Nearly Approximate Transitivity (AT) for Circulant Matrices

David Handelman

Abstract. By previous work of Giordano and the author, ergodic actions of **Z** (and other discrete groups) are completely classified measure-theoretically by their dimension space, a construction analogous to the dimension group used in C*-algebras and topological dynamics. Here we investigate how far from approximately transitive (AT) actions can be that derive from circulant (and related) matrices. It turns out not very: although non-AT actions can arise from this method of construction, under very modest additional conditions, approximate transitivity arises. KIn addition, if we drop the positivity requirement in the isomorphism of dimension spaces, then all these ergodic actions satisfy an analogue of AT. Many examples are provided.

1 Introduction

Let (X, μ, T) (often abbreviated T) be a measure space with an invertible measurable transformation T (up to sets of measure zero), such that $\mu \circ T^k$ is absolutely continuous with respect to μ for all integers k. We also assume the action of T is ergodic, that is, any T-invariant set has measure zero or one.

Motivated by the fundamental results of Connes and Woods [CW] on the classification of such systems, particularly those that are approximately transitive, and by the ordered K_0-theoretic classification of C*-algebras initiated by Elliott [E], Giordano and I [GH] constructed a complete invariant for the measure theoretic classification of ergodic T, called a dimension space (together with a specified positive element thereof). This formed part of a more general construction, specialized to **Z**-actions, *i.e.*, single automorphisms.

Here we discuss a class of examples arising from (dual) abelian group actions on product type systems. These translate to circulant (and an obvious generalization called hemicirculant) matrix-valued random walks. Before we can state the results, we must describe the invariant in more detail.

We outline the construction given in [GH]. Let $A = \mathbf{R}[x, x^{-1}]$ be the usual Laurent polynomial ring. We use inner product notation to describe the coefficients of a polynomial p, that is, $p = \sum (p, x^k) x^k$. The ring A is equipped with the obvious partial ordering, $A^+ = \{ p \in A \mid (p, x^k) \geq 0 \text{ for all } k \}$, making it into a partially ordered ring. We impose the l^1-norm on A, $\|p\| = \sum |(p, x^k)|$, and, of course, the completion is $l^1(\mathbf{Z})$, with convolution-extending multiplication. The evaluation (or augmentation)

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map, from A to $\mathbf{R}$, given $p \mapsto p(1)$ is, of course, positive, and $\|p\| \geq |p(1)|$ with equality when $p \in A^+$.

Now form the space of columns of size n, denoted A^n. Equipped with the obvious (coordinatewise) positive cone, $(A^n)^+ = \{(p_1, p_2, \ldots, p_n)^T \in A^n \mid p_i \in A^+\}$, A^n becomes a partially ordered A-module, and multiplication by x (described by $(p_j)^T \mapsto (xp_j)^T$) is a positive invertible A-module transformation. In addition, the augmentation map extends to the obvious map $A^n \to \mathbf{R}^n$, which is again positive.

Now let $n(1), n(2), \ldots$, be an infinite sequence of positive integers, and let M_k be $n(k+1) \times n(k)$ matrices with entries from A^+. We can also evaluate each matrix entrywise using the augmentation map; the resulting real matrix $M_i(1)$ (obtained by evaluating each entry at $x = 1$) is, of course, nonnegative, and thus induces an order-preserving map $\mathbf{R}^{n(k)} \to \mathbf{R}^{n(k+1)}$. This gives rise to a (preliminary) direct limit construction.

$$A^{n(1)} \xrightarrow{M_1} A^{n(2)} \xrightarrow{M_2} A^{n(3)} \xrightarrow{M_3} \cdots \qquad\qquad G = \varinjlim M_k \colon A^{n(k)} \to A^{n(k+1)}$$

$$\mathbf{R}^{n(1)} \xrightarrow{M_1(1)} \mathbf{R}^{n(2)} \xrightarrow{M_2(1)} \mathbf{R}^{n(3)} \xrightarrow{M_3(1)} \cdots \qquad G(1) = \varinjlim M_k(1) \colon \mathbf{R}^{n(k)} \to \mathbf{R}^{n(k+1)}$$

The direct limit of the top row G is a partially ordered A-module, the direct limit of the bottom row is a partially ordered vector space, and the vertical map(s), augmentation, induce a positive onto map. Now we make another assumption on the matrices: that the column sums of all the $M_i(1)$ are one. This justifies the alternative name *matrix-valued random walks* [CW].

Now we must find a linear functional on the direct limit which will (eventually) translate to the measure μ. Elements of the j-th term in the top row will be indicated by (v, j), where $v \in A^{n(j)}$; the image of the latter in the direct limit will be denoted $[v, j]$, so that $[M_j v, j+1] = [v, j]$. Because we will be working almost exclusively with invariant measures, we are interested here only in x-invariant linear functionals, that is, positive linear functionals τ on G such that $\tau[xv, j] = \tau[v, j]$. By [GH, Lemma 2.3], these must factor through the vertical map $G \to G(1)$.

The direct limit $G(1)$ is itself a dimension group (over the reals, rather than the usual integers), and under weak conditions (*e.g.*, if all the $M_i(1)$ have no zero entries or if $\{n(i)\}$ is bounded and $G(1)$ is not the trivial group) will itself admit a positive linear functional known as a trace. In all our examples, such a ϕ will exist. Let $\phi \colon G(1) \to \mathbf{R}$ be a trace on $G(1)$; then $\tau := \phi \circ \rho$ is a positive invariant linear functional on G.

