbf{C}^{(1)}$ $(i = 1, 2)$ be two unitary characters of $k_v^\times$. We define the principal series representation $I(\chi_1, \chi_2)$ by

$$I(\chi_1, \chi_2) := \{f \colon \mathrm{GL}(2)_{k_v} \to \mathbf{C} \mid f(bh_1) = \chi_1(b_1)\chi_2(b_2)f(h_1),$$
$$\forall b = \left(\begin{smallmatrix} b_1 & * \\ 0 & b_2 \end{smallmatrix}\right) \in \mathrm{B}'_{k_v},\ \forall h_1 \in \mathrm{GL}(2)_{k_v}\}.$$

The action of $\mathrm{GL}(2)_{k_v}$ on $I(\chi_1, \chi_2)$ is given by right translation. We regard the disjoint union

$$I(\chi_1, \chi_2; s) := \bigsqcup_{s \in \mathbf{C}} I(\chi_1 |\cdot|^s, \chi_2 |\cdot|^{-s})$$

as a *vector bundle* on the base space $\mathbf{C}$.

Definition 2.1 (i) A function $f\colon \mathbf{C}\times \mathrm{GL}(2)_{\mathrm{k}_v} \to \mathbf{C}$ is said to be a *standard section* of $I(\chi_1,\chi_2;s)$ if the following two conditions hold:

(a) for each fixed $s\in\mathbf{C}$, the function $\mathrm{GL}(2)_{\mathrm{k}_v}\ni h_1\mapsto f(s,h_1)\in\mathbf{C}$ is a right K_v'-finite vector in the principal series representation $I(\chi_1|\cdot|^s,\chi_2|\cdot|^{-s})$;

(b) the functions $K_v'\ni u\mapsto f(s,u)\in\mathbf{C}$ are independent of $s\in\mathbf{C}$.

We denote the totality of standard sections by $\mathcal{I}^{std}(\chi_1,\chi_2;s)$.

(ii) A function $f\colon \mathbf{C}\times\mathrm{GL}(2)_{\mathrm{k}_v}\to\mathbf{C}\cup\{\infty\}$ is said to be a *holomorphic* (resp. *meromorphic*) *section* of $I(\chi_1,\chi_2;s)$ if there exist standard sections $f_i\in\mathcal{I}^{std}(\chi_1,\chi_2;s)$ and holomorphic (resp. meromorphic) functions $a_i(s)$ on $\mathbf{C}$ $(1\le i\le N)$ such that

$$f(s,h_1)=\sum_{i=1}^{N}a_i(s)f_i(s,h_1).$$

Next we introduce a useful class of meromorphic sections, which we call *Jacquet sections*. If v is a finite place (resp. a real place), then we define the space $\mathcal{S}(\mathrm{k}_v^2)$ of Schwartz–Bruhat functions on k_v^2 by

$$\mathcal{S}(\mathrm{k}_v^2):=\{\Phi\colon \mathrm{k}_v^2\to\mathbf{C}\mid \text{locally constant, the support of }\Phi\text{ is compact}\},$$

$$(\text{resp. }\mathcal{S}(\mathrm{k}_v^2):=\{\Phi(x,y)=P(x,y)\exp(-\pi(x^2+y^2))\mid P\in\mathbf{C}[X,Y]\}).$$

For each $\Phi\in\mathcal{S}(\mathrm{k}_v^2)$, we set

$$f_\Phi^{(v)}(s,h_1):=\chi_1(\det(h_1))\,|\det(h_1)|_v^s\int_{\mathrm{k}_v^\times}\Phi((0,t)h_1)(\chi_1\cdot\chi_2^{-1})(t)|t|^{2s}d^\times t.$$

Proposition 2.2 *The integral $f_\Phi^{(v)}(s,h_1)$ converges absolutely for $\mathrm{Re}(s)>0$ and defines a meromorphic section of $I(\chi_1,\chi_2;s)$ by analytic continuation. If $\chi_1\cdot\chi_2^{-1}=|\cdot|^\lambda$ for some $\lambda\in\sqrt{-1}\mathbf{R}$, then $f_\Phi^{(v)}(s,h_1)$ is holomorphic except for having possible simple poles at*

$$s\in\frac{-\lambda}{2}+\frac{\pi\sqrt{-1}\mathbf{Z}}{\log q_v}.$$

Otherwise, $f_\Phi^{(v)}(s,h_1)$ is a holomorphic section.

We call $f_\Phi^{(v)}(s,h_1)$ the Jacquet section attached to Φ. We denote the totality of holomorphic sections (resp. that of meromorphic sections, that of Jacquet sections) by $\mathcal{I}^{\mathrm{holo}}(\chi_1,\chi_2;s)$, (resp. $\mathcal{I}^{\mathrm{mero}}(\chi_1,\chi_2;s)$, $\mathcal{I}^J(\chi_1,\chi_2;s)$).

2.2 Local Intertwining Operators

In this subsection, we recall some basic facts on local intertwining integrals for $\mathrm{GL}(2)_{\mathrm{k}_v}$.

Proposition 2.3 *For a meromorphic section $f(s,h_1)\in\mathcal{I}^{\mathrm{mero}}(\chi_1,\chi_2;s)$, we define the local intertwining integral $M_vf(s,h_1)$ by*

$$M_vf(s,h_1):=\int_{\mathrm{k}_v}f(s,w_0\begin{pmatrix}1&x\\0&1\end{pmatrix}h_1)dx,\qquad w_0:=\begin{pmatrix}0&1\\-1&0\end{pmatrix}.$$

If $\mathrm{Re}(s) > 1/2$*, then the integral* $M_v f(s, h_1)$ *converges absolutely except for the poles of* $f(s, h_1)$*. Moreover,* $M_v f(s, h_1)$ *defines a meromorphic section of* $I(\chi_2, \chi_1; 1-s)$ *by analytic continuation.*

By Proposition 2.3, we have a linear map

$$M_v \colon \mathcal{I}^{mero}(\chi_1, \chi_2; s) \to \mathcal{I}^{mero}(\chi_2, \chi_1; 1-s).$$

In order to describe the behavior of Jacquet sections under the intertwining operator M_v, we define the Fourier transform $\widehat{\Phi} \in S(\mathrm{k}_v^2)$ of a Schwartz–Bruhat function $\Phi \in S(\mathrm{k}_v^2)$ by

$$\widehat{\Phi}(u, v) := \int_{\mathrm{k}_v^2} \Phi(x, y)\psi_v(-yu + xv)dxdy.$$

Moreover we set

$$\widetilde{f_{\widehat{\Phi}}}(s, h_1) := \chi_2(\det(h_1))|\det(h_1)|_v^{1-s} \int_{\mathrm{k}_v^\times} \widehat{\Phi}((0,t)h_1)(\chi_2 \cdot \chi_1^{-1})(t)|t|^{2-2s}d^\times t.$$

Then we have the following.

Proposition 2.4 **([J-2, pp. 14-15])** *The image* $M_v f_\Phi$ *of a Jacquet section* $f_\Phi \in \mathcal{I}^J(\chi_1, \chi_2, s)$ *is given by*

$$(\chi_1 \cdot \chi_2^{-1})(-1) \times \gamma(2s-1, \chi_1 \cdot \chi_2^{-1}, \psi_v)^{-1} \times \widetilde{f_{\widehat{\Phi}}}(s, h_1). \tag{2.1}$$

Taking (2.1) into consideration, we define the normalized intertwining operator M_v^* by

$$M_v^* = (\chi_1 \cdot \chi_2^{-1})(-1)\gamma(2s-1, \chi_1 \cdot \chi_2^{-1}, \psi_v)M_v \colon \mathcal{I}^J(\chi_1, \chi_2; s) \to \mathcal{I}^J(\chi_2, \chi_1; 1-s).$$

We need some examples of local Jacquet sections:

Example (i) Let v be a finite place such that ψ_v is unramified (*i.e.,* $d_v = 0$). Suppose that the characters χ_i ($i = 1, 2$) are both unramified (*i.e.,* the restriction of χ_i to $\mathfrak{O}_v^\times$ is trivial). Let $\Phi_v^0(x, y) \in \mathcal{S}(\mathrm{k}_v^2)$ be the characteristic function of $\mathfrak{O}_v^2$. We define a Jacquet section $f_{J,0}^{(v)}(s, h_1)$ by $f_{J,0}^{(v)}(s, h_1) := f_{\Phi_v^0}^{(v)}(s, h_1)$. Then we have

$$f_{J,0}^{(v)}(s, u) = L(2s, \chi_1 \cdot \chi_2^{-1}) = [1 - \chi_1 \cdot \chi_2^{-1}(\varpi_v)q_v^{-2s}]^{-1}, \qquad u \in K_v'.$$

Since $\widehat{\Phi}_v^0 = \Phi_v^0$, we know that $M_v^* f_{\Phi_v^0}(s, h_1) = \widetilde{f}_{\Phi_v^0}(s, h_1)$.

(ii) Suppose that $\mathrm{k}_v \cong \mathbf{R}$. We define a basis $\{\Phi_m(x, y) \mid m \in \mathbf{Z}\}$ of $\mathcal{S}(\mathrm{k}_v^2)$ as follows:

$$\Phi_m(x, y) := \begin{cases} (-x\sqrt{-1} + y)^m \exp(-\pi(x^2 + y^2)) & \text{if } m \geq 0; \\ (x\sqrt{-1} + y)^{-m} \exp(-\pi(x^2 + y^2)) & \text{if } m \leq 0. \end{cases}$$

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We define Jacquet sections $f^{(v)}_{J,m}(s,h_1)$ and $\widetilde{f}^{(v)}_{J,m}(s,h_1)$ by $f^{(v)}_{J,m}(s,h_1) := f^{(v)}_{\Phi_m}(s,h_1)$ and $\widetilde{f}^{(v)}_{J,m}(s,h_1) := \widetilde{f}^{(v)}_{\Phi_m}(s,h_1)$. Then it is easy to see that

$$f^{(v)}_{J,m}\left(s, \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\right) = \\ e^{\sqrt{-1}m\theta}\Gamma_{\mathbf{R}}(2s+|m|+c_v) \times \begin{cases} 1 & \text{if } (-1)^m = (\chi_1\cdot\chi_2^{-1})(-1); \\ 0 & \text{otherwise.} \end{cases}$$

Here the constant $c_v \in \sqrt{-1}\mathbf{R}$ is defined by $(\chi_1\cdot\chi_2)(t) = t^{c_v}$ $(t>0)$. Since $\widehat{\Phi}_m = \Phi_m$ $(m \geq 0)$ and $\widehat{\Phi}_m = (-1)^m\Phi_m$ $(m \leq 0)$, we know that

$$[M_v^* f^{(v)}_{J,m}](s,h_1) = \begin{cases} \widetilde{f}^{(v)}_{J,m}(s,h_1) & \text{if } m \geq 0; \\ (-1)^m \widetilde{f}^{(v)}_{J,m}(s,h_1) & \text{if } m \leq 0. \end{cases}$$

2.3 Eisenstein Series

In this subsection, we fix a unitary idele class character $\omega\colon \mathbf{A}_k^\times/k^\times \to \mathbf{C}^{(1)}$ and denote the restriction of ω to $k_v^\times$ by $\omega_v\colon k_v^\times \to \mathbf{C}^{(1)}$. Let $S_{\omega,\psi}$ be a finite set of places of k consisting of (i) all the real places and (ii) all the finite places v such that ω_v is ramified or $d_v > 0$. We define the space $\mathcal{S}(\mathbf{A}_k^2)$ of Schwartz–Bruhat functions on $\mathbf{A}_k^2$ as the restricted tensor product $\bigotimes' \mathcal{S}(k_v^2)$ with respect to $\{\Phi_v^0 | v \notin S_{\omega,\psi}\}$. For any $\Phi \in \mathcal{S}(\mathbf{A}_k^2)$, we set

(2.2)

$$f_\Phi(s,h_1) := |\det(h_1)|^s_{\mathbf{A}_k} \int_{\mathbf{A}_k^\times} \Phi((0,t)h_1)\omega(t)|t|^{2s}_{\mathbf{A}_k} d^\times t, \qquad (s,h_1) \in \mathbf{C}\times \mathrm{GL}(2)_{\mathbf{A}_k}.$$

