A NEW GROUP ALGEBRA FOR LOCALLY COMPACT GROUPS II

JOHN ERNEST

In an earlier work, we defined and described a new group algebra $\mathfrak{A}(G)$, which is a von Neumann algebra containing the group G **(3)**. In this paper we continue this study be relating the lattice of normal subgroups of the group G to the lattice of central projections of the group algebra $\mathfrak{A}(G)$. More precisely, we shall exhibit a one-to-one mapping ϕ of the lattice of closed normal subgroups of G into the lattice of central projections of $\mathfrak{A}(G)$, having the property that if $N_1 \subset N_2$, then $\phi(N_2) \leqslant \phi(N_1)$. Further, the correspondence is such that if N is a closed normal subgroup of G and E is its corresponding central projection in $\mathfrak{A}(G)$, then the group algebra $\mathfrak{A}(G/N)$ of the quotient group G/N is just the induced von Neumann algebra $\mathfrak{A}(G)_E$.

In order to obtain this result, we require a different axiomatic description of the group algebra $\mathfrak{A}(G)$ than was given in **(3)**. In Section 1 we give this new set of axioms and show that it is equivalent to that given in **(3)**. In Section 2 we use this axiomatic description to obtain the main result, Theorem 2.1, described in the previous paragraph. In Section 3 we prove that the group algebra $\mathfrak{A}(G)$ is a Type I von Neumann algebra if and only if the group G is of Type I. We use this fact to illustrate the sense in which the elementary operation of taking quotient groups of locally compact groups parallels the elementary operation of induction for von Neumann algebras.

1. A new set of axioms for the group algebra $\mathfrak{A}(G)$**.** Let G denote a separable locally compact group. The group algebra $\mathfrak{A}(G)$ was defined in **(3)** as follows. Let $\mathfrak{H}$ denote an infinite-dimensional separable complex Hilbert space. Let G^c denote the collection of strongly continuous unitary representations of G, with representation space $\mathfrak{H}$.

DEFINITION 1.1. *Let $\mathfrak{A}(G)$ denote the set of all maps J on G^c, whose values are operators on the space $\mathfrak{H}$, satisfying the following two properties:*

(1) $\operatorname{Sup}\{\|J(L)\| : L \in G^c\} < +\infty$;

(2) *if M and N are elements of G^c and if U is an isometric linear mapping of the representation space of $M \oplus N$ onto $\mathfrak{H}$, then*

$$J(U(M \oplus N)U^{-1}) = U(J(M) \oplus J(N))U^{-1}.$$

The $*$-algebra operations in the set $\mathfrak{A}(G)$ are then defined pointwise.

The author has found that, in practice, Condition (2) is often inconvenient

Received May 11, 1964.

https://doi.org/10.4153/CJM-1965-060-0 Published online by Cambridge University Press

and unpleasant to verify. This is partially owing to the fact that G^c is not the most natural "concrete dual." For one thing, it does not contain the finite-dimensional representations. It is perhaps interesting to note that the group algebra $\mathfrak{A}(G)$ may be defined and studied without ever considering finite-dimensional representations. However, in many contexts, and in particular in the computations of this paper, finite-dimensional representations do arise. Furthermore, G^c is not closed under the operations which give structure to the dual. For these reasons we shall consider in this paper a new concrete dual.

Let G^s denote the set of all strongly continuous unitary representations of G, with representation space some closed subspace of $\mathfrak{H}$. Clearly $G^c \subset G^s$. But G^s contains the finite-dimensional representations of G. Further, G^s is closed under some crucial representation theoretic operations. More specifically, if L is contained in G^s, then every subrepresentation of L is contained in G^s. Here the term "subrepresentation" is used in the concrete sense of being the restriction of L to an invariant subspace. We use the notation $M \leqslant L$ to denote the fact that M is a subrepresentation of L, in this concrete sense. The superscript s in the notation G^s is to emphasize the fact that G^s is closed under the operation of taking subrepresentations. If L is an element of G^s, $\mathfrak{H}(L)$ will denote the representation space of L. Finally, addition of representations may be defined within G^s. Specifically, if $L \in G^s, M \in G^s$, and $\mathfrak{H}(L) \perp \mathfrak{H}(M)$, then $L \oplus M \in G^s$.

PROPOSITION 1.2. *Each J in the group algebra $\mathfrak{A}(G)$ has a unique extension to all of G^s such that:*

(1) *$J(L)$ is an operator on the representation space $\mathfrak{H}(L)$ of L, for all L in G^s;*

(2) *if L and M are elements of G^s and $M \leqslant L$, then $J(M)$ is the restriction of $J(L)$ to the representation space $\mathfrak{H}(M)$ of M.*

Proof. If M is an element of G^s, let L denote an element of G^c for which $M \leqslant L$. It is clear that there is at least one such L. Indeed let I denote the identity representation of G, on the space $\mathfrak{H} - \mathfrak{H}(M)$, and let $L = M \oplus I$. We define $J(M)$ to be the restriction of $J(L)$ to $\mathfrak{H}(M)$. We must show that $J(M)$ is well defined.