Now we complete G with respect to the L^1 norm induced by τ; (see [GoH]). On elements of any partially ordered abelian group J with a trace τ, define the (possibly pseudo-) norm

$$\|j\| = \inf\{\tau(j_1) + \tau(j_2) \mid j_1, j_2 \in J^+, j = j_1 - j_2\}.$$

In the case of G, the completion is of the form $H := L^1(Y, \mu)$ and the action of multiplication by x is an invertible positive isometry thereon. We can thus regard the

completion H as an ordered $l^1(\mathbf{Z})$-module. Moreover, μ is determined by the trace τ: τ extends to the completion, and $\int_Y f\, d\mu = \tau(f)$.
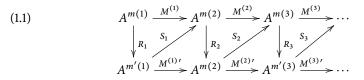
The action of x on Y (induced by the action of x on the H) is ergodic if and only if ρ, the trace on the vector space $G(1)$, is extremal as a trace thereon [GH, Proposition 3.8]. Although the Y is impossibly complicated to work with directly, we can easily decide when the action will be ergodic. Not only does this construction yield an ergodic $\mathbf{Z}$-action, but all ergodic $\mathbf{Z}$-actions with invariant measure can be realized by this method of construction. (This is part of a more general result of Elliott annd Giordano [EG], that every ergodic amenable action of a discrete group can be constructed using $\mathbf{R}G$ in place of $A = \mathbf{R}[x, x^{-1}] = \mathbf{R}[\mathbf{Z}]$, with suitable Radon–Nikodym derivatives in the non-invariant case.)

Although the space (Y, μ) is normally far too complicated to work with, we can decide relatively easily whether two such systems are isomorphic. Here isomorphism means that the corresponding $\mathbf{Z}$-actions are measure-theoretically conjugate. The ordered A-module, together with its associated trace (measure), is known as the *dimension space* associated with (M_j). If H and H' with all their structures (that is, with the trace, with the ordered A-module structure) are isomorphic, then the actions are isomorphic; this is almost tautological. However, when both are ergodic, there is a criterion which sometimes allows us to determine isomorphism.

Suppose H is constructed from the sequence (M_j) and H' is constructed from the sequence (M'_k). Each comes with its own L^1-norm induced by the ergodic invariant trace (measure). There is a notion of approximate intertwining that is equivalent to isomorphism of the dimension spaces, as follows.

First, we may form telescopings of (M_j) and (M'_k), that is, two strictly increasing sequences $1 \le m(1) < m(2) < m(3) < \cdots$ and $1 \le m'(1) < m'(2) < m'(3) < \cdots$ from which we define new sequences $(M^{(j)} = M_{m(j+1)-1}M_{m(j+1)-2}\cdots M_{m(j+1)}M_{m(j)})$ and $(M^{(k)\prime} = M_{m'(k+1)-1}M_{m(k+1)-2}\cdots M_{m'(k+1)}M_{m'(k)})$. Now suppose we have sequences of rectangular matrices R_k and S_k with entries from A^+ such that $S_k R_k$ and $M^{(k)}$ have the same dimensions, and $R_{k+1}S_k$ and $M^{(k)\prime}$ have the same dimensions. We do not insist on the equalities, $S_k R_k = M^{(k)}$ and $R_{k+1}S_k = M^{(k)\prime}$, but, instead, that they be close in the following sense.

We can consider the difference $M^{(k)} - S_k R_k$ as a map $A^{n(k)} \to A^{n(k+1)}$. Each of its columns corresponds to an element of G, and we take its norm in G (the L^1-norm induced by the trace). Then we take the maximum over all the columns (this corresponds to the $1-1$ norm, as a map between L^1-spaces). Denote the resulting norm via $\|M^{(k)} - S_k R_k\|$. We similarly define $\|R_{k+1}S_k - M^{(k)\prime}\|$ (the notation does not reflect the dependence on the choice of dimension space; this would be too cumbersome). Then we say the resulting diagram

$$(1.1) \qquad
\begin{array}{ccccccc}
A^{m(1)} & \xrightarrow{M^{(1)}} & A^{m(2)} & \xrightarrow{M^{(2)}} & A^{m(3)} & \xrightarrow{M^{(3)}} & \cdots \\
\big\downarrow{\scriptstyle R_1} & {\scriptstyle S_1}\nearrow & \big\downarrow{\scriptstyle R_2} & {\scriptstyle S_2}\nearrow & \big\downarrow{\scriptstyle R_3} & {\scriptstyle S_3}\nearrow & \\
A^{m'(1)} & \xrightarrow{M^{(1)\prime}} & A^{m(2)} & \xrightarrow{M^{(2)\prime}} & A^{m'(3)} & \xrightarrow{M^{(3)\prime}} & \cdots
\end{array}$$

is *approximately intertwining* if

$$\sum_k \|M^{(k)} - S_k R_k\| < \infty \quad \text{and} \quad \sum_k \|R_{k+1} S_k - M^{(k)\prime}\| < \infty.$$

Finally, the two-dimensional spaces (respectively, the ergodic transformations) are isomorphic if and only if there exists an approximately intertwining diagram between them [GH, Theorem 3.1]. We can also refine this to ensure that R and S are norm one. Despite the complications, computations are frequently quite simple, especially dealing with circulant and related matrices.

Order-preserving A-module maps between dimension spaces are implemented by the one-sided version of the preceding (no S involved).

The dimension space (equivalently, the transformation, or the sequence of matrices (M_j)) is said to be AT(n) if it can be represented as a sequence with matrix sizes at most n (this is not the actual definition, but is equivalent to it [GH, pp. 32–33]), and it is AT (or *approximately transitive*) if it is AT(1) (again, this is not the actual definition, but is equivalent to it).