We call $f_\Phi(s,h_1)$ the global Jacquet section attached to Φ. Then the totality $\mathcal{I}^J(1,\omega^{-1};s)$ of global Jacquet sections can be identified with the restricted tensor product $\otimes'_v \mathcal{I}^J(1,\omega_v^{-1};s)$ with respect to $\{f^{(v)}_{J,0} \mid v \notin S_{\omega,\psi}\}$. For a global Jacquet section $f \in \mathcal{I}^J(1,\omega^{-1};s)$, we define the Eisenstein series $E(s,h_1;f)$ on $\mathrm{GL}(2)_{\mathbf{A}_k}$ by

$$E(s,h_1;f) := \sum_{\gamma \in \mathrm{B}'_k\backslash \mathrm{GL}(2)_k} f(s,\gamma h_1).$$

This converges absolutely and uniformly on every compact subset in $\mathrm{Re}(s) > 1$ except for the poles of $f(s,h_1)$. For each $h_1 \in \mathrm{GL}(2)_{\mathbf{A}_k}$, $E(s,h_1;f)$ is continued to a meromorphic function on the whole s-plane and defines an automorphic form on $\mathrm{GL}(2)_{\mathbf{A}_k}$ except for its poles. Similarly, we set

$$E(s,h_1;Mf) := \sum_{\gamma \in \mathrm{B}'_k\backslash \mathrm{GL}(2)_k} Mf(s,\gamma h_1), \quad \forall f(s,h_1) \in \mathcal{I}^J(1,\omega^{-1};s),$$

where M stands for the global intertwining operator defined by

$$Mf(s,h_1) := \int_{\mathbf{A}_k} f(s, w_0 \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} h_1)dx.$$

Note that $M = \bigotimes_v' M_v = \bigotimes_v' M_v^*$. Then $E(s, h_1; Mf)$ converges absolutely for $\mathrm{Re}(s) < 0$ except for the poles of $Mf(s, h_1)$ and is continued to a meromorphic function on the whole s-plane. The most basic analytic properties of the Eisenstein series $E(s, h_1; f)$ are summarized as follows:

Proposition 2.5 **(c.f. [J-2, p. 119], [J-S, Lemma 4.2])** (i) *For a Schwartz–Bruhat function $\Phi \in \mathcal{S}(\mathbf{A}_k^2)$ we have $E(s, h_1; f_\Phi) = E(s, h_1; Mf_\Phi)$.*

(ii) *If $\omega \neq |\cdot|^\lambda$ for any $\lambda \in \sqrt{-1}\mathbf{R}$, then the Eisenstein series $E(s, h_1; f_\Phi)$ is an entire function in $s \in \mathbf{C}$.*

(iii) *If $\omega = |\cdot|^\lambda$ for some $\lambda \in \sqrt{-1}\mathbf{R}$, then there exists a constant $c > 0$ such that*

$$E(s, h_1; f_\Phi) = \frac{c|\det(h_1)|^{(s+\lambda/2)-1}\widehat{\Phi}(0)}{(s+\lambda/2)-1} - \frac{c|\det(h_1)|^{-(s+\lambda/2)}\Phi(0)}{(s+\lambda/2)} + R(s, h_1).$$

Here $R(s, h_1)$ is an entire function in $s \in \mathbf{C}$ for each $h_1 \in \mathrm{GL}(2)_{\mathbf{A}_k}$.

3 Zeta Integrals and the Basic Identity

In this section, we recall Novodvorsky's zeta integrals, which express the L-functions for generic cusp forms on GSp(2) × GL(2). Basic references are [Bu, Section 3], [No-1, No-2], and [So-1].

3.1 Global Zeta Integrals

Recall that Π (resp. σ) is a cuspidal automorphic representation of $\mathrm{G}_{\mathbf{A}_k}$ (resp. of $\mathrm{GL}(2)_{\mathbf{A}_k}$). Set $\omega := \omega_\Pi \cdot \omega_\sigma$. For a pair $(F, \varphi) \in \Pi \times \sigma$ of cusp forms and a global Jacquet section $f_\Phi \in \mathcal{I}^J(1, \omega^{-1}; s)$, we define the global zeta integrals $Z(s, F \otimes \varphi, f_\Phi)$ and $\widetilde{Z}(s, F \otimes \varphi, f_\Phi)$ by

$$Z(s, F \otimes \varphi, f_\Phi) := \int_{\mathrm{Z}_{\mathbf{A}_k}\mathrm{H}_k \backslash \mathrm{H}_{\mathbf{A}_k}} F(h)\varphi(h_2)E(s, h_1; f_\Phi)dh \quad \text{and}$$

$$\widetilde{Z}(s, F \otimes \varphi, f_\Phi) := \int_{\mathrm{Z}_{\mathbf{A}_k}\mathrm{H}_k \backslash \mathrm{H}_{\mathbf{A}_k}} \widetilde{F}(h)\widetilde{\varphi}(h_2)\omega(\det(h_1))E(s, h_1; Mf_\Phi)dh,$$

respectively. The two integrals $Z(s, F \otimes \varphi, f_\Phi)$ and $\widetilde{Z}(s, F \otimes \varphi, f_\Phi)$ converge absolutely except for the poles of the Eisenstein series and define meromorphic functions in $s \in \mathbf{C}$. In view of Proposition 2.5, we have the following.

Proposition 3.1 (i) *We have the functional equation*

$$\widetilde{Z}(s, F \otimes \varphi, f_\Phi) = Z(s, F \otimes \varphi, f_\Phi).$$

(ii) *If $\omega \neq |\cdot|^\lambda$ for any $\lambda \in \sqrt{-1}\mathbf{R}$, then the zeta integral $Z(s, F \otimes \varphi, f_\Phi)$ is an entire function in $s \in \mathbf{C}$.*

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(iii) *If* $\omega = |\cdot|^{\lambda}$ *for some* $\lambda \in \sqrt{-1}\mathbf{R}$*, then* $Z(s, F \otimes \varphi, f_{\Phi})$ *is holomorphic except for possible simple poles at* $s = 1 - \lambda/2, -\lambda/2$*. Moreover we have*

$$\mathrm{Res}_{s=1-\lambda/2} Z(s, F \otimes \varphi, f_{\Phi}) = c \times \widehat{\Phi}(0) \times \int_{\mathrm{Z}_{\mathbf{A}_k} \mathrm{H}_k \backslash \mathrm{H}_{\mathbf{A}_k}} F(h)\varphi(h_2)dh;$$

$$\mathrm{Res}_{s=-\lambda/2} Z(s, F \otimes \varphi, f_{\Phi}) = -c \times \Phi(0) \times \int_{\mathrm{Z}_{\mathbf{A}_k} \mathrm{H}_k \backslash \mathrm{H}_{\mathbf{A}_k}} F(h)\varphi(h_2)dh.$$

Here the constant $c > 0$ *is the same as that in Proposition* 2.5.

3.2 Local Zeta Integrals and the Basic Identity

Now we suppose that the cusp forms F and φ are decomposable in the restricted tensor product $\Pi = \bigotimes' \Pi_v$ and $\sigma = \bigotimes' \sigma_v$, respectively. Then the local multiplicity one property of Whittaker models ([Sha, Theorem 3.1], [Wa, Theorem 8.8(1)], [Ro, Theorem 3]) implies that $\mathcal{W}_F$ (resp. $\mathcal{W}_\varphi$) is decomposed into a product of local Whittaker functions:

$$\mathcal{W}_F(g) = \prod_v \mathcal{W}_F^{(v)}(g_v), \qquad \text{for } g = (g_v) \in \mathrm{G}_{\mathbf{A}_k},$$

$$\mathcal{W}_\varphi(h_2) = \prod_v \mathcal{W}_\varphi^{(v)}(h_{2,v}), \qquad \text{for } h_2 = (h_{2,v}) \in \mathrm{GL}(2)_{\mathbf{A}_k}.$$

Here $\mathcal{W}_F^{(v)}(g_v)$ (resp. $\mathcal{W}_\varphi^{(v)}(h_{2,v})$) belongs to the local Whittaker model $\mathsf{Wh}(\Pi_v, \psi_v)$ (resp. $\mathsf{Wh}(\sigma_v, \psi_v)$). Moreover we take a global Jacquet section $f \in \mathcal{I}^J(1, \omega^{-1}; s)$ of the form

$$f(s, h_1) = \prod_v f^{(v)}(s, h_{1,v}), \qquad f^{(v)} \in \mathcal{I}^J(1, \omega_v^{-1}; s).$$

For each place v of k, we define the two local zeta integrals $Z^{(v)}(s, \mathcal{W}_F^{(v)}, \mathcal{W}_\varphi^{(v)}, f^{(v)})$ and $\widetilde{Z}^{(v)}(s, \mathcal{W}_F^{(v)}, \mathcal{W}_\varphi^{(v)}, f^{(v)})$ by

$$Z^{(v)}(s, \mathcal{W}_F^{(v)}, \mathcal{W}_\varphi^{(v)}, f^{(v)}) := \int_{\mathrm{Z}_{k_v} \mathrm{N}^{\mathrm{H}}{}_{k_v} \backslash \mathrm{H}_{k_v}} \mathcal{W}_F^{(v)}(h_v)\mathcal{W}_\varphi^{(v)}(h_{2,v}) f^{(v)}(s, h_{1,v}) dh_v$$

and

$$\begin{aligned}\widetilde{Z}^{(v)}(s, \mathcal{W}_F^{(v)}, \mathcal{W}_\varphi^{(v)}, f^{(v)}) &:= \int_{\mathrm{Z}_{k_v} \mathrm{N}^{\mathrm{H}}{}_{k_v} \backslash \mathrm{H}_{k_v}} \mathcal{W}_F^{(v)}(h_v)\mathcal{W}_\varphi^{(v)}(h_{2,v}) M_v^* f^{(v)}(s, h_{1,v}) dh_v \\ &= \int_{\mathrm{Z}_{k_v} \mathrm{N}^{\mathrm{H}}{}_{k_v} \backslash \mathrm{H}_{k_v}} \widetilde{\mathcal{W}}_F^{(v)}(h_v)\widetilde{\mathcal{W}}_\varphi^{(v)}(h_{2,v}) \omega_v(\det(h_{1,v})) M_v^* f^{(v)}(s, h_{1,v}) dh_v,\end{aligned}$$

where we set

$$\widetilde{\mathcal{W}}_F^{(v)}(g_v) := \omega_\Pi(\nu(g_v))^{-1} \mathcal{W}_F^{(v)}(g_v), \qquad \widetilde{\mathcal{W}}_\varphi^{(v)}(h_{2,v}) := \omega_\sigma(\det(h_{2,v}))^{-1} \mathcal{W}_\varphi^{(v)}(h_{2,v}).$$

Here we remark that $\omega_v(\det(h_{1,v})) M_v^* f^{(v)}(s, h_{1,v}) \in \mathcal{I}^J(1, \omega_v; 1 - s)$.

By an unfolding procedure, we have the following basic identity (cf. [Bu, Section 3]).

Proposition 3.2 (i) *Suppose that the local zeta integral* $Z^{(v)}(s, \mathcal{W}_F^{(v)}, \mathcal{W}_\varphi^{(v)}, f^{(v)})$ *at each real place* v *converges absolutely for* $\mathrm{Re}(s) > e_\infty$. *Then the integral*

$$\int_{Z_{\mathbb{A}_k} N^H{}_{\mathbb{A}_k} \backslash H_{\mathbb{A}_k}} \mathcal{W}_F(h) \mathcal{W}_\varphi(h_2) f(s, h_1) dh$$

converges absolutely for $\mathrm{Re}(s) > \max\{3, e_\infty\}$ *(except for the poles of* f*) and is equal to* $Z(s, F \otimes \varphi, f)$.

(ii) *Suppose that the local zeta integral* $\widetilde{Z}^{(v)}(s, \mathcal{W}_F^{(v)}, \mathcal{W}_\varphi^{(v)}, f^{(v)})$ *at each real place* v *converges absolutely for* $\mathrm{Re}(s) < e'_\infty$. *Then the integral*

$$\int_{Z_{\mathbb{A}_k} N^H{}_{\mathbb{A}_k} \backslash H_{\mathbb{A}_k}} \mathcal{W}_F(h) \mathcal{W}_\varphi(h_2) M f(s, h_1) dh$$

converges absolutely for $\mathrm{Re}(s) < \min\{-2, e'_\infty\}$ *(except for the poles of* Mf*) and is equal to* $\widetilde{Z}(s, F \otimes \varphi, f)$.