Suppose L and L' are two elements of G^c for which $M \leqslant L$ and $M \leqslant L'$. Let E denote the projection on $\mathfrak{H}$, with range space $\mathfrak{H}(M)$. Then $\mathfrak{A}(L)_E = \mathfrak{A}(L')_E = \mathfrak{A}(M)$. (For any representation L, $\mathfrak{A}(L)$ denotes the von Neumann algebra generated by the range of L.) We then consider two homomorphisms ϕ and ϕ' of $\mathfrak{A}(G)$ into $\mathfrak{A}(M)$, defined as the composite of the following homomorphisms:

$$\phi\colon \mathfrak{A}(G) \xrightarrow{L} \mathfrak{A}(L) \to \mathfrak{A}(L)_E = \mathfrak{A}(M),$$

$$\phi'\colon \mathfrak{A}(G) \xrightarrow{L'} \mathfrak{A}(L') \to \mathfrak{A}(L')_E = \mathfrak{A}(M).$$

https://doi.org/10.4153/CJM-1965-060-0 Published online by Cambridge University Press

Here we have used the fact (**3**, Theorem 8.3) that every representation L of G may be extended uniquely to all of $\mathfrak{A}(G)$.

Notice that ϕ and ϕ' are identical on the group elements. Indeed $\phi(x) = M_x = \phi'(x)$ for all x in G. Since G generates $\mathfrak{A}(G)$ (**3**, Theorem 7.2), we have that ϕ and ϕ' are identical on $\mathfrak{A}(G)$.

Hence for all J in $\mathfrak{A}(G)$, $\phi(J) = \phi(J')$ implies that the restriction of $J(L)$ to the range of E is equal to the restriction of $J(L')$ to the range of E. Thus $J(M)$ is well defined. The reader may now easily verify that (1) and (2) are satisfied. Clearly, Condition (2) determines the extension of J to G^s completely and hence the extension is unique with respect to this property.

COROLLARY 1.3. *Each element J in $\mathfrak{A}(G)$, considered as a mapping on G^s, satisfies the following properties:*

(1) *$J(L) \in \mathfrak{A}(L)$ for all L in G^s,*

(2) *if L and M are in G^s and $\mathfrak{H}(L) \perp \mathfrak{H}(M)$, then $J(L \oplus M) = J(L) \oplus J(M)$;*

(3) *if L and M are in G^s and U is a linear isometry of $\mathfrak{H}(L)$ onto $\mathfrak{H}(M)$ such that $L = U^{-1}MU$, then $J(L) = U^{-1}J(M)U$.*

Proof. (1) Suppose L is in G^s, and $L \leqslant L'$, where $L' \in G^c$. Let E denote the projection of $\mathfrak{H}$ onto $\mathfrak{H}(L)$. If T is an operator on $\mathfrak{H}$ which commutes with E, let T_E denote the restriction of T to the range of E. Then by (**3**), Theorem 4.1, $J(L') \in \mathfrak{A}(L')$ and hence

$$J(L) = J(L')_E \in \mathfrak{A}(L')_E = \mathfrak{A}(L).$$

(2) Let E $[F]$ denote the projection of $\mathfrak{H}(L) \oplus \mathfrak{H}(M)$ onto $\mathfrak{H}(L)$ $[\mathfrak{H}(M)]$. Then using part 2 of Proposition 1.2 we have

$$J(L \oplus M) = J(L \oplus M)_E \oplus J(L \oplus M)_F = J(L) \oplus J(M).$$

(3) Suppose first that L and M are both finite-dimensional representations in G^s and that U is a linear isometry of $\mathfrak{H}(L)$ onto $\mathfrak{H}(M)$ such that $L = U^{-1}MU$. Let E and F denote the projections of $\mathfrak{H}$ onto $\mathfrak{H}(L)$ and $\mathfrak{H}(M)$ respectively. Suppose L' is an element of G^c such that $L \leqslant L'$. Let U' denote an extension of the isometry U to an isometry of $\mathfrak{H}$ onto $\mathfrak{H}$. Let $M' = U'L'U'^{-1}$. Then $M \leqslant M'$ and $L' = U'^{-1}M'U'$. Further, $J(L') = U'^{-1}J(M')U'$ by (**3**), Theorem 4.1. Hence

$$J(L) = J(L')_E = (U'^{-1}J(M')U')_E = U^{-1}J(M')_F\, U = U^{-1}J(M)U.$$

If L and M are infinite-dimensional, this reasoning does not apply since the orthogonal complements of $\mathfrak{H}(L)$ and $\mathfrak{H}(M)$ may have different dimension and hence the isometry U may not be extendable to all of $\mathfrak{H}$. According to (**3**), Theorem 8.3, every separable infinite-dimensional unitary representation of G may be extended uniquely to a normal $*$-representation of $\mathfrak{A}(G)$. Since $U^{-1}L_x\, U = M_x$ for all x in G, we have that $U^{-1}L_J\, U = M_J$ for all J in $\mathfrak{A}(G)$. However, if $L \leqslant L'$, for some L'in G^c, then L_J is the restriction of $L'_J = J(L')$

to $\mathfrak{H}(L)$, i.e., $L_J = J(L)$. Similarly, $M_J = J(M)$. Thus the equation $U^{-1}L_J U = M_J$ becomes $U^{-1}J(L)U = J(M)$.