In this paper, we typically deal with dimension spaces arising from sequences (M_j), where the M_j all have left eigenvector $\mathbf{1}_n = (1, 1, \ldots, 1)$ (with constant n), and we use the resulting ρ to obtain the invariant linear functional τ. We first need to investigate when this is ergodic, which we do in Section 2. We also show that for this class of examples, the matrix of squares (M_j^2) (and higher, varying powers) is AT. (There are very difficult examples of this type arising from $n = 2$ for which (M_j) is not AT (but is necessarily AT(2).)

1.1 Statement of Results

Section 2 provides the preliminaries on sequences of hemicirculant matrices (a generalization of circulant matrices defined for finite abelian groups), and a surprising result that if they are squares or (higher powers), then the resulting sequence, if ergodic, yields an AT action. Section 3 deals with a strong property that guarantees isomorphism of the dynamical system with the sequence of traces, called hollowness. As a special case, we show that if

$$M_j = \begin{pmatrix} 1 & x^{g(n)} \\ x^{g(n)} & 1 \end{pmatrix},$$

then the dynamical system to which it corresponds is hollow if g satisfies an even-term recurrence relation, and is not hollow (but still AT) if it satisfies an odd-term recurrence relation.

Section 4 deals with tensor products. We can construct tensor products of dimension modules in a natural way, thereby obtaining a construction of new dynamical systems (which always preserves ergodicity); however, it is not clear how to obtain these dynamically (without reference to dimension modules). In any event, under very modest conditions on the hemicirculant matrices, (M_j^2) and $(M_j \otimes M_j)$ yield conjugate systems, and this is used in a later example.

Section 5 recalls a class of numerical invariants from [H], and uses it to show that if (M_j^2) is not hollow, then M_j is not isomorphic to its sequence of traces (the latter is automatically AT, the former generically). Combined with Section 4, we obtain a

sequence of hemicirculant matrices (M_j) such that $(M_j \otimes M_j^T)$ and $(M_j M_j^T)$ do not yield conjugate systems, in contrast to the main result of Section 4.

The brief Section 6 explains how sequences of hemicirculant matrices arise naturally from the dual action of product type actions of finite abelian groups on some W*-algebras.

Section 7 deals with a general problem: given an ergodic dynamical system (X, μ, T) with corresponding dimension module, how do we construct the dimension module for a power of T, that is, (X, μ, T^n) (assuming T^n is ergodic)? This is very closely related to systems of circulant matrices. Among other things, we show that if T is AT and T^k is ergodic, then $T^k \otimes T^k$ is AT, a somewhat mystifying result (since T^k need not be AT).

2 Circulant Matrices and Their Relatives

Let H be a finite abelian group of order n, and form the group algebra $V = \mathbf{R}H$, treating it as a vector space with basis $\{e_g\}_{g \in H}$, on which $\mathbf{R}H$ (a different copy) acts as a commutative algebra of endomorphisms via $m_g(e_h) = e_{gh}$. Then each m_g is represented (with respect to the basis $\{e_g\}$) as a size n permutation matrix (arising from the regular representation of H), and we identify $\sum_g q_g m_g$ with the matrix that represents it, where $q_g \in A = \mathbf{R}[x, x^{-1}]$. The matrix representations of m_g and m_h have disjoint supports (that is, for each coordinate, at most one of them has a 1 in that position), and $\sum_H m_g$ is represented by the matrix all of whose entries are 1.

For $M = \sum_H q_g m_g$, a matrix with entries from A, all of its entries lie in A^+ if and only if all q_g belong to A^+ (this comes from the disjointness of the supporting entries); the collection of such will be denoted $(AH)^+$, viewed as a subset of the $n \times n$ matrix algebra over A.

We can write down explicitly the common eigenvectors for the elements of AH. For each α in $\widehat{H}$ (the dual group of H), define a vector v_α in $V \otimes \mathbf{C} = \mathbf{C}H$ via

$$v_\alpha = \frac{1}{|H|} \sum_{g \in H} \alpha(g) e_g$$

(the normalization is to ensure that $\|v_\alpha\| = 1$ in the appropriate l^1-like norm). We check immediately that $m_g v_\alpha = \alpha(g^{-1}) v_\alpha$, so that each v_α is an eigenvector, they are distinct (and can be separated by the m_g), and form an orthonormal set, hence constitute a basis for $\mathbf{C}H$. For any $M = \sum_H q_g m_g$ in AH, its α-th eigenvalue, $\lambda_\alpha(M)$, is in $\mathbf{C}[x, x^{-1}]$, and is given by the eigenvalue corresponding to v_α, that is,

$$\lambda_\alpha(M) = \sum_{g \in H} q_g \alpha(g^{-1}).$$

For each α, the assignment $\lambda_\alpha \colon AH \to \mathbf{C}[x, x^{-1}]$ is an A-algebra homomorphism (that is, additive, multiplicative, and compatible with multiplication by elements of A). A special choice occurs if α is the trivial character, denoted χ_0, in which case we denote the corresponding eigenvector by $v_0 = \frac{1}{|H|} \sum e_g$, and let $\lambda_0(M) = \sum q_g$.

This defines right eigenvectors; left eigenvectors w_α are defined analogously, but with complex conjugation. The corresponding left eigenvector is given by

$$w_\alpha = \sum E_g \alpha(g^{-1}),$$

where we are using E_g to denote basis elements of AH viewed as a *right AH* module. In particular, $w_\alpha = |H| \overline{v_\alpha}^T$ (note the complex conjugate); by construction, $w_\alpha v_\alpha = 1$, and as an operator $w_\alpha \colon AH \to A$ (via $w_\alpha(e_g) = \alpha(g^{-1})$), in the operator l^1-norm, $\|w_\alpha\| = 1$.