The above proposition implies that

$$Z(s, F \otimes \varphi, f) = \prod_v Z^{(v)}(s, \mathcal{W}_F^{(v)}, \mathcal{W}_\varphi^{(v)}, f^{(v)});$$
$$\widetilde{Z}(s, F \otimes \varphi, f) = \prod_v \widetilde{Z}^{(v)}(s, \mathcal{W}_F^{(v)}, \mathcal{W}_\varphi^{(v)}, f^{(v)}).$$

3.3 The Local Functional Equation at the Finite Places

D. Soudry [So-1, Section 2] proved the following local functional equations at the finite places.

Proposition-Definition 3.3 *Suppose that* v *is a finite place of* k. *Let* Π_v *(resp.* σ_v*) be an irreducible admissible representation of* G_{k_v} *(resp. of* $GL(2)_{k_v}$*) which has a local Whittaker model* $\mathrm{Wh}(\Pi_v, \psi_v)$ *(resp.* $\mathrm{Wh}(\sigma_v, \psi_v)$*).*

(i) *There exists a unique polynomial* $P_v(X) \in \mathbf{C}[X]$ *such that* $P_v(0) = 1$ *and that*

$$\mathbf{C}\text{-span}\{Z^{(v)}(s, W, W', f^{(v)}) \mid W \in \mathrm{Wh}(\pi_v, \psi_v), W' \in \mathrm{Wh}(\sigma_v, \psi_v), f^{(v)} \in \mathcal{I}^J(1, \omega_v^{-1}; s)\} = P_v(q_v^{-s})^{-1} \mathbf{C}[q_v^{-s}, q_v^s].$$

We define the L-factor $L(s, \Pi_v \times \sigma_v)$ *for the pair* (Π_v, σ_v) *by*

$$L(s, \Pi_v \times \sigma_v) := P_v(q_v^{-s})^{-1}.$$

(ii) *There exists a monomial* $\epsilon(s, \Pi_v \times \sigma_v, \psi_v) = a q_v^{-bs}$ $(a \in \mathbf{C}^\times, b \in \mathbf{Z})$ *in* q_v^{-s} *such that the equation*

$$\frac{\widetilde{Z}^{(v)}(s, W, W', f)}{L(1 - s, \Pi_v^\vee \times \sigma_v^\vee)} = \epsilon(s, \Pi_v \times \sigma_v, \psi_v) \times \frac{Z^{(v)}(s, W, W', f)}{L(s, \Pi_v \times \sigma_v)}$$

holds for all $(W, W', f) \in \mathrm{Wh}(\Pi_v, \psi_v) \times \mathrm{Wh}(\sigma_v, \psi_v) \times \mathcal{I}^J(1, \omega_v^{-1}; s)$.

3.4 Unramified Computation

In this subsection, we recall the computation of the local zeta integrals at the unramified places (cf. [Bu, G-PS-R]). Let $S_{\Pi,\sigma,\psi}$ be a finite set of places of k satisfying the following condition: if $v \notin S_{\Pi,\sigma,\psi}$, then v is a finite place such that ψ_v is unramified and the representation of Π_v (resp. σ_v) has a vector fixed by K_v (resp. by K'_v). Then we have $S_{\omega,\psi} \subset S_{\Pi,\sigma,\psi}$. Fix a place $v \notin S_{\Pi,\sigma,\psi}$. Since we have assumed that Π_v and σ_v have a local Whittaker model, the representations Π_v and σ_v are equivalent to some unramified principal series representations ([Ba-M, Li, Re]). Let $W^0 \in \mathrm{Wh}(\Pi_v, \psi_v)$ (resp. ${W'}^0 \in \mathrm{Wh}(\sigma_v, \psi_v)$) be the K_v-fixed local Whittaker function normalized so that $W^0(I_4) = 1$ (resp. ${W'}^0(I_2) = 1$). Then the local zeta integral $Z^{(v)}(s, W^0, {W'}^0, f^{(v)}_{J,0})$ equals

$$L(2s,\omega_v) \times \sum_{k,l\geq 0} W^0(\mathrm{diag}(\varpi_v^{k+l}, \varpi_v^l, \varpi_v^{-k}, 1)){W'}^0(\mathrm{diag}(\varpi_v^l, 1))\omega_v(\varpi_v^k) q_v^{-(2k+l)s+2(k+l)}.$$

Suppose that the Satake parameter A_v (resp. B_v) of Π_v (resp. of σ_v) is given by

$$A_v = \mathrm{diag}(\alpha_0\alpha_1, \alpha_0\alpha_2, \alpha_1^{-1}, \alpha_2^{-1}) \in \widehat{\mathrm{G}} := \mathrm{GSp}(2, \mathbf{C}),$$
$$B_v = \mathrm{diag}(\beta_1, \beta_2) \in \mathrm{GL}(2, \mathbf{C}),$$

where $\widehat{\mathrm{G}}$ (resp. $\mathrm{GL}(2,\mathbf{C})$) is the dual group of G (resp. of GL(2)) (cf. Subsection 4.3 below). Then, by virtue of Kato–Casselman–Shalika formulae [C-S], we have

$$W^0(\mathrm{diag}(\varpi_v^{k+l}, \varpi_v^l, \varpi_v^{-k}, 1)) = \begin{cases} \alpha_0^l \times q_v^{(-4k-3l)/2} \times \dfrac{\sum_{\sigma\in\mathfrak{S}_2}\sum_{\epsilon_j=\pm1} \mathrm{sgn}(\sigma)\epsilon_1\epsilon_2 \alpha_{\sigma(1)}^{\epsilon_1(k+l+2)}\alpha_{\sigma(2)}^{\epsilon_2(k+1)}}{(\alpha_1+\alpha_1^{-1}-\alpha_2-\alpha_2^{-1})(\alpha_1-\alpha_1^{-1})(\alpha_2-\alpha_2^{-1})} & \text{if } k,l\geq 0; \\ 0 & \text{otherwise;} \end{cases}$$

$${W'}^0(\mathrm{diag}(\varpi_v^l, 1)) = \begin{cases} q_v^{-l/2} \times \dfrac{\beta_1^{l+1}-\beta_2^{l+1}}{\beta_1-\beta_2} & \text{if } l \geq 0; \\ 0 & \text{otherwise.} \end{cases}$$

By inserting these explicit formulae, we obtain

$$Z^{(v)}(s, W^0, {W'}^0, f^{(v)}_{J,0}) = [\det(I_8 - A_v \otimes B_v \cdot q_v^{-s})]^{-1};$$
$$\widetilde{Z}^{(v)}(s, W^0, {W'}^0, f^{(v)}_{J,0}) = [\det(I_8 - A_v^\vee \otimes B_v^\vee \cdot q_v^{-(1-s)})]^{-1}.$$

Here $A_v^\vee$ (resp. $B_v^\vee$) stands for the Satake parameter of the unramified principal series representation $\Pi_v^\vee$ (resp. of $\sigma_v^\vee$), which is given by $A_v^\vee = \omega_\Pi(\varpi_v)^{-1} \cdot A_v \in \mathrm{GSp}(2,\mathbf{C})$ (resp. $B_v^\vee = \omega_\sigma(\varpi_v)^{-1} \cdot B_v \in \mathrm{GL}(2,\mathbf{C})$). Note that $A_v^\vee$ (resp. $B_v^\vee$) is conjugate to A_v^{-1} (resp. B_v^{-1}) in $\mathrm{GSp}(2,\mathbf{C})$ (resp. in $\mathrm{GL}(2,\mathbf{C})$). In view of the Iwasawa decomposition, for each $W \in \mathrm{Wh}(\Pi_v, \psi_v)$ (resp. $W' \in \mathrm{Wh}(\sigma_v, \psi_v)$), there exists a Laurent polynomial $P_W \in \mathbf{C}[Y_1^{\pm}, Y_2^{\pm}]$ (resp. $P_{W'} \in \mathbf{C}[Y_1^{\pm}]$) and $\epsilon > 0$ such that

$$W(\mathrm{diag}(y_1y_2, y_1, y_2^{-1}, 1)) = P_W(|y_1|_v, |y_2|_v)W^0(\mathrm{diag}(y_1y_2, y_1, y_2^{-1}, 1)), \text{ and}$$
$$W'(\mathrm{diag}(y_1, 1)) = P_{W'}(|y_1|_v){W'}^0(\mathrm{diag}(y_1, 1)),$$

for all $y_i \in \mathrm{k}_v^\times$ with $|y_i| < \epsilon$ $(i = 1, 2)$. From these asymptotic expansions, we conclude that

$$L(s, \Pi_v \times \sigma_v) = [\det(I_8 - A_v \otimes B_v \cdot q_v^{-s})]^{-1},$$
$$L(s, \Pi_v^\vee \times \sigma_v^\vee) = [\det(I_8 - A_v^\vee \otimes B_v^\vee \cdot q_v^{-s})]^{-1}.$$

Hence we have $\epsilon(s, \Pi_v \times \sigma_v, \psi_v) = 1 (\forall v \notin S_{\Pi,\sigma,\psi})$.

4 The L- and ϵ-Factors at the Real Places Via the Langlands Parameters

In this section, we fix a real place v of k. Recall that Π_v (resp. σ_v) is the local component at v of a cuspidal automorphic representation Π (resp. σ) satisfying A.1 and A.2 in Subsection 1.3. Define three complex numbers c_v', c_v'', and $c_v \in \mathbf{C}$ by

$$\Pi_v(zI_4) = z^{c_v'} \ (z > 0); \quad \sigma_v(zI_2) = z^{c_v''} \ (z > 0); \quad c_v := c_v' + c_v''.$$

Denote the Langlands parameter of Π_v (resp. σ_v) by $\phi[\Pi_v] \colon W_{\mathbf{R}} \to \mathrm{GSp}(2, \mathbf{C})$ (resp. $\phi[\sigma_v] \colon W_{\mathbf{R}} \to \mathrm{GL}(2, \mathbf{C})$). Then we have the eight-dimensional representation

$$\phi[\Pi_v] \otimes \phi[\sigma_v] \colon W_{\mathbf{R}} \to \mathrm{GL}(8, \mathbf{C}).$$

Denote the corresponding L- and ϵ-factors by $L(s, \Pi_v \times \sigma_v)$ and $\epsilon(s, \Pi_v \times \sigma_v, \psi_v)$, respectively. We shall write down the Langlands parameters for Π_v and σ_v and determine the local factors $L(s, \Pi_v \times \sigma_v)$ and $\epsilon(s, \Pi_v \times \sigma_v, \psi_v)$ explicitly.

4.1 The Representations of the Weil Group $W_{\mathbf{R}}$

The Weil group $W_{\mathbf{R}}$ for the field $\mathbf{R}$ is given by $W_{\mathbf{R}} := \mathbf{C}^\times \sqcup \mathbf{C}^\times \cdot j \subset \mathbf{H}^\times$. Here we regard $W_{\mathbf{R}}$ as a subgroup of the multiplicative group $\mathbf{H}^\times$ of the Hamilton quaternion algebra $\mathbf{H} = \mathbf{C} \oplus \mathbf{C} \cdot j$ ($j^2 = -1$, jzj^{-1} equals the complex conjugate of z, $z \in \mathbf{C}$). We recall all the irreducible continuous representations of $W_{\mathbf{R}}$ and the corresponding L- and ϵ-factors.

(i) *Characters.* A character of $W_{\mathbf{R}}$ is given by ϕ_μ^+ or ϕ_μ^-, $(\mu \in \mathbf{C})$, where

$$\phi_\mu^{\pm}(z) = |z|^{2\mu} = |z|_{\mathbf{C}}^{\mu}, \quad \phi_\mu^{\pm}(j) = \pm 1.$$

We define the corresponding L- and ϵ-factors by

$$L(s, \phi_\mu^+) := \Gamma_{\mathbf{R}}(s + \mu), \qquad \epsilon(s, \phi_\mu^+, \psi_\infty) := 1,$$
$$L(s, \phi_\mu^-) := \Gamma_{\mathbf{R}}(s + \mu + 1), \qquad \epsilon(s, \phi_\mu^-, \psi_\infty) := \sqrt{-1}.$$

The set of characters of the Weil group $W_{\mathbf{R}}$ parameterizes the set of characters of $\mathbf{R}^\times$ via

$$\phi_\mu^+ \leftrightarrow [\mathbf{R}^\times \ni x \mapsto |x|^\mu \in \mathbf{C}^\times], \qquad \phi_\mu^- \leftrightarrow [\mathbf{R}^\times \ni x \mapsto |x|^\mu \mathrm{sgn}(x) \in \mathbf{C}^\times].$$

(ii) *Two-dimensional representations.* We define two-dimensional representations $\phi_{\mu,N}\colon W_{\mathbf{R}} \to \mathrm{GL}(2,\mathbf{C})$ ($\mu \in \mathbf{C}, N \in \mathbf{Z}_{\geq 0}$) as follows

$$\phi_{\mu,N}(re^{\sqrt{-1}\theta}) = \begin{pmatrix} r^{2\mu-N}e^{-\sqrt{-1}N\theta} & 0 \\ 0 & r^{2\mu-N}e^{+\sqrt{-1}N\theta} \end{pmatrix},$$
$$\phi_{\mu,N}(j) = \begin{pmatrix} 0 & (-1)^N \\ 1 & 0 \end{pmatrix}.$$

The representation $\phi_{\mu,N}$ is irreducible for $N > 0$, while $\phi_{\mu,0}$ is equivalent to a sum of two characters ϕ_μ^+ and ϕ_μ^-. The characters $\phi_\mu^\pm$ together with the two dimensional representations $\phi_{\mu,N}$ ($N > 0$) make up all the irreducible continuous representations of $W_{\mathbf{R}}$. The L- and ϵ-factors corresponding to the representations $\phi_{\mu,N}$ ($N > 0$) are given by

$$L(s,\phi_{\mu,N}) := \Gamma_{\mathbf{C}}(s+\mu), \quad \epsilon(s,\phi_{\mu,N},\psi_\infty) := (\sqrt{-1})^{N+1}.$$

Here we note the following.