DEFINITION 1.4. *Let* $\mathfrak{B}(G)$ *denote the set of all maps* J *on* G^s, *satisfying the following properties:*

(1) *for each* L *in* G^s, $J(L)$ *is a bounded operator on* $\mathfrak{H}(L)$;

(2) $\sup\{||J(L)||: L \in G^s\} < +\infty$;

(3) *if* $L, M \in G^s$ *and* $L \leqslant M$, *then* $J(L)$ *is the restriction of* $J(M)$ *to* $\mathfrak{H}(L)$,

(4) *if* L *and* M *are elements of* G^s *and* U *is a linear isometry of* $\mathfrak{H}(L)$ *onto* $\mathfrak{H}(M)$ *such that* $L = U^{-1}MU$, *then* $J(L) = UJ(M)U^{-1}$.

$\mathfrak{B}(G)$ may be given a $*$-algebra structure by defining the $*$-algebra operations pointwise.

PROPOSITION 1.5. *In the above definition, Axiom* (3) *may be replaced by:*

(3′) *if* $L \in G^c$, $M \in G^c$ *and* $\mathfrak{H}(L) \perp \mathfrak{H}(M)$, *then* $J(L \oplus M) = J(L) \oplus J(M)$.

Proof. This is left to the reader.

We have just proved (Proposition 1.2) that every element J in $\mathfrak{A}(G)$ may be extended uniquely to all of G^s to give an element of $\mathfrak{B}(G)$. We next show that every element J in $\mathfrak{B}(G)$, when restricted to G^c, gives an element of $\mathfrak{A}(G)$.

PROPOSITION 1.6. *If* J *is an element of* $\mathfrak{B}(G)$, *then the restriction of* J *to* G^c *is an element of* $\mathfrak{A}(G)$.

Proof. We suppose that J is an element of $\mathfrak{B}(G)$. We must show that the restriction of J to G^c satisfies Axioms (1) and (2) of Definition 1.1. Axiom (1) follows immediately from Property (2) of Definition 1.4.

We next verify Axiom (2). We assume that M and N are elements of G^c and that U is a linear isometry of $\mathfrak{H}(M \oplus N)$ onto $\mathfrak{H}$. Let U_1 and U_2 denote linear isometries of $\mathfrak{H}$ onto two subspaces $\mathfrak{H}_1$ and $\mathfrak{H}_2$ of $\mathfrak{H}$, such that $\mathfrak{H}_1 \oplus \mathfrak{H}_2 = \mathfrak{H}$. Let $M_1 = U_1 M U_1^{-1}$ and $N_2 = U_2 N U_2^{-1}$. Then M_1 and N_2 are elements of G^s. Let $U_3 = (U_1 \oplus U_2)U^{-1}$. Then U_3 is an isometry of $\mathfrak{H}$ onto $\mathfrak{H}$. We next use Axioms (3′) and (4) of Definition 1.4 to verify that:

$$\begin{aligned} U_3 J(U(M \oplus N)U^{-1})U_3^{-1} &= J(U_3 U(M \oplus N)U^{-1}U_3^{-1}) \\ &= J(U_1 M U_1^{-1} \oplus U_2 N U_2^{-1}) \\ &= J(U_1 M U_1^{-1}) \oplus J(U_2 N U_2^{-1}) \\ &= U_1 J(M) U_1^{-1} \oplus U_2 J(N) U_2^{-1} \\ &= U_3 U(J(M) \oplus J(N))U^{-1}U_3^{-1}. \end{aligned}$$

Hence

$$J(U(M \oplus N)U^{-1}) = U(J(M) \oplus J(N))U^{-1}.$$

Propositions 1.2 and 1.6 together set up a one-to-one correspondence between $\mathfrak{A}(G)$ and $\mathfrak{B}(G)$ which is clearly a $*$-algebra isomorphism. Thus the considerations of this section may be summarized in the following theorem.

THEOREM 1.7. *$\mathfrak{A}(G)$ and $\mathfrak{B}(G)$ are isomorphic $*$-algebras. Thus Definitions* 1.1 *and* 1.4 *may be looked upon as alternative, but equivalent, axiomatic descriptions of the group algebra $\mathfrak{A}(G)$.*

2. The group algebra of quotient groups. The purpose of this section is to prove the following theorem.

THEOREM 2.1. *Let G denote a separable locally compact group. There is a one-to-one mapping ϕ of the lattice of closed normal subgroups of G into the lattice of central projections in $\mathfrak{A}(G)$ such that, if N_1 and N_2 are closed normal subgroups of G and $N_1 \subset N_2$, then $\phi(N_2) \leqslant \phi(N_1)$. The group algebra $\mathfrak{A}(G/N)$ of the quotient group G/N is isomorphic to the induced von Neumann algebra $\mathfrak{A}(G)_{\phi(N)}$.*

We shall prove this result by proving a series of propositions, which converges to Theorem 2.1.

PROPOSITION 2.2. *Let G be a separable locally compact group. With each closed normal subgroup N of G, we associate the projection $E = \phi(N)$, defined by:*

$$E = \sup\{F\colon F \text{ is a projection in } \mathfrak{A}(G) \text{ and } xF = F \text{ for all } x \text{ in } N\}.$$

Then E is a central projection. Further, if N_1 and N_2 are closed normal subgroups of G such that $N_1 \subset N_2$, then $\phi(N_2) \leqslant \phi(N_1)$.