Examples (a) *Circulant matrices.* If $H = \mathbf{Z}_n$, the cyclic group of size n, then AH consists precisely of the circulant matrices of size n (with entries from A), where the generator $g = [1]$ maps to the cyclic permutation matrix whose first row is $(0\,1\,0\ldots0)$.

(b) $H = \mathbf{Z}_2 \times \mathbf{Z}_2$. In this case, $n = 4$, and AH consists of all matrices of the form

$$\begin{pmatrix} a_0 & a_1 & a_2 & a_3 \\ a_1 & a_0 & a_3 & a_2 \\ a_2 & a_3 & a_0 & a_1 \\ a_3 & a_2 & a_1 & a_0 \end{pmatrix} = a_0 \mathrm{I} + a_1 \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} + a_2 \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} + a_3 \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

where a_i belong to A.

We call a matrix *hemicirculant with respect to H* (or *H-hemicirculant*) if it is of the form $M = \sum_H q_g m_g$. Suppose that (M_j) is a sequence of hemicirculant non-negative matrices (with respect to the same fixed H), that is, in $(AH)^+$, such that $\lambda_0(M_j)(1) = 1$, so that the eigenvalue at v_0, when evaluated at $x = 1$, is 1, *i.e.*, $\sum q_g(1) = 1$. We wish to study the dimension space arising out of the direct limit $\lim M_j \colon A^n \to A^n$, where we take the invariant functional arising from the left eigenvector $(1\,1\cdots1)$, followed by evaluation at $x = 1$.

We must first check when this yields an ergodic dimension space. This boils down to weak ergodicity (in the context of matrices with real entries) of hemicirculant matrices (this should be in the literature, but I was not able to find any relevant references, except when n is prime). Necessary and sufficient conditions are given in Lemma 2.1.

Let C_j be hemicirculant matrices with nonnegative *real* entries, and whose column sums are all 1. Weak ergodicity of the sequence (C_j) is equivalent to the limiting dimension group $\lim C_j \colon \mathbf{R}^n \to \mathbf{R}^n$ having unique trace, and also to projective and actual convergence of the products to a rank one matrix (this is the usual definition, although in general, order matters; here, the matrices mutually commute), which in this case must be $v_0 w_0$.

Write $C_j = \sum_{g \in H} c_{gj} m_g$ where $0 \le c_{gj}$ and $\sum_{g \in H} c_{gj} = 1$ for all j. It is easy to check that if n is prime (so, in particular, the C_j are circulant matrices), then weak ergodicity is equivalent to $\sum_j (1 - \max_g \{c_{gj}\}) = \infty$. For other choices of H (even cyclic ones), this criterion is merely necessary, but not sufficient (the mass may accumulate on a proper subgroup of H). While the criterion becomes messier and messier depending on how far H is from cyclic, it is still computable, and more importantly, it is used in our subsequent AT results.

Fix H, and let α and β be two unequal elements of $\widehat{H}$. Define $S_{H,\alpha,\beta}$ (or $S_{\alpha,\beta}$ if there is no ambiguity about H) via

$$S_{H,\alpha,\beta} = \left\{ (g,h) \in H \times H \mid \alpha(g^{-1})\beta(h^{-1}) \ne \alpha(h^{-1})\beta(g^{-1}) \right\}.$$

Since $\alpha(g^{-1})$ is just the complex conjugate of $\alpha(g)$, we could have just written the defining condition as $\alpha(gh^{-1}) \ne \beta(gh^{-1})$, or even $\alpha\beta^{-1}(gh^{-1}) \ne 1$; in particular, whether (g,h) belongs to $S_{\alpha,\beta}$ depends only on gh^{-1} and $\alpha\beta^{-1}$.

The statement in the following criterion involves a centring (otherwise it becomes even more awkward). The condition that c_{0j} be maximal can be arranged by multiplying each term by a suitable element of H; this does not affect weak ergodicity. The criteria in (iii) boil down to one computation for each prime divisor of $n = |H|$ if H is cyclic, but hordes of them in the noncyclic case, *e.g.*, if $H = (\mathbf{Z}_p)^k$ with prime p, then the number of maximal subgroups is $p^k - 1 = n - 1$. It may be possible to restrict the maximal subgroups to a reasonable number.

Lemma 2.1 *(Weak ergodicity criterion). Suppose for each j that $C_j = \sum_{g \in H} c_{gj} m_g$ is a real hemicirculant matrix with respect to H, where $0 \le c_{gj}$ and $\sum_{g \in H} c_{gj} = 1$ for all j. Assume in addition that $c_{0j} = \max_{g \in H}\{c_{gj}\}$ for all j. The following are equivalent.*

  (i) *The limit dimension group $\lim C_j \colon \mathbf{R}^n \to \mathbf{R}^n$ is a simple dimension group with unique trace;*
 (ii) *for every $\alpha \ne \chi_0$, for all j_0, $\prod_{j \ge j_0} \lambda_\alpha(C_j) = 0$;*
(iii) *for every $\alpha \ne \chi_0$, for all j_0, $\sum_j \sum_{\alpha(g) \ne 1} c_{gj} = \infty$;*
 (iv) *for each maximal proper subgroup K of H, $\sum_j \sum_{g \notin K} c_{gj} = \infty$;*
  (v) *for all $\alpha \ne \beta$ in $\widehat{H}$, $\sum_j \sum_{(g,h) \in S_{\alpha,\beta}} c_{gj} c_{hj} = \infty$.*

Proof The equivalence of (i)–(iii) is clear, and the equivalence of (iii) with (iv) stems from every proper subgroup being contained in a maximal proper subgroup. Let $n = |H|$.