***Lemma 4.1* (tensor products)** *We have the following equivalences:*

$$\phi_{\mu_1,N_1} \otimes \phi_{\mu_2}^\pm \cong \phi_{\mu_1+\mu_2,N_1},$$
$$\phi_{\mu_1,N_1} \otimes \phi_{\mu_2,N_2} \cong \phi_{\mu_1+\mu_2,N_1+N_2} \oplus \phi_{\mu_1+\mu_2-N_2,N_1-N_2}$$

if $N_1 \geq N_2$.

4.2 L- and ϵ-Factors at the Real Places

In what follows, we shall separate our argument into two cases (a) and (b) depending on types of the representations Π_v in assumption A.2. Moreover each case is divided into two subcases (x-i) and (x-ii) ($x = a, b$) corresponding to types of the representations σ_v.

Case (a): the case where $\Pi_v \cong D_{(\lambda_1,\lambda_2)}[c_v']$ with $1-\lambda_1 \leq \lambda_2 \leq 0$. Then the composition of $\phi[\Pi_v]$ with the natural inclusion $\iota\colon \mathrm{GSp}(2,\mathbf{C}) \hookrightarrow \mathrm{GL}(4,\mathbf{C})$ is given by

$$\iota \circ \phi[\Pi_v] \cong \phi_{\mu_1,N_1} \oplus \phi_{\mu_2,N_2} \tag{4.1}$$

with

$$\mu_1 = (c_v' + \lambda_1 - \lambda_2 - 1)/2, \quad N_1 = \lambda_1 - \lambda_2 - 1;$$
$$\mu_2 = (c_v' + \lambda_1 + \lambda_2 - 1)/2, \quad N_2 = \lambda_1 + \lambda_2 - 1,$$

which will be proved in the next subsection.

Case (a-i): the case of $\sigma_v \cong D_l[c_v'']$. The Langlands parameter $\phi[\sigma_v]$ of σ_v is given by

$$\phi[\sigma_v] = \phi_{\mu_3,N_3}, \qquad \text{with} \quad \mu_3 = (c_v'' + l - 1)/2, \quad N_3 = l - 1. \tag{4.2}$$

By Lemma 4.1, the L- and ϵ-factor for $\Pi_v \times \sigma_v$ is given by

$$\begin{aligned} L(s, \Pi_v \times \sigma_v) &= \Gamma_{\mathbf{C}}(s + \tfrac{c_v+\lambda_1-\lambda_2+l-2}{2})\Gamma_{\mathbf{C}}(s + \tfrac{c_v+|\lambda_1-\lambda_2-l|}{2}) \\ &\quad \times \Gamma_{\mathbf{C}}(s + \tfrac{c_v+\lambda_1+\lambda_2+l-2}{2})\Gamma_{\mathbf{C}}(s + \tfrac{c_v+|\lambda_1+\lambda_2-l|}{2}), \\ \epsilon(s, \Pi_v \times \sigma_v, \psi_v) &= \begin{cases} 1 & \text{if } |l - \lambda_1| \geq -\lambda_2; \\ (-1)^{l+\lambda_1-\lambda_2} & \text{if } |l - \lambda_1| \leq -\lambda_2. \end{cases} \end{aligned}$$

Case (a-ii): the case of $\sigma_v \cong I_{\mathrm{B}_{\mathbf{R}}'}^{\mathrm{GL}(2)_{\mathbf{R}}}(\epsilon_1, \epsilon_2; c_v'', \nu)$. The Langlands parameter $\phi[\sigma_v]$ of σ_v is given by

$$\phi[\sigma_v] = \phi_{\mu_4}^{\epsilon_1} \oplus \phi_{\mu_5}^{\epsilon_1} \qquad \mu_4 = (c_v'' + \nu)/2, \quad \mu_5 = (c_v'' - \nu)/2. \tag{4.3}$$

Here we identify ϵ_i with the signature $\pm$ of $\epsilon_i(-1)$. By Lemma 4.1, the L- and ϵ-factors for $\Pi_v \times \sigma_v$ are given by

$$\begin{aligned} L(s, \Pi_v \times \sigma_v) &= \Gamma_{\mathbf{C}}(s + \tfrac{c_v+\lambda_1-\lambda_2+\nu-1}{2})\Gamma_{\mathbf{C}}(s + \tfrac{c_v+\lambda_1-\lambda_2-\nu-1}{2}) \\ &\quad \times \Gamma_{\mathbf{C}}(s + \tfrac{c_v+\lambda_1+\lambda_2+\nu-1}{2})\Gamma_{\mathbf{C}}(s + \tfrac{c_v+\lambda_1+\lambda_2-\nu-1}{2}), \\ \epsilon(s, \Pi_v \times \sigma_v, \psi_v) &= 1. \end{aligned}$$

Case (b): the case where $\Pi_v \cong I(P_1; \sigma_{n,\pm}, c, \nu_1)$. Then the composition of $\phi[\Pi_v]$ and the natural inclusion $\iota\colon \mathrm{GSp}(2, \mathbf{C}) \hookrightarrow \mathrm{GL}(4, \mathbf{C})$ is given by

$$\iota \circ \phi[\Pi_v] \cong \phi_{\mu_1,N_1} \oplus \phi_{\mu_2,N_2} \tag{4.4}$$

with

$$\mu_1 = (c_v' + n + \nu_1 - 1)/2, \quad \mu_2 = (c_v' + n - \nu_1 - 1)/2, \quad N_1 = N_2 = n - 1,$$

which will be proved in the next subsection, too.

Case (b-i): the case of $\sigma_v \cong D_l[c_v'']$. Then Lemma 4.1 plus (4.2) and (4.4) implies that

$$\begin{aligned} L(s, \Pi_v \times \sigma_v) =&\Gamma_{\mathbf{C}}(s + \tfrac{c_v+n+l+\nu_1-2}{2})\Gamma_{\mathbf{C}}(s + \tfrac{c_v+|n-l|+\nu_1}{2}) \\ &\times \Gamma_{\mathbf{C}}(s + \tfrac{c_v+n+l-\nu_1-2}{2})\Gamma_{\mathbf{C}}(s + \tfrac{c_v+|n-l|-\nu_1}{2}), \\ \epsilon(s, \Pi_v \times \sigma_v, \psi_v) =&1. \end{aligned}$$

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Case (b-ii): the case of $\sigma_v \cong I_{B'_{\mathbf{R}}}^{\mathrm{GL}(2)_{\mathbf{R}}}(\epsilon_1, \epsilon_2; c''_v, \nu)$. Then Lemma 4.1 plus (4.3) and (4.4) implies that

$$\begin{aligned} L(s, \Pi_v \times \sigma_v) =&\Gamma_{\mathbf{C}}(s + \tfrac{c_v+n+\nu_1+\nu-1}{2})\Gamma_{\mathbf{C}}(s + \tfrac{c_v+n-\nu_1+\nu-1}{2}) \\ &\times \Gamma_{\mathbf{C}}(s + \tfrac{c_v+n+\nu_1-\nu-1}{2})\Gamma_{\mathbf{C}}(s + \tfrac{c_v+n-\nu_1-\nu-1}{2}), \end{aligned}$$

$$\epsilon(s, \Pi_v \times \sigma_v, \psi_v) =1.$$

4.3 Langlands Parameters of Π_v

In this subsection, we shall prove (4.1) and (4.4). Until the end of this subsection, we identify $\mathsf{G}_{\mathsf{k}_v}$ with $\mathrm{GSp}(2, \mathbf{R})$ and write c in place of c'_v. We fix a maximal torus T (resp. $\widehat{\mathsf{T}}$) of G (resp. of $\widehat{\mathsf{G}} = \mathrm{GSp}(2, \mathbf{C})$) as $\mathsf{T} := \{t = \mathrm{diag}(t_0t_1, t_0t_2, t_1^{-1}, t_2^{-1}) \mid t_i \in \mathrm{GL}(1)\}$ (resp. $\widehat{\mathsf{T}} := \{a = \mathrm{diag}(a_0a_1, a_0a_2, a_1^{-1}, a_2^{-1}) \mid a_i \in \mathbf{C}^\times\}$). Define a basis $\{e_i \mid 0 \le i \le 2\}$ (resp. $\{f'_i \mid 0 \le i \le 2\}$) of the character group $X^*(\mathsf{T})$ (resp. $X^*(\widehat{\mathsf{T}})$) by $e_i(t) = t_i$ (resp. $f'_i(a) = a_i$). We denote by $\{f_i \mid 0 \le i \le 2\}$ (resp. $\{e'_i \mid 0 \le i \le 2\}$) the basis of the cocharacter group $X_*(\mathsf{T})$ (resp. $X^*(\widehat{\mathsf{T}})$) dual to $\{e_i \mid 0 \le i \le 2\}$ (resp. $\{f'_i \mid 0 \le i \le 2\}$). The identification $X^*(\mathsf{T})$ with $X_*(\widehat{\mathsf{T}})$ is given by

$$e_0 = 2e'_0 - e'_1 - e'_2, \quad e_1 = -e'_0 + e'_1 + e'_2, \quad e_2 = -e'_0 + e'_1.$$

If we denote the Lie algebra of $\mathsf{T}_{\mathbf{R}}$ by $\mathfrak{t}$, then there is a natural identification

$$\mathfrak{t}^*_{\mathbf{C}} := \mathrm{Hom}_{\mathbf{R}}(\mathfrak{t}, \mathbf{C}) \cong X^*(\mathsf{T}) \otimes \mathbf{C} \cong X_*(\widehat{\mathsf{T}}) \otimes \mathbf{C}.$$

We first prove (4.4). By [Bo, 11.3], we have

$$\phi[I(P_1; \sigma_{n,\pm}, c, \nu_1)](w) = \left(\begin{array}{c|c} A(w) & \\ \hline & D(w) \end{array}\right) \qquad (\forall w \in W_{\mathbf{R}}) \tag{4.5}$$

with

$$A(w) = \phi_{\mu_1,N_1}(w), \quad D(w) = \phi^{\pm}_{-\nu_1}(w) \det(\phi_{\mu_1,N_1}(w))\, {}^t\phi_{\mu_1,N_1}(w)^{-1},$$

where (μ_1, N_1) is as in (4.4). By Lemma 4.1, we have (4.4). We also note that the infinitesimal character of the representation $I(P_1; \sigma_{n,\pm}, c, \nu_1)$ is given by

$$\tfrac{1}{2}(c + \nu_1 + n - 1)e_0 + \nu_1 e_1 + (n - 1)e_2.$$

Next we prove (4.1). Note that the infinitesimal character of $D_{(\lambda_1,\lambda_2)}[c]$ is given by

$$\begin{aligned} &\tfrac{1}{2}(c + \lambda_1 - 1 - \lambda_2)e_0 + (\lambda_1 - 1)e_1 + (-\lambda_2)e_2 = \\ &\qquad ce'_0 + \tfrac{1}{2}(-c + \lambda_1 - 1 - \lambda_2)e'_1 + \tfrac{1}{2}(-c + \lambda_1 - 1 + \lambda_2)e'_2. \end{aligned}$$