Proof. The last statement is obvious from the definition of ϕ. Thus it is sufficient to verify that E is a central projection. Consider the von Neumann algebra $\mathfrak{A}(G)$ as acting on some Hilbert space, say H cf. (**3**, Section 6). For each closed normal subgroup N of G, let

$$\mathfrak{E}(N) = \{\psi\colon \psi \in H \text{ and } x\psi = \psi \text{ for all } x \text{ in } N\}.$$

One may easily verify that $\mathfrak{E}(N)$ is a closed subspace of H. Note further that $\mathfrak{E}(N)$ is invariant under the action of G. Indeed, suppose that $y \in G$ and $\psi \in \mathfrak{E}(N)$. Then for each x in N,

$$x(y\psi) = y(y^{-1}xy\psi) = y\psi.$$

Thus $y\psi \in \mathfrak{E}(N)$. Hence the projection, say E_0, with range $\mathfrak{E}(N)$, is contained in $\mathfrak{A}(G)'$, the commutator of $\mathfrak{A}(G)$. Further, $\mathfrak{E}(N)$ is invariant under $\mathfrak{A}(G)'$. Indeed if $T \in \mathfrak{A}(G)'$ and $\psi \in \mathfrak{E}(N)$, then $x(T\psi) = Tx\psi = T\psi$ for all x in N. Hence $T\psi \in \mathfrak{E}(N)$. Thus $E_0 \in \mathfrak{A}(G)$. Note that if F is any projection on H, for which $xF = F$ for all x in N, then $F \leqslant E_0$. Thus $E = E_0$ and E is a central projection with range $\mathfrak{E}(N)$.

PROPOSITION 2.3. *Let G be a separable locally compact group and let $E = \phi(G)$ denote the central projection of $\mathfrak{A}(G)$ given by Proposition* 2.2. *Then E is a minimal projection in $\mathfrak{A}(G)$.*

Proof. We first note that E has the property that, for every T in $\mathfrak{A}(G)$, $TE = \alpha(T)E$ for some complex number $\alpha(T)$. According to (**3**, Theorem 7.2),

https://doi.org/10.4153/CJM-1965-060-0 Published online by Cambridge University Press

the elements of G generate $\mathfrak{A}(G)$. Let $\sum \alpha_i x_i$ denote any linear combination of elements of G. Then

$$(\sum \alpha_i x_i)E = \sum \alpha_i x_i E = (\sum \alpha_i)E.$$

Thus the stated property holds for a $*$-algebra dense in $\mathfrak{A}(G)$. Suppose that T_λ is a net of elements of $\mathfrak{A}(G)$, converging weakly to T in $\mathfrak{A}(G)$, and suppose that each T_λ has the stated property. Then $T_\lambda E = \alpha_\lambda E$ converges weakly to TE. Thus $TE = \alpha E$ for some complex number α.

We next use this property to show that E is minimal. Suppose that F is a projection in $\mathfrak{A}(G)$ and $F \leqslant E$. By the previous paragraph we have $F = FE = \alpha(F)E$ for some complex number $\alpha = \alpha(F)$. Since F is a projection, it follows that either $\alpha = 0$ or $\alpha = 1$. Thus either $F = 0$ or $F = E$.

LEMMA 2.4. *Let N_1 and N_2 denote two closed normal subgroups of a separable locally compact group G. Suppose that $E = \phi(N_1) = \phi(N_2)$, where ϕ is defined in Proposition 2.2. Then there is a closed normal subgroup N of G such that $N_1 \subset N$, $N_2 \subset N$, and $\phi(N) = E$.*

Proof. Let $N = \{x: x \in G \text{ and } xE = E\}$. The reader may verify that N is a closed subgroup of G. Further, if $x \in N$ and $y \in G$, then

$$(yxy^{-1})E = yxEy^{-1} = yEy^{-1} = E.$$

Thus N is a normal subgroup. Since N leaves every element in the range of E fixed, we have $E \leqslant \phi(N)$. On the other hand,

$$\begin{aligned}\phi(N) &= \operatorname{Sup}\{F: F \text{ is a projection in } \mathfrak{A}(G) \text{ and } xF = F \text{ for all } x \in N\}\\ &\leqslant \operatorname{Sup}\{F: F \text{ is a projection in } \mathfrak{A}(G) \text{ and } xF = F \text{ for all } x \in N_1\} = E.\end{aligned}$$

Thus $\phi(N) = E$.

PROPOSITION 2.5. *The mapping ϕ of the lattice of closed normal subgroups of G into the lattice of central projections in $\mathfrak{A}(G)$, defined in Proposition 2.2, is one-to-one.*

Proof. We first recall the representation of $\mathfrak{A}(G)$ as a von Neumann algebra, given in (**3**, Section 6). The Hilbert space on which $\mathfrak{A}(G)$ acts is $H = \sum \oplus \mathfrak{H}_L$, where $\mathfrak{H}_L = \mathfrak{H}$ is the representation space of L, for each L in G^c. Thus a vector ψ in H is a vector-valued function $\{\psi_L\}$ on G^c such that

$$\sum_{L \in G^c} ||\psi_L||^2 < +\infty.$$

$\mathfrak{A}(G)$ acts on H by the rule $J\{\psi_L\} = \{J(L)\psi_L\}$ for all J in $\mathfrak{A}(G)$.