We prove (iii) implies (v). Form the element $\gamma = \alpha\beta^{-1}$ of $\widehat{H}$; this is not the trivial character. For each j, we may find g_j not in the kernel of γ such that $c_{g_j j} \ge \sum_{\gamma(g) \ne 1} c_{gj}/n$; this forces $\sum_j c_{g_j j} = \infty$, and since $c_{0,j} \ge 1/n$ as a consequence of the hypotheses, we have $\sum_j c_{g_j j} c_{0,j} = \infty$.

We note that $(g,h) \in S_{\alpha,\beta}$ if gh^{-1} does not belong to the kernel of γ. Hence $(g_j, 1) \in S_{\alpha,\beta}$, and thus $\sum_j \sum_{g \in S_{\alpha,\beta}} c_{gj} c_{h,j} \ge \sum_j c_{g_j j} c_{0,j}$, and the latter diverges.

Now (v) implies (iii). Let γ be a nontrivial element of $\widehat{H}$. With $\alpha = \gamma$ and $\beta = \chi_0$ (the trivial character), find $(g_j, h_j) \in S_{\alpha,\beta}$ such that $c_{g_j j} c_{h_j j} \ge \sum_{S_{\alpha,\beta}} c_{gj} c_{hj}/n^2$ (this is possible since the sum is over fewer than n^2 elements). Since $\gamma(gh^{-1}) \ne 1$, at least one of c_{g_j} or c_{h_j} is not in the kernel of γ. We are done, since $\sum_j \sum_{\gamma(g) \ne 1} c_{gj} \ge c_{g_j}$ or c_{h_j}, respectively. ∎

For elements α and β of $\widehat{H}$, and elements g and h, set $z = \alpha(g^{-1})\beta(h^{-1})$ and $y = \alpha(h^{-1})\beta(g^{-1})$. Let N denote the exponent of H (the smallest positive integer exceeding 1 such that $k^N = 1$ for all k in H), and set $\xi = e^{2\pi i/N}$. Then the values of all irreducible characters on elements of H lie in $\{\xi^j\}$. Obviously, $|z + y| \le 2$, with

equality occurring only if the arguments of z and of y are equal: in this case implying $z = y$ (since both are just powers of ξ). On the other hand, if $|z + y| < 2$, then $|z + y| < 2\cos \pi/N$ (not $2\cos 2\pi/N$). In particular, either $z = y$ or $|z + y| \le 2\cos \pi/N$, and so if (g, h) belongs to $S_{H,\alpha,\beta}$, then $|\alpha(g^{-1})\beta(h^{-1}) + \alpha(h^{-1})\beta(g^{-1})| \le 2\cos \pi/N$.

When we write (M_j) is an ergodic sequence of hemicirculant matrices (with respect to some finite group H), we mean that $M_j \in (AH)^+$, the column sums on evaluation at $x = 1$ are all 1, and the sequence $(C_j = M_j(1))$ is weakly ergodic. In particular, the trace obtained from the constant eigenvector yields an ergodic measure. The eigenvalues of M_j lie in $\mathbf{C}[x, x^{-1}] = A \otimes \mathbf{C}$; we impose the obvious l^1-norm on the latter *i.e.*, the sum of the absolute values of the coefficients.

Lemma 2.2 *Suppose that $(M_j = \sum_{g \in H} q_{gj} m_g)$ is an ergodic sequence of $n \times n$ hemicirculant matrices with respect to the finite abelian group H. Then for all $\alpha \ne \beta$ in $\widehat{H}$ and all $j_0 \ge 1$,*

$$\lim_{d \to \infty} \left\| \lambda_\alpha \Big(\prod_{j=j_0}^{j_0+d} M_j \Big) \lambda_\beta \Big(\prod_{j=j_0}^{j_0+d} M_j \Big) \right\| = 0.$$

Proof Temporarily drop the subscript j. Then

$$\lambda_\alpha(M) \cdot \lambda_\beta(M) = \sum_{g \in H} q_g \alpha(g^{-1}) \cdot \sum_{h \in H} q_h \beta(h^{-1})$$

$$= \frac{1}{2} \sum_{(g,h) \in S_{\alpha,\beta}} q_g q_h \cdot (\alpha(g^{-1})\beta(h^{-1})$$

$$+ \alpha(h^{-1})\beta(g^{-1})) + \sum_{(H \times H) \smallsetminus S_{\alpha,\beta}} q_g q_h \alpha(g^{-1})\beta(h^{-1}),$$

so

$$\|\lambda_\alpha(M) \cdot \lambda_\beta(M)\| \le \cos \frac{\pi}{N} \sum_{(g,h) \in S_{\alpha,\beta}} q_g q_h(1) + \sum_{(H \times H) \smallsetminus S_{\alpha,\beta}} q_g q_h(1)$$

$$= \Big(\sum_g q_g(1) \Big)^2 - \Big(1 - \cos \frac{\pi}{N} \Big) \sum_{(g,h) \in S_{\alpha,\beta}} q_g q_h(1)$$

$$= 1 - 2\sin^2 \frac{\pi}{N} \sum_{(g,h) \in S_{\alpha,\beta}} q_g q_h(1).$$

Restoring the j, by the earlier lemma, with $c_{gj} = q_{gj}(1)$, ergodicity implies that $\sum_j \sum_{S_{\alpha,\beta}} q_{gj} q_{hj}(1) = \infty$. Thus $\prod \|\lambda_\alpha(M_j)\lambda_\beta(M_j)\| \to 0$ (in the strongest possible sense). Since λ_α is multiplicative on hemicirculant matrices, the result follows. ∎