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If $1-\lambda_1<\lambda_2<0$, then we know from [Bo, 11.2] that

$$\phi[\Pi_\nu](w)=\left(\begin{array}{cc|cc} a_1 & & b_1 & \\ & a_2 & & b_2 \\ \hline c_1 & & d_1 & \\ & c_2 & & d_2 \end{array}\right)\quad (\forall w\in W_{\mathbf{R}})$$

with $\left(\begin{smallmatrix} a_i & b_i \\ c_i & d_i\end{smallmatrix}\right)=\phi_{\mu_i,N_i}(w)$, $(i=1,2)$. Here (μ_i,N_i) are as in (4.1). In order to treat the case where $1-\lambda_1=\lambda_2$ or $\lambda_2=0$, we have to recall (generalized) principal series representations induced from parabolic subgroups of $\mathrm{G}_{\mathbf{R}}$ other than P_1. First, let $P_2=M_2N_2$ be the Siegel parabolic subgroup of $\mathrm{G}_{\mathbf{R}}$, where we set

$$M_2:=\left\{\left(\begin{array}{c|c} t_0 m & \\ \hline & {}^t m^{-1}\end{array}\right)\mid (t_0,m)\in\mathbf{R}^\times\times \mathrm{GL}(2)_{\mathbf{R}}\right\},$$
$$N_2:=\{n(0,x_1,x_2,x_3)\mid x_i\in\mathbf{R}\}.$$

Then the P_2-principal series representation $I(P_2;\epsilon\otimes D_n[\nu],c)$ is realized on the space of all C^∞-functions $f\colon \mathrm{GSp}(2,\mathbf{R})\to V_n[\nu]$ satisfying

$$f\left(\left(\begin{array}{c|c} t_0 m & \\ \hline & {}^t m^{-1}\end{array}\right)n_2 g\right)=\epsilon(t_0/|t_0|)|t_0|^{(c+\nu+3)/2}D_n[\nu+3](m)f(g)$$
$$\forall(t_0,m,n_2,g)\in\mathbf{R}^\times\times\mathrm{GL}(2)_{\mathbf{R}}\times N_2\times \mathrm{G}_{\mathbf{R}}.$$

Here ϵ is a character of $\{\pm1\}$ and $V_n[\nu]$ stands for the representation space of $D_n[\nu]$. The infinitesimal character of $I(P_2;\epsilon\otimes D_n[\nu],c)$ is given by

$$\tfrac12(c+\nu)e_0+\tfrac12(\nu+1-n)e_1+\tfrac12(\nu+n-1)e_2.$$

By [Bo, 11.3], the Langlands parameter of an arbitrary direct summand of the unitary P_2-principal series representation $I(P_2;\epsilon\otimes D_n[\nu],c)$ with $c,\nu\in\sqrt{-1}\mathbf{R}$ and $n\geq 2$ is given by

$$(4.6)\qquad W_{\mathbf{R}}\ni w\mapsto\left(\begin{array}{cc|cc} \phi^{\epsilon}_{(c+\nu)/2}(w) & & & \\ & a_2 & & b_2 \\ \hline & & \phi^{(-1)^n\epsilon}_{(c-\nu)/2}(w) & \\ & c_2 & & d_2\end{array}\right)\in\mathrm{GSp}(2,\mathbf{C})$$

with

$$\begin{pmatrix} a_2 & b_2\\ c_2 & d_2\end{pmatrix}=\phi^{\epsilon}_{(c+\nu)/2}(w)\,{}^t\phi_{(\nu+n-1)/2,n-1}(w)^{-1}.$$

Here we identify ϵ with the sign of $\epsilon(-1)$. Next we fix a minimal parabolic subgroup P_0 of $\mathrm{G}_{\mathbf{R}}$ as $P_0=\mathrm{T}_{\mathbf{R}}\mathrm{N}_{\mathbf{R}}$. For a character $\chi\colon\mathrm{T}_{\mathbf{R}}\to\mathbf{C}^\times$, the principal series representation $I(P_0;\chi)$ of $\mathrm{G}_{\mathbf{R}}$ is realized on the space of all C^∞-functions $f\colon\mathrm{GSp}(2,\mathbf{R})\to\mathbf{C}$ satisfying

$$f(tng)=\chi(t)|t_0|^{3/2}|t_1|^2|t_2|^1f(g)$$
$$\forall(t,n,g)\in\mathrm{T}_{\mathbf{R}}\times\mathrm{N}_{\mathbf{R}}\times\mathrm{G}_{\mathbf{R}},\quad t=\mathrm{diag}(t_0t_1,t_0t_2,t_1^{-1},t_2^{-1}).$$

If the character χ is of the form $\chi(t) = \prod_{i=0}^{2} \epsilon_i(t_i/|t_i|)|t_i|^{\nu_i}$ with characters ϵ_i on $\{\pm 1\}$ and $\nu = (\nu_0, \nu_1, \nu_2) \in \mathbf{C}^3$, then we also write $I(P_0; \epsilon_0, \epsilon_1, \epsilon_2, \nu)$ in place of $I(P_0; \chi)$. The infinitesimal character of $I(P_0; \epsilon_0, \epsilon_1, \epsilon_2, \nu)$ is given by

$$\nu_0 e_0 + \nu_1 e_1 + \nu_2 e_2.$$

By [Bo, 11.3] again, the Langlands parameter of an arbitrary direct summand of the unitary principal series representation $I(P_0; \epsilon_0, \epsilon_1, \epsilon_2, \nu)$ with $\nu_i \in \sqrt{-1}\mathbf{R}$ is given by

$$W_{\mathbf{R}} \ni w \mapsto \phi_{\nu_0}^{\epsilon_0}(w)\mathrm{diag}(1, \phi_{-\nu_2}^{\epsilon_2}(w), \phi_{-\nu_1-\nu_2}^{\epsilon_1\epsilon_2}(w), \phi_{-\nu_1}^{\epsilon_1}(w)) \in \mathrm{GSp}(2, \mathbf{C}). \tag{4.7}$$

Here ϵ_i is identified with the sign of $\epsilon_i(-1)$.

Now we shall determine the Langlands parameter of $D_{(\lambda_1,\lambda_2)}[c]$ when $1 - \lambda_1 = \lambda_2$ or $\lambda_2 = 0$. Since the representation $D_{(\lambda_1,\lambda_2)}[c]$ is tempered, it is a direct summand of one of the following unitary (generalized) principal series representations:

- $I(P_1; \sigma_{n,\pm}, c, \nu_1)$ with $\nu_1 \in \sqrt{-1}\mathbf{R}$ and $n \geq 2$;
- $I(P_2; \epsilon \otimes D_n[\nu], c)$ with $\nu \in \sqrt{-1}\mathbf{R}$ and $n \geq 2$;
- $I(P_0; \epsilon_0, \epsilon_1, \epsilon_2, \nu)$ with $\nu_i \in \sqrt{-1}\mathbf{R}$ and $c = 2\nu_0 - \nu_1 - \nu_2$.

(i) Suppose that $\lambda_2 = 0$ and $\lambda_1 > 1$. Then by comparing the infinitesimal characters and the actions of $-I_4 \in G_{\mathbf{R}}$, we know that $D_{(\lambda_1,0)}[c]$ is a direct summand of $I(P_1; \sigma_{\lambda_1,+}, c, 0)$.

(ii) Suppose that $\lambda_2 < 0$ and $\lambda_2 = 1 - \lambda_1$. Then by comparing the infinitesimal characters, we know that $D_{(\lambda_1,1-\lambda_1)}[c]$ is a direct summand of $I(P_2; \epsilon \otimes D_{2\lambda_1-1}[0], c)$ (ϵ is arbitrary).

(iii) Finally we suppose that $(\lambda_1, \lambda_2) = (1, 0)$. Then the representation $D_{(1,0)}[c]$ is a direct summand of $I(P_0; \chi)$ with $\chi(-I_4) = -1$ and $\nu = (c/2, 0, 0)$ (ϵ_0 is arbitrary).

Hence, the assertion (4.1) in the case of $1 - \lambda_1 = \lambda_2$ or $\lambda_2 = 0$ follows from (4.5), (4.6), and (4.7).

5 Computation of the Local Zeta Integrals at the Real Places

In this section, we prove our main results by computing the local zeta integrals $Z^{(v)}(s, W, W', f)$ at each real place v of k.

5.1 Reduction to a Real Local Problem

By virtue of the results recalled in Section 3, we can easily reduce our main result (Theorem 1.1) to the following proposition, through a typical argument in the theory of zeta integrals (see [Co, G-Sh] for example).

Proposition 5.1 *Let v be a fixed real place of* k. *Suppose that Π_v is an irreducible admissible representation of $G_{k_v} \cong \mathrm{GSp}(2, \mathbf{R})$ satisfying assumption* A.2, *and let* $\mathrm{Wh}(\Pi_v, \psi_v)$ *be its local Whittaker model. Let σ_v be an arbitrary irreducible admissible representation of $\mathrm{GL}(2)_{k_v}$ having a local Whittaker model $\mathrm{Wh}(\sigma_v, \psi_v)$. Then there* *exists a triplet* $(W, W', f) \in \mathrm{Wh}(\Pi_v, \psi_v) \times \mathrm{Wh}(\sigma_v, \psi_v) \times \mathcal{I}^J(1, \omega_v^{-1}; s)$ *such that*

$$Z^{(v)}(s, W, W', f) = L(s, \Pi_v \times \sigma_v) \tag{5.1}$$

and

$$\frac{\widetilde{Z}^{(v)}(s, W, W', f)}{L(1-s, \Pi_v \times \sigma_v)} = \epsilon(s, \Pi_v \times \sigma_v, \psi_v) \frac{Z^{(v)}(s, W, W', f)}{L(s, \Pi_v \times \sigma_v)}. \tag{5.2}$$

Here the local factors $L(s, \Pi_v \times \sigma_v)$ *and* $\epsilon(s, \Pi_v \times \sigma_v, \psi_v)$ *are those given in the previous section.*

We shall prove Proposition 5.1 by using the explicit formulae of the local Whittaker functions to compute the local zeta integrals $Z^{(v)}(s, W, W', f)$. Without any loss of generality, we may assume that $c_v' = c_v'' = 0$. Then we have

$$\widetilde{Z}^{(v)}(s, W, W', f_{J,m}^{(v)}) = Z^{(v)}(1-s, W, W', f_{J,m}^{(v)}) \times \begin{cases} 1 & \text{if } m \geq 0; \\ (-1)^m & \text{if } m \leq 0. \end{cases} \tag{5.3}$$

5.2 A Computational Lemma

The following lemma will be used repeatedly:

Lemma 5.2 *For* $\alpha_i \in \mathbf{C}$ $(1 \leq i \leq 5)$, $l, \nu \in \mathbf{C}$, *and a polynomial* $P(s_1) \in \mathbf{C}[s_1]$, *we put*

$$\begin{aligned} I(s; \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5; \nu; P(s_1)) &= \frac{\Gamma_{\mathbf{R}}(2s + |\alpha_5|)}{(2\pi\sqrt{-1})^3} \int_0^\infty d^\times y_1 \int_0^\infty d^\times y_2 \, y_1^{s-2} \, y_2^{2s-2} \\ &\times \int_{L_1} ds_1 \int_{L_2} ds_2 \, (4\pi^3 y_1 y_2^2)^{(-s_1+\alpha_1)/2} (4\pi y_1)^{(-s_2+\alpha_2)/2} \\ &\times \Gamma\Big(\frac{s_1+s_2+\alpha_3}{2}\Big) \Gamma\Big(\frac{s_1+s_2+\alpha_4}{2}\Big) \Gamma\Big(\frac{s_1}{2}\Big) \Gamma\Big(-\frac{s_2}{2}\Big) \, P(s_1) \\ &\times \int_{L_3} ds_3 \, (4\pi y_1)^{-s_3} \frac{\Gamma(s_3 + (\nu+1)/2)\Gamma(s_3 + (-\nu+1)/2)}{\Gamma(s_3 + 1 - \alpha_2/2)} \end{aligned}$$

and

$$\begin{aligned} &J(s; \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5; l; P(s_1)) \\ &= \frac{\Gamma_{\mathbf{R}}(2s + |\alpha_5|)}{(2\pi\sqrt{-1})^2} \times \int_0^\infty d^\times y_1 \int_0^\infty d^\times y_2 \, y_1^{s-2} \, y_2^{2s-2} \, e^{-4\pi y_1} (4\pi y_1)^{l/2} \\ &\times \int_{L_1} ds_1 \int_{L_2} ds_2 \, (4\pi^3 y_1^2 y_2)^{(-s_1+\alpha_1)/2} (4\pi y_1)^{(-s_1+\alpha_2)/2} \\ &\times \Gamma\Big(\frac{s_1+s_2+\alpha_3}{2}\Big) \Gamma\Big(\frac{s_1+s_2+\alpha_4}{2}\Big) \Gamma\Big(\frac{s_1}{2}\Big) \Gamma\Big(-\frac{s_2}{2}\Big) \, P(s_1). \end{aligned}$$