Let N_1 and N_2 denote two normal subgroups of G. The condition that $\phi(N_1) = \phi(N_2)$ is equivalent to the condition that, for every L in G^c and every ψ in $\mathfrak{H}$, $L_x \psi = \psi$ for all x in N_1, if and only if $L_x \psi = \psi$ for all x in N_2.

Let N_1 and N_2 denote two distinct closed normal subgroups of G. According to Lemma 2.4, there is no loss of generality if we assume that $N_1 \subset N_2$. We must show that $\phi(N_1)$ and $\phi(N_2)$ are distinct. According to the previous paragraph, to do this we need only exhibit a strongly continuous unitary representation L of G, on a separable Hilbert space $\mathfrak{H}$, and a vector in ψ such that $L_x \psi = \psi$ for all x in N_1 and $L_y \psi \neq \psi$ for some y in N_2. We next exhibit such a representation.

Form the quotient group G/N_1 and let $\tilde{L}$ denote its regular representation. Define the representation L of G by $L_x = \tilde{L}_{\tilde{x}}$, for all x in G, where $\tilde{x}$ denotes the N_1-coset containing x. It may easily be verified that L is a strongly continuous unitary representation of G. Let y denote any element of N_2 such that $y \notin N_1$. Then $\tilde{y}$ is distinct from the identity element of G/N_1. Since the regular representation $\tilde{L}$ of G/N_1 is faithful, it follows that $L_y = \tilde{L}_{\tilde{y}} \neq I$. Thus there exists a vector ψ in $\mathfrak{H}(L)$ such that $L_y \psi \neq \psi$. However, for all x in N_1, $L_x = \tilde{L}_{\tilde{x}} = I$ and hence $L_x \psi = \psi$ for all x in N_1.

Remark 2.6. $(G/N)^c$ may be embedded as a subset of G^c as follows. Suppose that $\tilde{L}$ is an element of $(G/N)^c$. Define L in G^c by $L_x = \tilde{L}_{\tilde{x}}$ for all x in G, where $\tilde{x}$ denotes the N-coset containing x. A trivial verification shows that L is a strongly continuous unitary representation of G, with representation space $\mathfrak{H}$. Identifying under this embedding, we may henceforth assume that $(G/N)^c$ is a subset of G^c. $(G/N)^c$ is then exactly the set of those elements L in G^c for which $L_x = I$ for all x in N. Similarly $(G/N)^s$ may be embedded as a subset of G^s.

J. M. G. Fell has defined a topology in G^c; cf. **(5)**. G^c is given the smallest (i.e., weakest) topology such that the maps $L \to (L_x \psi, \phi)$ are continuous, for all x in G and ψ, ϕ in $\mathfrak{H}$. In this topology, $(G/N)^c$ is a closed subset of G^c. Indeed suppose that $\{L^\lambda\}$ is a net of elements converging to an element L in G^c and suppose that each L^λ is an element in $(G/N)^c$. Then for each x in N, $L^\lambda{}_x = I$ converges weakly to L_x. Thus $L_x = I$ for every x in N. Hence $L \in (G/N)^c$.

Following the terminology of Mackey **(8)**, we say that two representations L and M are *disjoint* if no subrepresentation of L is equivalent to any subrepresentation of M.

PROPOSITION 2.7. *Each element L of G^s may be decomposed uniquely in the form*

$$L = L^{(1)} \oplus L^{(2)},$$

where $L^{(1)}$ and $L^{(2)}$ are elements of G^s such that $L^{(1)}$ is contained in $(G/N)^s$ and $L^{(2)}$ and M are disjoint for every M in $(G/N)^s$.

Proof. Let $\mathfrak{K}^{(1)}$ denote the closed linear subspace of $\mathfrak{H}(L)$ defined by

$$\mathfrak{K}^{(1)} = \{\psi : \psi \in \mathfrak{H}(L) \text{ and } L_x \psi = \psi \text{ for all } x \text{ in } N\}.$$

For each ψ in $\mathfrak{K}^{(1)}$, y in G, and x in N, we have

$$L_x(L_y \psi) = L_y L_{y^{-1}xy} \psi = L_y \psi.$$

Thus $\mathfrak{K}^{(1)}$ is an invariant subspace of L. Let $\mathfrak{K}^{(2)}$ denote the orthogonal complement of $\mathfrak{E}^{(1)}$ in $\mathfrak{H}(L)$. Let $L^{(1)}$ and $L^{(2)}$ denote the restrictions of L to the invariant subspaces $\mathfrak{E}^{(1)}$ and $\mathfrak{E}^{(2)}$ respectively. Clearly $L^{(1)}{}_x = I$ for all x in N and hence $L^{(1)} \in (G/N)^s$.