Theorem 2.3 *Suppose that (M_j) is an ergodic sequence of $n \times n$ hemicirculant matrices. Let $f : \mathbf{N} \to \mathbf{N}$ be a function such that $f(j) \ge 2$ for all j. Then $(M_j^{f(j)})$ is AT. In particular, let $0 = n(1) < n(2) < \cdots$ be an infinite sequence of positive integers such that for all $\alpha \ne \beta \in \widehat{H}$, $\sum_i \|\lambda_\alpha(M^{(i)})\lambda_\beta(M^{(i)})\| < \infty$, where $M^{(i)} = \prod_{j=n(i)}^{n(i+1)-1} M_j$. Write $M^{(i)} M^{(i+1)} = \sum_{g \in H} p_{gi} m_g$.*

(i) *(M_j^2) is isomorphic to the AT sequence $(\mathrm{tr}(M^{(i)} M^{(i+1)}) = |H| p_{0i})$.*

(ii) *The automorphism of the dimension space of (p_{0i}) induced by m_g (for $g \in H$) is implemented by the map $[f, i] \mapsto [p_{g^{-1}i}f, i + 1]$ (for $f \in A$).*

Proof If M is a hemicirculant matrix such that $\|\lambda_\alpha(M)\lambda_\beta(M)\| < \epsilon/p^2$ (as would arise as a product of suitable M_j, by the previous lemma), write

$$M = \sum_{\widehat{H}} \lambda_\alpha(M)v_\alpha w_\alpha = \sum_g q_g m_g.$$

Define $V = \sum \lambda_\alpha(M)v_\alpha$ and $W = \sum \lambda_\alpha(M)w_\alpha$. We note

$$\|M^2 - VW\| = \left\| \lambda_\alpha(M)^2 \sum v_\alpha w_\alpha - \left(\sum \lambda_\alpha(M)v_\alpha \right) \left(\sum \lambda_\beta(M)v_\beta \right) \right\|$$
$$= \left\| \sum_{\alpha \neq \beta} \lambda_\alpha(M)\lambda_\beta(M)v_\alpha v_\beta \right\|$$
$$< \frac{(p^2 - p)\epsilon}{p^2} < \epsilon.$$

From the displayed computation, $\|M^2 - VW\| < \epsilon$. Now we show that both V and W have entries in A^+. Since $W = |H|\overline{V^T}$, it suffices to show this for V.

$$V = \sum \lambda_\alpha(M)v_\alpha = \frac{1}{|H|} \sum_\alpha \sum_g e_g \alpha(g)\lambda_\alpha(M)$$
$$= \frac{1}{|H|} \sum_{\alpha,g,h} e_g \alpha(g)\alpha(h^{-1})q_h = \frac{1}{|H|} \sum_{\alpha,g,h} e_g q_h \alpha(gh^{-1})$$
$$= \frac{1}{|H|} \sum_g e_g \sum_h q_h \sum_\alpha \alpha(gh^{-1}) = \sum_g e_g q_g \in (AH)^+.$$

A similar computation yields $W = \sum q_g e_{g^{-1}}$. Since (M_j) is ergodic, for all j_0, there exists d such that $\max_{\alpha \neq \beta} \|\lambda_\alpha(\prod_{j_0}^{j_0+d} M_j)\lambda_\beta(\prod_{j_0}^{j_0+d} M_j)\| < \epsilon/p^2$, and it follows that (M_j^2) is AT. For higher powers $(f(j))$, the argument is similar.

Now this process can be done for any telescoping $\{n(i)\}$ such that

$$\sum_i \|\lambda_\alpha(M^{(i)})\lambda_\beta(M^{(i)})\| < \infty,$$

i.e., $M^{(i)} = M$. Such a telescoping exists by the previous lemma. Then define $V^{(i)}$ and $W^{(i)}$ as above. The approximate factorization yields an isomorphism between the dimension spaces of $(M_j^2) = ((M^{(i)})^2)$ and $(W^{(i+1)}V^{(i)})$, with maps given by $[z, i] \mapsto [W^{(i)} \cdot z, i]$ $(z \in A^n)$ and in the reverse direction via $[f, i] \mapsto [f V^{(i)}, i+1]$ $(f \in A)$. We expand $W^{(i+1)}V^{(i)} = \frac{1}{|H|} \sum q_{gi} q_{g^{-1},i+1}$. This is exactly $\frac{1}{|H|} \operatorname{tr} M^{(i)}M^{(i+1)} = p_{0i}$.

To determine the effect of the automorphism induced by g in H, we begin with the element $[1, i]$ in the dimension space of (p_{0i}). Under the map $V^{(i)}$, this is sent to $[V^{(i)}, i + 1]$ (of the dimension space of (M_j^2)). Now g acts on this directly by multiplication, yielding the element $[\sum_{h \in H} q_{hi} e_{hg}, i + 1]$, Then $W^{(i+1)}$ sends this to $[\sum_{h \in H} q_{hg^{-1},i}q_{h^{-1},i+1}, i + 1]$, but this is just $[p_{g^{-1}i}, i + 1]$. Since the map at the i-th level is uniquely determined by its effect on $[1, i]$ (and the automorphism exists by the isomorphism), the automorphism is induced as indicated. ∎

In particular, the diagonal entries of VW are respectively $q_{g^{-1}}q_g$; this is unsurprising, as $\operatorname{tr} M^2 = |H| \sum_g q_g q_{g^{-1}}$ (this entails that $|H| q_g q_{g^{-1}}$ are close in l^1 as g varies). The (g, h) entry of VW is $q_g q_{h^{-1}}$, so VW is not generally itself a hemicirculant matrix and M does not (usually) commute with $VW \sim M^2$. It does not even seem possible to perturb V to V' and W to W' (with error bounded by a fixed multiple of $\sum_{\alpha \neq \beta} \|\lambda_\alpha(M)\lambda_\beta(M)\|$) so that $V'W'$ commutes with M.