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Here the paths L_j $(1 \le j \le 3)$ of integration are the vertical line from $\sigma_j - \sqrt{-1}\infty$ to $\sigma_j + \sqrt{-1}\infty$ with $\sigma_j \in \mathbf{R}$ satisfying

$$\sigma_1 + \sigma_2 + \mathrm{Re}(\alpha_i) > 0 \quad (i = 3, 4), \quad \sigma_1 > 0 > \sigma_2,$$
$$\sigma_3 + \mathrm{Re}(\nu + 1)/2 > 0, \quad \sigma_3 + \mathrm{Re}(-\nu + 1)/2 > 0.$$

(i) *The integral $I(s; \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5; \nu; P(s_1))$ converges absolutely for $\mathrm{Re}(s) \gg 0$ and is equal to*

$$\pi^{-4s+4-|\alpha_5|/2} 2^{-2s+4} P(2s + \alpha_1 - 2)\Gamma(s + \tfrac{\alpha_1 - 2}{2})\Gamma(s + \tfrac{|\alpha_5|}{2})$$
$$\times \Gamma(s + \tfrac{\alpha_1+\alpha_2+\alpha_3+\nu-3}{2})\Gamma(s + \tfrac{\alpha_1+\alpha_2+\alpha_4+\nu-3}{2})\Gamma(s + \tfrac{\alpha_1+\alpha_2+\alpha_3-\nu-3}{2})\Gamma(s + \tfrac{\alpha_1+\alpha_2+\alpha_4-\nu-3}{2})$$
$$\times \Gamma(2s + \alpha_1 + \alpha_2 + \tfrac{\alpha_3+\alpha_4}{2} - 3)^{-1}.$$

(ii) *The integral $J(s; \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5; l; P(s_1))$ converges absolutely for $\mathrm{Re}(s) \gg 0$ and is equal to*

$$\pi^{-4s+2-|\alpha_5|/2} 2^{-2s+4} P(2s + \alpha_1 - 2)\Gamma(s + \tfrac{\alpha_1 - 2}{2})\Gamma(s + \tfrac{|\alpha_5|}{2})$$
$$\times \Gamma(s + \tfrac{\alpha_1+\alpha_3-2}{2})\Gamma(s + \tfrac{\alpha_1+\alpha_4-2}{2})\Gamma(s + \tfrac{\alpha_1+\alpha_2+\alpha_3+l-4}{2})\Gamma(s + \tfrac{\alpha_1+\alpha_2+\alpha_4+l-4}{2})$$
$$\times \Gamma(2s + \tfrac{2\alpha_1+\alpha_2+\alpha_3+\alpha_4+l}{2} - 3)^{-1}.$$

Proof This can be easily obtained by using the Barnes' lemma [W-W, p.289]. ■

5.3 Proof of Proposition 5.1, Case (a)

Let us start with recalling the explicit formulae for some Whittaker functions in $\mathsf{Wh}(\Pi_\nu, \psi_\nu)$ given in [Mo-3]. Fix a maximal compact subgroup K_ν^{H} of the identity component of $\mathsf{H}_{\mathsf{k}_\nu}$ as follows:

$$K_\nu^{\mathsf{H}} := \{r(\theta_1, \theta_2) \mid \theta_i \in \mathbf{R}\}, \qquad r(\theta_1, \theta_2) := (r_{\theta_1}, r_{\theta_2}) \in \mathsf{H}_{\mathsf{k}_\nu}.$$

This is also a Cartan subgroup of the identity component K_ν^0 of K_ν. We say that a vector $\xi \in \Pi_\nu$ is *of type* (q_1, q_2) $(q_i \in \mathbf{Z})$ if it satisfies

$$\Pi_\nu(r(\theta_1, \theta_2))\xi = \exp(q_1\theta_1 + q_2\theta_2)\xi, \qquad \forall \theta_i \in \mathbf{R}.$$

We use similar terminology for a cusp form F or a local Whitaker function. Fix a basis $\{v_k \in \Pi_\nu \mid 0 \le k \le d = \lambda_1 - \lambda_2\}$ of the minimal K_ν^0-type in $D_{(-\lambda_2, -\lambda_1)}$ as in [Mo-3]. Then v_k is of type $(-\lambda_1 + k, -\lambda_2 - k)$. We denote by $W_{v_k} \in \mathsf{Wh}(\Pi_\nu, \psi_\nu)$ the Whittaker function corresponding to v_k. Put

$$\widetilde{y} := \mathrm{diag}(y_1 y_2, y_1, y_2^{-1}, 1) \in \mathsf{G}_{\mathsf{k}_\nu}, \qquad y_i \in \mathsf{k}_\nu^\times.$$

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We know from [Mo-3, Proposition 7] that, for $y_i > 0$,

$$(5.4)\quad W_{\nu_k}(\tilde{y}) = C_1 \times \binom{d}{k} \times (2\sqrt{-1})^{-k} \frac{e^{-2\pi y_1}}{(2\pi\sqrt{-1})^2} \times \int_{L_1} ds_1 \int_{L_2} ds_2 (4\pi^3 y_1^2 y_2)^{(-s_1+\alpha_1)/2} (4\pi y_1)^{(-s_2+\alpha_2)/2} \times \Gamma\Big(\frac{s_1+s_2+\alpha_3}{2}\Big)\Gamma\Big(\frac{s_1+s_2+\alpha_4}{2}\Big)\Gamma\Big(\frac{s_1}{2}\Big)\Gamma\Big(-\frac{s_2}{2}\Big) \times (s_1)_k$$

with

$$(5.5)\qquad \alpha_1 = \lambda_1 + 1 - k; \quad \alpha_2 = \lambda_2 + k; \quad \alpha_3 = -2\lambda_2 + 1; \quad \alpha_4 = 1.$$

Here $C_1 \in \mathbf{C}^\times$ is a non-zero constant and the paths of integration are taken as in Lemma 5.2. On the other hand we have

$$(5.6)\qquad W_{\nu_k}(\mathrm{diag}(y_1y_2, y_1, y_2^{-1}, 1)) = 0, \qquad \forall y_1 < 0, \quad \forall y_2 \in \mathrm{k}_\nu^\times.$$

Case (a-ii): Suppose that $\sigma_\nu \cong I_{B'_{\mathbf{R}}}^{\mathrm{GL}(2)_{\mathbf{R}}}(\epsilon_1, \epsilon_2; 0, \nu)$. For an integer m satisfying $(-1)^m = (\epsilon_1 \cdot \epsilon_2)(-1)$, let $W'_m \in \mathrm{Wh}(\sigma_\nu, \psi_\nu)$ be the local Whittaker function characterized by

$$(5.7)\qquad W'_m(h_2 r_\theta) = \exp(2\pi\sqrt{-1}m\theta) W'_m(h_2), \qquad \forall h_2 \in \mathrm{GL}(2)_{\mathrm{k}_\nu}, \forall\theta \in \mathbf{R},$$

up to a constant multiple. Then it is well known that, for $y_1 > 0$,

$$(5.8)\ W'_m\left(\begin{smallmatrix} y_1 & 0 \\ 0 & 1\end{smallmatrix}\right) = (-C_2) \times W_{m/2,\nu/2}(4\pi y_1) = C_2 \times \frac{e^{2\pi y_1}}{2\pi\sqrt{-1}} \int_{L_3} \frac{\Gamma(s_3 + (\nu+1)/2)\Gamma(s_3 + (-\nu+1)/2)}{\Gamma(s_3 + 1 - m/2)} (4\pi y_1)^{-s_3}\, ds_3,$$

where $W_{m/2,\nu/2}$ stands for the classical Whittaker function [W-W, Chapter 16]. Here and (5.10) below the path of integration L_3 is taken as in Lemma 5.2 and C_2 is a non-zero constant. We choose an integer $0 \le k \le d$ such that $(-1)^{\lambda_2+k} = (\epsilon_1 \cdot \epsilon_2)(-1)$ and $\lambda_1 \ge k$. Then, by using (5.6) and (5.7), we have

$$Z^{(\nu)}(s, W_{\nu_k}, W'_{\lambda_2+k}, f_{\Phi_{\lambda_1-k}}) = \int_0^\infty d^\times y_1 \int_0^\infty d^\times y_2 \int_0^{2\pi} \frac{d\theta_1}{2\pi} \int_0^{2\pi} \frac{d\theta_2}{2\pi} W_{\nu_k}(\tilde{y} r(\theta_1, \theta_2)) \times W'_{\lambda_2+k}\Big(\begin{pmatrix} y_1 & 0 \\ 0 & 1\end{pmatrix} r_{\theta_2}\Big) f_{\Phi_{\lambda_1-k}}\Big(s, \begin{pmatrix} y_1y_2 & 0 \\ 0 & y_2^{-1}\end{pmatrix} r_{\theta_1}\Big)(y_1y_2)^{-2}.$$

By inserting (5.4) and (5.8), we find that

$$Z^{(\nu)}(s, W_{\nu_k}, W'_{\lambda_2+k}, f_{\Phi_{\lambda_1-k}}) = C \times I(s, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5; \nu; (s_1)_k)$$

for some $C \in \mathbf{C}^\times$, where α_i $(1 \le i \le 4)$ are as in (5.5) and $\alpha_5 = \lambda_1 - k$. Finally we apply Lemma 5.2 to get

$$Z^{(v)}(s, W_{v_k}, W'_{\lambda_2+k}, f_{\Phi_{\lambda_1-k}}) = C' \times L(s, \Pi_v \times \sigma_v), \qquad C' \in \mathbf{C}^\times. \tag{5.9}$$

By (5.9) and (5.3), Proposition 5.1 follows in Case (a-ii).

Case (a-i): Suppose that $\sigma_v \cong D_l[0]$. For each $m \in \mathbf{Z}$ with $m \ge l$ and $m \equiv l \pmod 2$, let us denote by $W'_m \in \mathrm{Wh}(\Pi_v, \psi_v)$ the Whittaker function characterized by (5.7), up to a constant multiple. Then we have

$$W'_m\begin{pmatrix} y_1 & \\ & 1 \end{pmatrix} = C_2 \times \frac{e^{2\pi y_1}}{2\pi\sqrt{-1}} \int_{L_3} \frac{\Gamma(s_3 + l/2)\Gamma(s_3 + (2-l)/2)}{\Gamma(s_3 + 1 - m/2)} (4\pi y_1)^{-s_3}\, ds_3, \tag{5.10}$$

for $y_1 > 0$ with some constant $C_2 \in \mathbf{C}^\times$. Note that

$$W'_l(\mathrm{diag}(y_1, 1)) = C_2 \times e^{-2\pi y_1}(4\pi y_1)^{l/2} \text{ for } y_1 > 0.$$

If $l \le \lambda_1$, then it follows from Lemma 5.2(ii) that

$$Z^{(v)}(s, W_{v_{l-\lambda_2}}, W'_l, f_{\Phi_{\lambda_1+\lambda_2-l}}) = C \times L(s, \Pi_v \times \sigma_v), \quad \text{for some } C \in \mathbf{C}^\times. \tag{5.11}$$

This proves Proposition 5.1 for the case $l \le \lambda_1$. Next we suppose that $l > \lambda_1$. In this case we need Whittaker functions $W \in \mathrm{Wh}(\Pi_v, \psi_v)$ *not* belonging to the minimal K_v^0-type of $D_{(-\lambda_2, -\lambda_1)} (\subset \Pi_v)$ in order to get a non-vanishing local zeta integral. Put $\mathfrak{g}_v := \mathrm{Lie}(G_{\mathrm{k}_v})$ and define two elements $X_{(\pm 1, -1)} \in \mathfrak{g}_{v,\mathbf{C}} := \mathfrak{g}_v \otimes_{\mathrm{k}_v} \mathbf{C}$ by

$$X_{(\pm1,-1)} := \left(\begin{array}{cc|cc} & 1 & & -\sqrt{-1} \\ \mp 1 & & -\sqrt{-1} & \\ \hline & \pm\sqrt{-1} & & \pm 1 \\ \pm\sqrt{-1} & & -1 & \end{array}\right).$$

Moreover we set $X_{(0,-2)} := [X_{(1,-1)}, X_{(-1,-1)}] \in \mathfrak{g}_{v,\mathbf{C}}$. We define a $\mathbf{C}$-linear action of the universal enveloping algebra $U(\mathfrak{g}_v)$ of $\mathfrak{g}_{v,\mathbf{C}}$ on $W \in \mathrm{Wh}(\Pi_v, \psi_v)$ by

$$W(g; X_1 \cdot X_2 \cdots X_n) = \frac{d}{dt_1}\Big|_{t_1=0} \frac{d}{dt_2}\Big|_{t_2=0} \cdots \frac{d}{dt_n}\Big|_{t_n=0} W(ge^{t_1X_1}e^{t_2X_2}\cdots e^{t_nX_n}),\ X_i \in \mathfrak{g}_v.$$

In the same manner, we define

$$W'(h_2; X_1 \cdots X_n), \qquad h_2 \in GL(2)_{\mathrm{k}_v}, X_i \in \mathrm{Lie}(GL(2)_{\mathrm{k}_v}) \otimes \mathbf{C}$$

for $W' \in \mathrm{Wh}(\sigma_v, \psi_v)$. Set $r = l - \lambda_1$. Then, by using Lemma 5.3 below, we find that

$$Z^{(v)}(s, W_{v_d}(\,\cdot\,; (X_{(-1,-1)})^r), W'_l, f^{(v)}_{J,\lambda_2+r}) = C \times L(s, \Pi_v \times \sigma_v) \tag{5.12}$$

for some $C \in \mathbf{C}^\times$. Now Proposition 5.1 in Case (a-i) follows from (5.11), (5.12), and (5.3).