Next suppose there exists a non-trivial representation M in $(G/N)^s$ such that $L^{(2)}$ is not disjoint from M. Then $L^{(2)}$ contains a subrepresentation $M^{(2)}$ which is equivalent to some subrepresentation of M. Since $M \in (G/N)^s$, considered as a subset of G^s, we have $M_x = I$ for all x in N. Hence $M^{(2)}{}_x = I$ for all x in N. Let ψ denote a non-zero vector in $\mathfrak{H}(M^{(2)}) \subset \mathfrak{H}(L^{(2)})$. Then $L^{(2)}{}_x \psi = M^{(2)}{}_x \psi = \psi$ for all x in N. Hence $\psi \in \mathfrak{E}^{(1)}$, which contradicts the fact that ψ is a non-zero vector in $\mathfrak{H}(L^{(2)}) = \mathfrak{E}^{(2)}$, the orthogonal complement of $\mathfrak{E}^{(1)}$ in $\mathfrak{H}(L)$.

In the statement and proof of our next proposition, we shall use Definition 1.4 as our axiomatic description of the group algebra $\mathfrak{A}(G)$; cf. Theorem 1.7. Each J in $\mathfrak{A}(G)$ is thus a map on G^s. For each J in $\mathfrak{A}(G)$, let $\tilde{J}$ denote the restriction of J to the subset $(G/N)^s$ of G^s; cf. Remark 2.6. Clearly $\tilde{J}$ is an element of $\mathfrak{A}(G/N)$.

PROPOSITION 2.8. *The mapping $J \rightarrow \tilde{J}$ is a norm-decreasing strong (σ-strong, weak and σ-weak) continuous homomorphism of $\mathfrak{A}(G)$ onto $\mathfrak{A}(G/N)$.*

Proof. The map $J \rightarrow \tilde{J}$ is easily verified to be a norm-decreasing homomorphism. To see that it is continuous, suppose that J^λ is a net in $\mathfrak{A}(G)$ converging to J in one of the topologies of $\mathfrak{A}(G)$, say the strong topology. Then for all L in $(G/N)^c$, $\tilde{J}^\lambda(L) = J^\lambda(L)$ converges strongly to $J(L) = \tilde{J}(L)$. Thus, $\tilde{J}^\lambda$ converges to $\tilde{J}$ in the strong topology of $\mathfrak{A}(G/N)$.

We next verify that the mapping is onto. For this verification, we shall find it convenient to use Definition 1.4 as our definition of the group algebra $\mathfrak{A}(G)$; cf. Theorem 1.7. Suppose K is an element of $\mathfrak{A}(G/N)$. According to Proposition 2.7, each L in G^s may be expressed uniquely in the form $L = L^{(1)} \oplus L^{(2)}$, where $L^{(1)} \in (G/N)^s$ and $L^{(2)}$ and M are disjoint for every M in $(G/N)^s$. Define the mapping J on G^s by $J(L) = K(L^{(1)}) \oplus O$, where O denotes the zero operator on $\mathfrak{H}(L^{(2)})$. Clearly the restriction of J to $(G/N)^c$ is K. Thus it remains to show that J is an element of $\mathfrak{A}(G)$. We leave the verification of the first two axioms to Definition 1.4 to the reader.

We next verify that J satisfies Axiom (3) of Definition 1.4. Suppose L and M are elements of G^s and $L \leqslant M$. Then according to Proposition 2.7, $L = L^{(1)} \oplus L^{(2)}$ and $M = M^{(1)} \oplus M^{(2)}$, where $L^{(1)}$ and $M^{(1)}$ are elements of $(G/N)^s$, and $L^{(2)}$ and $M^{(2)}$ are disjoint from every element of $(G/N)^s$. Clearly $L \leqslant M$ implies that $L^{(1)} \leqslant M^{(1)}$ and $L^{(2)} \leqslant M^{(2)}$. Then $J(L) = K(L^{(1)}) \oplus O$, where O is the zero operator on $\mathfrak{H}(L^{(2)})$. Since $L^{(1)} \leqslant M^{(1)}$, $K(L^{(1)})$ is the restriction of $K(M^{(1)})$ to $\mathfrak{H}(L^{(1)})$. Thus $J(L) = K(L^{(1)}) \oplus O$ is the restriction of the operator $J(M) = K(M^{(1)}) \oplus O$ acting on the space

$$\mathfrak{H}(M) = \mathfrak{H}(M^{(1)}) \oplus \mathfrak{H}(M^{(2)})$$

to the subspace $\mathfrak{H}(L) = \mathfrak{H}(L^{(1)}) \oplus \mathfrak{H}(L^{(2)})$.

We next verify that J satisfies Axiom (4) of Definition 1.4. Suppose L and M are elements of G^s and that U is a linear isometry of $\mathfrak{H}(L)$ onto $\mathfrak{H}(M)$ such that $L = U^{-1}MU$. According to Proposition 2.7, $M = M^{(1)} \oplus M^{(2)}$, where $M^{(1)} \in (G/N)^s$ and $M^{(2)}$ is disjoint from $(G/N)^s$. Let $U = U_1 \oplus U_2$ such that

$$\begin{aligned} L = U^{-1}MU &= (U_1 \oplus U_2)^{-1}(M^{(1)} \oplus M^{(2)})(U_1 \oplus U_2) \\ &= (U_1^{-1}M^{(1)}U_1) \oplus (U_2^{-1}M^{(2)}U_2). \end{aligned}$$