As mentioned in [GH, p. 32], there is a weaker equivalence relation than what we have called isomorphism (positive A-module isomorphism with positive inverse, both of which are isometries). If M and M' are complete $l^1(\mathbf{Z})$-modules, we say they are *neutrally isomorphic* if there is an $A = l^1(\mathbf{Z})$-module isometry with isometric inverse from M to M'. In other words, positivity has been left out of the definition.

Following [GH, p. 32], we say a dimension space (or its sequence of maps), M, is WAT (weakly approximately transitive) if there exists a complete $l^1(\mathbf{Z})$-module N given as the completion of the direct limit of the sequence of maps of the form $\times p_i \colon l^1(\mathbf{Z}) \to l^1(\mathbf{Z})$ where p_i are in $l^1(\mathbf{Z})$ (and can be assumed to be polynomials).

If $\mathcal{M}$ is given as the completion of the (order) direct limit, $\lim M_j \colon A^{n(j)} \to A^{n(j+1)}$ (where $n(j)$ are positive integers, m_j are $n(j+1) \times n(j)$ matrices with entries in A^+ such that after evaluating all the polynomial entries at $x = 1$, the resulting real matrices are column stochastic), then necessary and sufficient for $\mathcal{M}$ to be WAT is that there exist a telescoping $t(1) < t(2) < \cdots$ such that on defining

$$M^{(i)} = M_{t(i+1)-1} \cdot M_{t(i+1)-2} \cdots M_{t(i)},$$

there exist for each i, a row W_i and a column V_i with entries in A, for which

$$\sum_i \|M^{(i)} - V_i W_i\| < \infty.$$

In that statement, if we insist that the entries of V_i and W_i belong to A^+, then we have a characterization of AT. There is no requirement that $W_i V_i$ belong to A^+, although this would be desirable.

Now we make a slight weakening of the definition to permit complex coefficients. We say M is WATC (weakly approximately transitive over the complexes), if we permit the entries of V_i and W_i to be in $A \otimes \mathbf{C} = \mathbf{C}[x^{\pm 1}]$ (equivalently, we deal with module isomorphisms of $M \otimes \mathbf{C}$).

Some invariants for dimension spaces intended for distinguishing isomorphism classes actually turn out to be neutral isomorphism invariants: in particular, the mass cancellation-type invariants in [H], under some circumstances, as we will see later.

We will show that if M is the dimension space arising from an ergodic sequence of hemicirculant matrices, then M is WATC. If additionally, the matrix size is two, then M is WAT. This contrasts with the situation for AT, since an example is known (size two) of an ergodic sequence of circulant matrices which is not AT. (In particular, WAT does not imply AT.)

Lemma 2.4 *Let $n > 1$ be an integer, and let $0 \leq j_0 < d$ be integers. Set $T = [j_0, d] \cap \mathbf{Z}$, and let $Z_{k,l}$ $((k, l) \in \{0, 1, 2, \ldots, n-1\} \times \{0, 1, 2, \ldots, n-1\}$ and $k \neq l)$ be pairwise disjoint subsets of T. Then there exist subsets $U_0, U_1, \ldots, U_{n-1}$ of T such that for all $k \neq l$, $Z_{k,l} \subseteq U_k \cap U_l^c$.*

Proof In the unit disk, let $C_0, C_1, \ldots, C_{p-1}$ be rays through the origin with angles $0, \pi/n, 2\pi/n, \ldots, (p-1)\pi/n$, respectively. This yields n open segments partitioning the disk less the union of the rays. Let $A_{0,+}$ be the open upper half-disk, and for each t in $\{0, 1, 2, \ldots, n-1\}$, define $A_{t,+}$ to be $A_{0,+}$ rotated by $t\pi/n$. Then C_t is in the boundary of $A_{t,+}$. Similarly let $A_{t,-}$ be the complement of $A_{t,+}$ less C_t. Map each $Z_{k,l}$ bijectively into $A_{k,+} \cap A_{l,-}$ (this is a nonempty sector). Obviously, we can do this so that the map is bijective on the union of the $Z_{k,l}$. Call the map f, and set $U_t = f^{-1}(A_{t,+})$. ∎

Proposition 2.5 *Suppose that (M_j) is an ergodic sequence of hemicirculant matrices with respect to the finite abelian group H. Then the corresponding dimension space is WATC. If $H = \mathbf{Z}_2^k$ for some k, then the dimension space is WAT.*

Proof Let $n = |H|$ and $Y = \{0, 1, \ldots, n-1\}$. Let

$$F \longrightarrow H \to Y \quad \text{and} \quad E \colon (Y \times Y) \smallsetminus \Delta \longrightarrow \{1, 2, \ldots, n^2 - n\}$$

be bijections. Given $\epsilon > 0$, by ergodicity, there exist positive integers

$$j_0 = d_1 < d_2 < \cdots < d_{n^2-n+1} = d$$

such that on setting (for unequal α and β in $\widehat{H}$),

$$Z_{F(\alpha),F(\beta)} = \left[d_{E(F(\alpha),F(\beta))}, d_{E(F(\alpha),F(\beta))+1} \right) \cap \mathbf{Z},$$

$$\left\| \prod_{j \in Z_{F(\alpha),F(\beta)}} \lambda_\alpha(M_j)\lambda_\beta(M_j) \right\| < \frac{\epsilon}{n^2}.$$

By the preceding lemma, there exist subsets $U_{F(\alpha)}$ (α ranging over $\widehat{H}$) of $[j_0, d) \cap \mathbf{Z}$ such that $Z_{F(\alpha),F(\beta)} \subseteq U_{F(\alpha)} \cap U_{F(\beta)}^c$.