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Lemma 5.3 *Let* $W \in \mathrm{Wh}(\Pi_v, \psi_v)$ *and* $W' \in \mathrm{Wh}(\sigma_v, \psi_v)$ *be local Whittaker functions satisfying* $W(g; X_{(1,-1)}) = 0$ *and* $W'(h; X_{(0,-2)}) = 0$*. Suppose that there exist* $q_1, q_2 \in \mathbf{Z}$ *and* $r \in \mathbf{Z}_{\geq 0}$ *such that*

$$W(g(r(\theta_1,\theta_2))) = \exp(2\pi\sqrt{-1}(q_1\theta_1 + q_2\theta_2))W(g), \quad \forall g \in \mathrm{G}_{\mathrm{k}_v}, \forall \theta_i \in \mathbf{R};$$
$$W'(h_2 r_{\theta_2}) = \exp(2\pi\sqrt{-1}((-q_2 + r)\theta_2))W'(h_2), \quad \forall h_2 \in \mathrm{GL}(2)_{\mathrm{k}_v}, \forall \theta_2 \in \mathbf{R}.$$

Moreover we suppose that the support of W *is contained in the identity component of* $\mathrm{G}_{\mathrm{k}_v}$*. Then the local zeta integral* $Z^{(v)}(s, W(\,\cdot\,; (X_{(-1,-1)})^r), W', f^{(v)}_{J,-q_1+r})$ *equals*

$$\int_0^\infty d^\times y_1 \int_0^\infty d^\times y_2\, W(\widetilde{y})W'(\mathrm{diag}(y_1,1)) f^{(v)}_{J,-q_1+r}(s, \mathrm{diag}(y_1y_2, y_2^{-1})) \times (-4\pi\sqrt{-1}y_2)^r (y_1y_2)^{-2}. \tag{5.13}$$

Proof Since $X_{(1,-1)} + X_{(-1,-1)}$ belongs to $\mathrm{Lie}(\mathrm{N}_{\mathrm{k}_v}) \otimes \mathbf{C}$, we have

$$(-4\pi\sqrt{-1}y_2)^r W(\widetilde{y}) = W(\widetilde{y}; (X_{(1,-1)} + X_{(-1,-1)})^r).$$

Hence (5.13) is equal to $\sum_{i_1,i_2,\ldots,i_r=\pm 1} Z(s; i_1, i_2, \ldots, i_r)$ with

$$Z(s; i_1, i_2, \ldots, i_r) := \int_0^\infty d^\times y_1 \int_0^\infty d^\times y_2 W(\widetilde{y}; X_{(i_r,-1)} \cdots X_{(i_2,-1)} \cdot X_{(i_1,-1)}) \times W'(\mathrm{diag}(y_1,1)) f^{(v)}_{J,-q_1+r}(s, \mathrm{diag}(y_1y_2, y_2^{-1}))(y_1y_2)^{-2}.$$

Here we have used the assumption on the support of W. Hence all we have to show is that the integral $Z(s; i_1, i_2, \ldots, i_r)$ is zero unless $\vec{i} = (i_1, i_2, \ldots, i_r) = (-1, -1, \ldots, -1)$. Fix $\vec{i} \in \{\pm 1\}^r$ such that $\vec{i} \neq (-1, -1, \ldots, -1)$. We let $1 \leq p \leq n$ be the least integer such that $i_p = 1$ and use induction on p. If $p = 1$, then we have

$$W(\widetilde{y}; X_{(i_r,-1)} \cdots X_{(i_2,-1)} \cdot X_{(i_1,-1)}) = 0.$$

Hence the integral $Z(s; i_1, i_2, \cdots, i_r)$ becomes zero. Now we suppose that $p \geq 2$. By

$$\begin{aligned} &X_{(i_r,-1)} \cdots X_{(i_2,-1)} \cdot X_{(i_1,-1)} \\ &\quad = X_{(i_r,-1)} \cdots X_{(i_{p+1},-1)} \cdot X_{(1,-1)} \cdot (X_{(-1,-1)})^{p-1} \\ &\quad = X_{(i_r,-1)} \cdots X_{(i_{p+1},-1)} \cdot (X_{(-1,-1)} \cdot X_{(1,-1)} + X_{(0,-2)}) \cdot (X_{(-1,-1)})^{p-2} \\ &\quad = X_{(i_r,-1)} \cdots X_{(i_{p+1},-1)} \cdot X_{(-1,-1)} \cdot X_{(1,-1)} \cdot (X_{(-1,-1)})^{p-2} \\ &\qquad\qquad + X_{(0,-2)} \cdot X_{(i_r,-1)} \cdots X_{(i_{p+1},-1)} \cdot (X_{(-1,-1)})^{p-2} \end{aligned}$$

and the hypothesis of induction, we know that

$$
\begin{aligned}
&Z(s; i_1, i_2, \ldots, i_r) \\
&\quad = \int_0^\infty d^\times y_1 \int_0^\infty d^\times y_2 W(\widetilde{y}; X_{(0,-2)} \cdot X_{(i_r,-1)} \cdots X_{(i_{p+1},-1)} \cdot (X_{(-1,-1)})^{p-2}) \\
&\qquad\qquad \times W'(\operatorname{diag}(y_1, 1)) f(s, \operatorname{diag}(y_1 y_2, y_2^{-1})) \times (y_1 y_2)^{-2}.
\end{aligned}
$$

In order to compute this, we define a subgroup M_J^+ of $\mathsf{H}_{\mathsf{k}_v}$ by

$$
M_J^+ := \left\{ m = (m_1, m_2) \in \mathsf{H}_{\mathsf{k}_v} \;\middle|\; m_1 = \begin{pmatrix} y_1 y_2 & \\ & y_2^{-1} \end{pmatrix}, y_i > 0, m_2 \in \mathrm{GL}(2)_{\mathsf{k}_v} \right\}.
$$

Note that $\det(m_2) = y_1 > 0$ and that the function

$$
W(m; X_{(0,-2)} \cdot X_{(i_r,-1)} \cdots X_{(i_{p+1},-1)} \cdot (X_{(-1,-1)})^{p-2}) W'(m_2)
$$

on M_J^+ is right invariant under $M_J^+ \cap K_v^0$. Hence the integral $Z(s; i_1, i_2, \ldots, i_r)$ equals

$$
\begin{aligned}
&\int_{\mathsf{Z}_{\mathsf{k}_v}(M_J^+ \cap \mathsf{N}_{\mathsf{k}_v}) \backslash M_J^+} W(m; X_{(0,-2)} \cdot X_{(i_r,-1)} \cdots X_{(i_{p+1},-1)} \cdot (X_{(-1,-1)})^{p-2}) \\
&\qquad\qquad \times W'(m_2) f(s, m_1) \times (y_1 y_2^2)^{-1}\, dm \\
&\quad = \int_{\mathsf{Z}_{\mathsf{k}_v}(M_J^+ \cap \mathsf{N}_{\mathsf{k}_v}) \backslash M_J^+} W(m; X_{(i_r,-1)} \cdots X_{(i_{p+1},-1)} \cdot (X_{(-1,-1)})^{p-2}) \\
&\qquad\qquad \times W'(m_2; -X_{(0,-2)}) f(s, m_1) (y_1 y_2^2)^{-1}\, dm.
\end{aligned}
$$

where dm stands for the usual right invariant measure. Since $W'(m_2; X_{(0,-2)}) = 0$ by assumption we have $Z(s; i_1, i_2, \ldots, i_r) = 0$. ■

5.4 Proof of Proposition 5.1, Case (b)

Recall some explicit formulae of local Whittaker functions for $\Pi_v \cong I(P_1; \sigma_{n,\epsilon}, 0, \nu)$. We first suppose that $\epsilon(-1)^n = 1$. Then there exists a unique, up to constant multiple, local Whittaker function $W_{(-n,-n)} \in \mathsf{Wh}(\Pi_v, \psi_v)$ of type $(-n, -n)$ satisfying $W_{(-n,-n)}(g; X_{(1,-1)}) = 0$. By [Mi-O-1, Proposition 8.1], $W_{(-n,-n)}(\widetilde{y}) = 0$ if $y_1 < 0$ $(i = 0, 1)$. If $y_1 > 0$, then it follows from [Mo-2, Theorem 3] that

$$
\begin{aligned}
(5.14)\quad & W_{(-n,-n)}(\widetilde{y}) = \\
& C_1 \times \frac{e^{-2\pi y_1}}{(2\pi\sqrt{-1})^2} \times \int_{L_1} ds_1 \int_{L_2} ds_2\, (4\pi^3 y_1 y_2^2)^{(-s_1+\alpha_1)/2} (4\pi y_1)^{(-s_2+\alpha_2)/2} \\
&\qquad \times \Gamma\Big(\frac{s_1 + s_2 + \alpha_3}{2}\Big) \Gamma\Big(\frac{s_1 + s_2 + \alpha_4}{2}\Big) \Gamma\Big(\frac{s_1}{2}\Big) \Gamma\Big(-\frac{s_2}{2}\Big)
\end{aligned}
$$

with

$$\alpha_1 = n+1, \quad \alpha_2 = n, \quad \alpha_3 = \nu_1 - n + 1, \quad \alpha_4 = -\nu_1 - n + 1.$$

Here and in (5.15) below the paths of integration are taken as in Lemma 5.2 and C_1 is a non-zero constant.

Next we suppose that $\Pi_\nu \cong I(P_1; \sigma_{n,\epsilon}, 0, \nu)$ with $\epsilon(-1)^{n+1} = 1$. Then there exists a unique, up to constant multiple, local Whittaker function $W_{(1-n,-n)} \in \mathrm{Wh}(\Pi_\nu, \psi_\nu)$ of type $(1-n, -n)$ satisfying $W_{(1-n,-n)}(g; X_{(1,-1)}) = 0$. We also define

$$W_{(-n,1-n)}(g) \in \mathrm{Wh}(\Pi_\nu, \psi_\nu) \text{ by } W_{(-n,1-n)}(g) := W_{(1-n,-n)}(g; -\overline{X_{(1,-1)}}/2).$$

By [Mi-O-1, Proposition 8.2] $W_{(-n+i,-n+1-i)}(\widetilde{y}) = 0$ for $y_1 < 0$ $(i = 0, 1)$. For $y_1 > 0$, we can deduce from [Mi-O-2] the following explicit formulae:

$$\tag{5.15} \begin{aligned} W_{(-n+i,-n+1-i)}(\widetilde{y}) = {} & C_1 \times \frac{e^{-2\pi y_1}}{(2\pi\sqrt{-1})^2} \\ & \times \int_{L_1} ds_1 \int_{L_2} ds_2\, (4\pi^3 y_1 y_2^2)^{(-s_1+\alpha_1)/2} (4\pi y_1)^{(-s_2+\alpha_2)/2} \\ & \times \Gamma\Big(\frac{s_1+s_2+\alpha_3}{2}\Big)\Gamma\Big(\frac{s_1+s_2+\alpha_4}{2}\Big)\Gamma\Big(\frac{s_1}{2}\Big)\Gamma\Big(-\frac{s_2}{2}\Big)\, p_i(s) \end{aligned}$$

with

$$\alpha_1 = n+1-i; \quad \alpha_2 = n-1+i; \quad \alpha_3 = \nu_1 - n + 2;$$
$$\alpha_4 = -\nu_1 - n + 2; \quad p_1(s_1) = (-\sqrt{-1}s_1)/2; \quad p_0(s_1) = 1.$$

Here $C_1 \in \mathbf{C}^\times$ is a constant common to $i = 0, 1$. By using (5.14) and (5.15), we have the following.