Let $L^{(1)} = U_1^{-1}M^{(1)}U_1$ and $L^{(2)} = U_2^{-1}M^{(2)}U_2$. Then $L^{(1)} \in (G/N)^s$ and $L^{(2)}$ is disjoint from every element of $(G/N)^s$. Thus $L^{(1)}$ and $L^{(2)}$ are the subrepresentations of L given by Proposition 2.7. Hence we have

$$\begin{aligned} J(L) &= K(L^{(1)}) \oplus O \\ &= K(U_1^{-1}M^{(1)}U_1) \oplus O \\ &= U_1^{-1}K(M^{(1)})U_1 \oplus U_2^{-1}OU_2 \\ &= (U_1 \oplus U_2)^{-1}(K(M^{(1)}) \oplus O)(U_1 \oplus U_2) \\ &= U^{-1}(K(M^{(1)}) \oplus O)U \\ &= U^{-1}J(M)U. \end{aligned}$$

In this verification we have used the fact that K satisfies Axiom (4) of Definition 1.4. We have also used the same symbol O to denote both the zero operator on $\mathfrak{H}(L^{(2)})$ and the zero operator on $\mathfrak{H}(M^{(2)})$.

PROPOSITION 2.9. *Let G denote a separable locally compact group. If N is a closed normal subgroup of G, let E denote the corresponding central projection $\phi(N)$, given by Proposition 2.2. Then the group algebra $\mathfrak{A}(G/N)$ is isomorphic to the induced von Neumann algebra $\mathfrak{A}(G)_E$.*

Proof. Let ϕ_1 denote the homomorphism of $\mathfrak{A}(G)$ onto $\mathfrak{A}(G/N)$ given by Proposition 2.8. Let ϕ_2 denote the natural homomorphism of $\mathfrak{A}(G)$ onto $\mathfrak{A}(G)_E$; for the definition and properties of such homomorphisms, called inductions, see (**1**, Chapter 1, Section 2.1). We shall prove the proposition by proving that ϕ_1 and ϕ_2 have the same kernel.

To obtain this result we must once again consider the representation of $\mathfrak{A}(G)$ as a von Neumann algebra; cf. (**3**, Section 6). A vector ψ in the space H on which $\mathfrak{A}(G)$ acts is a vector-valued function on G^c, $L \to \psi_L$, where $\psi_L \in \mathfrak{H}$ for all L in G^c, and $\sum \|\psi_L\|^2 < +\infty$. The action of $\mathfrak{A}(G)$ on H is defined by $J\{\psi_L\} = \{J(L)\psi_L\}$. In particular, for x in G,

$$x\psi = x\{\psi_L\} = \{L_x \psi_L\}.$$

Thus the range of E is

$$\begin{aligned} \mathfrak{E} &= \{\psi : \psi \in H \text{ and } x\psi = \psi \text{ for all } x \text{ in } N\} \\ &= \{\{\psi_L\} : \{L_x \psi_L\} = \{\psi_L\} \text{ for all } x \text{ in } N\}. \end{aligned}$$

Let K_1 $[K_2]$ denote the kernal of ϕ_1 $[\phi_2]$. Then

$$\begin{aligned} K_1 &= \{J : J \in \mathfrak{A}(G) \text{ and } J(L) = 0 \text{ for all } L \text{ in } (G/N)^c\}, \\ K_2 &= \{J : J \in \mathfrak{A}(G) \text{ and } J\psi = 0 \text{ for all } \psi \text{ in } \mathfrak{E}\}. \end{aligned}$$

We first show that $K_2 \subset K_1$. Suppose $J \in K_2$ and $M \in (G/N)^c \subset G^c$. Then $M_x \psi = \psi$ for all x in N and ψ in $\mathfrak{H}$. Let ψ by any vector in $\mathfrak{H}$. Define the vector $\psi(M)$ in H as follows. $\psi(M)_L = \psi$ if $L = M$ and $\psi(M)_L = 0$ if $L \neq M$. Note that $x\psi(M) = \psi(M)$ for all x in N. Thus $\psi(M) \in \mathfrak{E}$. Since $J \in K_2$, we have $J\psi(M) = 0$. Looking at the Mth component of this vector in H, we conclude that $J(M)\psi = 0$. Since ψ is an arbitrary vector in $\mathfrak{H}$, we have that $J(M) = 0$. Thus J is contained in K_1.

We next show that K_1 is contained in K_2. Suppose L is an element of G^c. According to Proposition 2.7, $L = L^{(1)} \oplus L^{(2)}$, where $L^{(1)} \in (G/N)^s, L^{(2)} \in G^s$, and $L^{(2)}$ is disjoint from every M in $(G/N)^s$. Suppose ψ is a vector in $\mathfrak{H}$ such that $L_x \psi = \psi$ for all x in N. It follows from the proof of Proposition 2.7 that ψ is a vector in $\mathfrak{H}(L^{(1)})$. Suppose J is an element of K_1. By Proposition 1.2 and Corollary 1.3, J vanishes on $(G/N)^s$. By Corollary 1.3 we have

$$J(L) = J(L^{(1)}) \oplus J(L^{(2)}).$$

Thus if $\psi \in \mathfrak{H}$ and $L_x \psi = \psi$ for all x in N we have that $\psi \in \mathfrak{H}(L^{(1)})$ and hence $J(L)\psi = J(L^{(1)})\psi = 0$. Thus for each $\{\psi_L\}$ in $\mathfrak{E}$, we have

$$J\{\psi_L\} = \{J(L)\psi_L\} = 0.$$

Thus J is contained in K_2.