Proposition 5.4 *If we take a triplet* $(W, W', f) \in \mathrm{Wh}(\Pi_\nu, \psi_\nu) \times \mathrm{Wh}(\sigma_\nu, \psi_\nu) \times \mathcal{I}^J(1, \omega_\nu^{-1}; s)$ *as below, then the equations* (5.1) *and* (5.2) *hold.*

(i) *the case of* $\epsilon(-1)^n = 1$ *and* $\sigma_\nu \cong D_l[0]$*:*

$$(W(g), W'(h_2), f(s,h_1)) :=$$
$$\begin{cases} (W_{(-n,-n)}(g), W'_n(h_2), f^{(\nu)}_{J,n}(s,h_1)) & \text{if } l \le n \text{ and } l \equiv n \pmod 2; \\ (W_{(-n,-n)}(g; X_{(1,1)}), W'_{n-1}(h_2), f^{(\nu)}_{J,n-1}(s,h_1)) & \text{if } l \le n \text{ and } l \equiv n-1 \pmod 2; \\ (W_{(-n,-n)}(g; X^{l-n}_{(-1,-1)}), W'_l(h_2), f^{(\nu)}_{J,l}(s,h_1)) & \text{if } l \ge n; \end{cases}$$

(ii) *the case of* $\epsilon(-1)^{n+1} = 1$ *and* $\sigma_\nu \cong D_l[0]$*:*

$$(W(g), W'(h_2), f(s,h_1)) :=$$
$$\begin{cases} (W_{(1-n,-n)}(g), W'_n(h_2), f^{(\nu)}_{J,n-1}(s,h_1)) & \text{if } l \le n \text{ and } l \equiv n \pmod 2; \\ (W_{(-n,1-n)}(g), W'_{n-1}(h_2), f^{(\nu)}_{J,n}(s,h_1)) & \text{if } l \le n \text{ and } l \equiv n-1 \pmod 2; \\ (W_{(1-n,-n)}(g; X^{l-n}_{(-1,-1)}), W'_l(h_2), f^{(\nu)}_{J,l-1}(s,h_1)) & \text{if } l \ge n; \end{cases}$$

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(iii) *the case of* $\epsilon(-1)^n = 1$ *and* $\sigma_\nu \cong I_{\mathrm{B}'_{\mathbf{R}}}^{\mathrm{GL}(2)_{\mathbf{R}}}(\epsilon_1, \epsilon_2; 0, \nu)$*:*

$$\begin{aligned}&(W(g), W'(h_2), f(s,h_1))\\ &:= \begin{cases}(W_{(-n,-n)}(g), W'_n(h_2), f_{J,n}^{(\nu)}(s,h_1)) & \text{if } (-1)^n = (\epsilon_1\cdot\epsilon_2)(-1);\\ (W_{(-n,-n)}(g;X_{(1,1)}), W'_{n-1}(h_2), f_{J,n-1}^{(\nu)}(s,h_1)) & \text{if } (-1)^{n+1} = (\epsilon_1\cdot\epsilon_2)(-1);\end{cases}\end{aligned}$$

(iv) *the case of* $\epsilon(-1)^{n+1} = 1$ *and* $\sigma_\nu \cong I_{\mathrm{B}'_{\mathbf{R}}}^{\mathrm{GL}(2)_{\mathbf{R}}}(\epsilon_1, \epsilon_2; 0, \nu)$*:*

$$\begin{aligned}&(W(g), W'(h_2), f(s,h_1))\\ &\qquad := \begin{cases}(W_{(1-n,-n)}(g), W'_n(h_2), f_{J,n-1}^{(\nu)}(s,h_1)) & \text{if } (-1)^n = (\epsilon_1\cdot\epsilon_2)(-1);\\ (W_{(-n,1-n)}(g), W'_{n-1}(h_2), f_{J,n}^{(\nu)}(s,h_1)) & \text{if } (-1)^{n+1} = (\epsilon_1\cdot\epsilon_2)(-1).\end{cases}\end{aligned}$$

Here we set $X_{(1,1)} := \overline{X_{(-1,-1)}} \in \mathfrak{g}_\nu$.

Proof In Cases (i) and (iii), we note

$$W_{(-n,-n)}(\widetilde{y}; X_{(1,1)}) = (-4\pi\sqrt{-1}y_2)W_{(-n,-n)}(\widetilde{y})$$

and that $W_{(-n,-n)}(g; X_{(1,1)})$ is of type $(1-n, 1-n)$. Then we can compute the local zeta integral in each case by using Lemmas 5.2 and 5.3. ■

Hence Proposition 5.1 holds in Case (b), too.

Remark. From the computation above, we can obtain sufficient conditions for the L-function $L(s, \Pi\times\sigma)$ being entire. For example, in Case (a-i), $L(s,\Pi\times\sigma)$ is entire unless $l = \lambda_1 \pm \lambda_2$.

6 Application to the Spinor L-Function of Π

If we replace the cusp form φ in the zeta integral $Z(s, F\otimes\varphi, f)$ by an Eisenstein series, we get a product of two spinor L-functions of Π. This is quite analogous to the case of the original Rankin–Selberg convolution for $\mathrm{GL}(2)\times\mathrm{GL}(2)$ [Shm, (4.3), p. 799]. As a simple application of this "decomposition formula", we shall give another proof of the following:

Proposition 6.1 (**[Mo-3, A-S-2]**) *Suppose that the base field* k *is* $\mathbf{Q}$. *Let* Π *be a cuspidal automorphic representation of* $\mathrm{G}_{\mathbf{A}}$ *with trivial central character satisfying assumptions* A.1 *and* A.2. *Moreover we assume that the local component* Π_p *at each finite place* p *is unramified. Then the spinor* L*-function* $L(s,\Pi)$ *is continued to an entire function.*

Remark. This was already proved in [Mo-3] without any assumptions on ω_Π and Π_p ($p<\infty$). Furthermore Asgari and Shahidi [A-S-2] proved that the spinor L-function $L(s,\Pi)$ is entire for an arbitrary cuspidal automorphic representation Π with no conditions on the base field k and archimedean types.

Proof For a Schwartz-Bruhat function Φ^i on $\mathbf{A}^2$ ($i = 1, 2$), we define the global Jacquet section $f_{\Phi^i}(s, h_i) \in \mathcal{I}^J(s, 1, 1)$ as in (2.2). Consider the following integral

$$(6.1)\quad Z(s,\nu) \equiv Z(s,\nu;F;f_{\Phi^2},f_{\Phi^1}) := \int_{\mathrm{Z_A H_Q \backslash H_A}} F(h)E(\tfrac{\nu+1}{2},h_2;f_{\Phi^2})E(s,h_1;f_{\Phi^1})dh,$$

which converges absolutely on every compact subset in $\mathbf{C}^2$ not containing the poles of two Eisenstein series. The integral (6.1) can be considered as a special case of Andrianov's zeta integral for the spinor L-function of Π (cf. [PS, Ha]). Since the proof of the basic identity (Proposition 3.2) does not necessitate the cuspidality of φ, we have

$$(6.2)\qquad Z(s,\nu;F;f_{\Phi^2},f_{\Phi^1}) = \int_{\mathrm{Z_A N^H{}_A \backslash H_A}} \mathcal{W}_F(h)\mathcal{W}_{\Phi^2}(\tfrac{\nu+1}{2};h_2)f_{\Phi_1}(s,h_1)dh,$$

with

$$\mathcal{W}_{\Phi^2}(\tfrac{\nu+1}{2};h_2) = \int_{\mathbf{Q}\backslash\mathbf{A}} E(\tfrac{\nu+1}{2},\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}h_2;f_{\Phi_2})\psi(-x)dx$$
$$= \int_{\mathbf{A}} f_{\Phi^2}(\tfrac{\nu+1}{2},w_0\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}h_2)\psi(-x)dx.$$

Here the right-hand side of (6.2) converges absolutely for $\mathrm{Re}(s) - |\mathrm{Re}(\nu)|/2 \gg 0$. We shall compute the integral $Z(s,\nu)$ in the following situation:

(a) $\mathcal{W}_F$ is decomposable and $\mathcal{W}_F^{(p)}$ is the unramified local Whittaker function for each finite place $v = p < \infty$;
(b) the local Whittaker function $\mathcal{W}_F^{(\infty)}$ at the real place is of type $(-q_1, -q_2)$;
(c) $\Phi^i(x,y) = \prod_v \Phi^i_v(x_v,y_v)$ where Φ^i_p is the characteristic function of $\mathbf{Z}_p^2$ for $p < \infty$ and $\Phi^i_\infty = \Phi_{q_i}$ ($i = 1, 2$).

Then we have
$$\mathcal{W}_{\Phi^2}(\tfrac{\nu+1}{2};h_2) = \prod_v \mathcal{W}_{\Phi^2_v}(\tfrac{\nu+1}{2};h_{2,v})$$
with
$$\mathcal{W}_{\Phi^2_v}(\tfrac{\nu+1}{2};h_{2,v}) := \int_{\mathbf{Q}_v} f_{\Phi^2_v}(\tfrac{\nu+1}{2},w_0\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}h_{2,v})\psi_v(-x)dx.$$

The functions $\mathcal{W}_{\Phi^2_v}(\frac{\nu+1}{2};h_{2,v})$ are known as Jacquet's integrals (cf. [J-1, Sections 4 and 6]). For a finite place $p < \infty$, $\mathcal{W}_{\Phi^2_p}(\frac{\nu+1}{2};h_{2,p})$ equals the unramified Whittaker function with Satake parameter $\mathrm{diag}(p^{\nu/2},p^{-\nu/2})$ whose value at the identity is 1. On the other hand,

$$\mathcal{W}_{\Phi_{q_2}}\Big(\tfrac{\nu+1}{2};\begin{pmatrix}y_1 & \\ & 1\end{pmatrix}\Big) =$$
$$(\sqrt{-1})^{q_2}\pi^{-q_2/2}\Gamma(\frac{\nu+1+|q_2|}{2})\Gamma(\frac{\nu+1+q_2}{2})^{-1}W_{q_2/2,\nu/2}(4\pi y_1),\quad y_1 > 0.$$

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If we choose the local Whittaker function $\mathcal{W}_F^{(\infty)}$ of type $(-q_1, -q_2)$ ($q_i \in 2\mathbf{Z}$) as in Section 5 so as to get the correct Gamma factor, we have

$$Z(s, \nu; F; f_{\Phi_1}, f_{\Phi_2}) = C \times \frac{\Gamma(\frac{\nu+1+|q_2|}{2})}{\Gamma(\frac{\nu+1+q_2}{2})} L(s+\nu/2, \Pi)L(s-\nu/2, \Pi)$$

for some $C \in \mathbf{C}^\times$. Fix a real number $\kappa \gg 0$ so that the Euler product defining $L(s, \Pi)$ converges absolutely for $\mathrm{Re}(s) > \kappa - \epsilon$ for some $\epsilon > 0$. For a fixed $s_0 \in \mathbf{C}$, we can take $\kappa_1(\geq \kappa)$ so that $Z(\frac{s+\kappa_1}{2}, s-\kappa_1)$ is holomorphic at $s = s_0$. Since

$$Z(\tfrac{s+\kappa_1}{2}, s-\kappa_1) = C \times L(s, \Pi)L(\kappa_1, \Pi) \quad \text{for some } C \in \mathbf{C}^\times,$$

the spinor L-function $L(s, \Pi)$ is entire. ■

Remark. If we replace the cusp form φ by a constant function, then we get the following vanishing result, which is essentially obtained in [A-G-R, Proposition 2] in a more general situation.

Proposition 6.2 *Let F be a (not necessarily generic) cusp form on $\mathrm{G}_{\mathbf{A}}$ satisfying $F(zg) = F(g)$ ($\forall z \in \mathrm{Z}_{\mathbf{A}_k}$). Then we have*

$$\int_{\mathrm{Z}_{\mathbf{A}_k}\mathrm{H}_k\backslash\mathrm{H}_{\mathbf{A}_k}} F(h)dh = 0.$$

If the representation π_F of G_{k_v} generated by F for a real place v is equivalent to the one appearing in the assumption A.2, then this vanishing result can be proved by a local argument, too. In fact,

$$\int_{\mathrm{Z}_{\mathbf{A}_k}\mathrm{H}_k\backslash\mathrm{H}_{\mathbf{A}_k}} F'(hg_v)dh, \quad F' \in \pi_F, g_v \in \mathrm{G}_{k_v}$$

defines an element in $\mathrm{Hom}_{\mathfrak{g}_v, K_v}(\pi_F, C^\infty(\mathrm{H}_{k_v}\backslash\mathrm{G}_{k_v}))$. However this intertwining space is zero by [Mo-1, Theorem 6.5].

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