Remark. The isomorphism described in Proposition 2.9 and Theorem 2.1 is an isometry since both $\mathfrak{A}(G/N)$ and $\mathfrak{A}(G)_E$ are C^*-algebras; cf. (**9**, p. 311, Theorem 3). The fact that this isomorphism is a σ-weak and a σ-strong homeomorphism follows from (**3**, Theorem 6.1) and (**1**, p. 57, Corollary 1).

3. The group algebra of Type *I* groups. In this section we note that the group algebra $\mathfrak{A}(G)$ of a separable locally compact group G is a Type I von Neumann algebra if and only if G is a Type I group. Recall that a group is said to be *Type I* if the range of every strongly continuous unitary representation of G generates a Type I von Neumann algebra.

LEMMA 3.1. *Let G be a separable locally compact group. Then every normal cyclic representation of the group algebra $\mathfrak{A}(G)$ is separable.*

Proof. Let T denote a normal cyclic $*$-representation of $\mathfrak{A}(G)$ and let S denote a countable subset of G which is dense in G. By (**3**, Theorem 7.2), G, and hence S, generates $\mathfrak{A}(G)$. Let $\mathfrak{S}$ denote the set of all finite linear combinations, with coefficients whose real and imaginary parts are rational, of elements of S. Then $\mathfrak{S}$ is countable and weakly dense in $\mathfrak{A}(G)$. Thus $T\mathfrak{S} = \{T_x : x \in \mathfrak{S}\}$ is weakly dense in $\mathfrak{A}(T)$, the von Neumann algebra generated by the range of T. Let $\mathfrak{H}(T)$ denote the representation space of T and let ψ denote an element of $\mathfrak{H}(T)$ which is a cyclic vector for T. Then $T\mathfrak{S}\psi = \{T_x \psi : x \in \mathfrak{S}\}$ is a countable subset of $\mathfrak{H}(T)$ which is dense in $\mathfrak{A}(T)\psi$ and hence in $\mathfrak{H}(T)$. Thus $\mathfrak{H}(T)$ is separable.

https://doi.org/10.4153/CJM-1965-060-0 Published online by Cambridge University Press

COROLLARY 3.2. *Let G be a separable locally compact group. If every separable normal representation of $\mathfrak{A}(G)$ is Type I, then every normal representation of $\mathfrak{A}(G)$ is Type I.*

Proof. Every normal representation may be expressed as a direct sum of cyclic representations; cf., for example, (**9**, p. 241). Every direct summand, being cyclic and hence separable, is Type I. But the direct sum of Type I representations is Type I; cf. (**1**, p. 121, Proposition 1).

Remark. A completely similar line of reasoning gives the analogous result for separable locally compact groups. Thus if we call a separable locally compact group G Type I if all its strongly continuous unitary representations are Type I, we obtain the proposition that G is Type I if and only if all its separable strongly continuous unitary representations are Type I.

THEOREM 3.3. *Let G denote a separable locally compact group. Then G is a Type I group if and only if $\mathfrak{A}(G)$ is a type I von Neumann algebra.*

Proof. Suppose G is Type I. Then every separable strongly continuous unitary representation of G is Type I. By (**3**, Theorem 8.3), every separable normal representation of $\mathfrak{A}(G)$ is Type I. By (**3**, Theorem 6.1), $\mathfrak{A}(G)$ is Type I.

Conversely, suppose $\mathfrak{A}(G)$ is Type I. According to well-known facts about von Neumann algebras (**1**) every normal representation is the composition of an amplification, and induction, and a spatial isomorphism. Further, every amplification and induction of a Type I von Neumann algebra is Type I. Thus every normal representation of $\mathfrak{A}(G)$ is Type I. By (**3**, Theorem 8.3) it follows that every separable strongly continuous unitary representation of G is Type I. Thus G is Type I.

Remark. Another von Neumann algebra which is naturally associated with a locally compact group is the von Neumann algebra generated by the range of the regular representation. It is natural to conjecture, as did Irving Kaplansky in (**6**), that the analogue of the previous theorem holds for this von Neumann algebra. This has been shown to be false, in general, by George Mackey in (**7**). Professor Mackey shows that there exist non-type-I groups having Type I regular representations. Thus the previous theorem shows that the group algebra $\mathfrak{A}(G)$ dominates "the situation" in a much stronger sense than does the regular representation. "Type I-ness" of the group algebra $\mathfrak{A}(G)$ does imply that the group is Type I.

Remark. Theorem 2.1 attempts to show that the elementary operation of forming quotient groups, in the context of locally compact groups, parallels the elementary operation of induction, in the context of von Neumann algebras. Consider, for example, the well-known fact (**1**, p. 124, Proposition 4) that if a von Neumann algebra $\mathfrak{A}$ is Type I, then every induction $\mathfrak{A}_E$ is Type I. This fact, in the presence of Theorems 2.1 and 3.3, implies a parallel proposition for quotient groups.

PROPOSITION 3.4. *Let G denote a Type I separable locally compact group. Then for each closed normal subgroup N of G, the quotient group G/N is Type I.*

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University of Rochester
Rochester, New York

https://doi.org/10.4153/CJM-1965-060-0 Published online by Cambridge University Press