We may write $\prod_{j=j_0}^{d} M_j = \sum_{\alpha \in \widehat{H}} v_\alpha w_\alpha \prod_{j=j_0}^{d} \lambda_\alpha(M_j)$. Set

$$V = \sum_{\widehat{H}} v_\alpha \prod_{j \in U_{F(\alpha)}} \lambda_\alpha(M_j), \quad W = \sum_{\widehat{H}} w_\alpha \prod_{j \in U_{F(\alpha)}^c} \lambda_\alpha(M_j).$$

Then

$$VW = \sum_{\alpha \in \widehat{H}} v_\alpha w_\alpha \prod_{j=j_0}^{d} \lambda_\alpha(M_j) + \sum_{\alpha \neq \beta} v_\alpha w_\beta \prod_{j \in U_{F(\alpha)}} \lambda_\alpha(M_j) \prod_{j \in U_{F(\beta)}^c} \lambda_\beta(M_j)$$

$$= \prod_{j=j_0}^{d} M_j + \sum_{\alpha \neq \beta} v_\alpha w_\beta \prod_{j \in U_{F(\alpha)}} \lambda_\alpha(M_j) \prod_{j \in U_{F(\beta)}^c} \lambda_\beta(M_j).$$

Since $\|\lambda_\alpha(M_j)\| \leq 1$ in any event, we have

$$\left\| \prod_{j \in U_{F(\alpha)}} \lambda_\alpha(M_j) \prod_{j \in U_{F(\beta)}^c} \lambda_\beta(M_j) \right\| \leq \left\| \prod_{j \in U_{F(\alpha)} \cap U_{F(\beta)}^c} \lambda_\alpha(M_j)\lambda_\beta(M_j) \right\|$$

$$\leq \left\| \prod_{j \in Z_{F(\alpha),F(\beta)}} \lambda_\alpha(M_j)\lambda_\beta(M_j) \right\| < \frac{\epsilon}{n^2}.$$

Hence $\| VW - \prod_{j=j_0}^{d} M_j \| < \epsilon \| \sum v_\alpha w_\alpha \|/n^2 < \epsilon$.

If $H = \mathbf{Z}_2^k$, then all the entries of each term in the definitions of V and W are real. ∎

As an aside, an easy computation yields $WV = \operatorname{tr}(\prod_{j_0}^M M_j)$, which shows the former has coefficients in A^+. Unfortunately, this does not, by itself, force (A_j) to be neutrally isomorphic, even over $\mathbf{C}$, to an AT sequence, because the corresponding sequence is of the form $W^{(s+1)} V^{(s)}$ (the superscripts indicating the iteration of the process in the proof of the proposition), and there is no guarantee that these will have positive (or real) coefficients.

3   Hollowness of Some Circulant Sequences

A sequence (M_j), of hemicirculant matrices with entries from A^+ and for which the row sums of all the coefficients add to 1 is *hollow* if for all j_0, and all α in $\widehat{H} \smallsetminus \{\chi_0\}$, $\lim_{d\to\infty} \|\lambda_\alpha(\prod_{j_0}^{j_0+d} M_j)\| = 0$. Examples of hollow sequences are easy to construct, as we will see. An obvious example arises when $\prod_{j=j_0}^{j_0+d} \|\lambda_\alpha(M_j)\| \to 0$ for all $\alpha \neq \chi_0$, but there are more interesting ones for which this stronger condition fails. For example, if $H = \mathbf{Z}_n$ with $n \geq 3$, let P denote the standard cyclic permutation matrix of size n, and set $M_j = (\frac{1}{2}(I + x^{2^i} P))^2$. Then $\|\lambda_\alpha(M_j)\| = 1$ for all α. However, if we telescope in triples, we find $\|\lambda_\alpha(M_{3j} M_{3j+1} M_{3j+2})\| < 1$ when $\alpha \neq \chi_0$, and it easily follows that (M_j) is hollow.

Lemma 3.1   *If (M_j) is a hollow sequence of hemicirculant matrices, then the corresponding dimension space* $\lim M_j \colon A^n \to A^n$ *is AT. There is a telescoping*

$$d(1) < d(2) < \cdots$$

so that

$$\sum_l \Big\| \prod_{j=d(l)}^{d(l+1)-1} M_j - v_0 w_0 \prod_{j=d(l)}^{d(l+1)-1} \lambda_0(M_j) \Big\| < \infty.$$

In particular, the dimension space of (M_j) is isomorphic to that of the AT system, $(\lambda_0(M_j))$.

Proof   It suffices to show, given j' and $\epsilon > 0$, that there exists d such that

$$\Big\| \prod_{j=j'}^{j'+d} M_j - v_0 w_0 \prod_{j=j'}^{j'+d} \lambda_0(M_j) \Big\| < \epsilon.$$

For each $\alpha \neq \chi_0$, there exists d_α such that $\|\prod_{j=j'}^{j'+d} \lambda_\alpha(M_j)\| < \epsilon/(n-1)$. If $d = \sup_{\widehat{H} \smallsetminus \{\chi_0\}} d_\alpha$, then

$$\Big\| \prod_{j=j'}^{j'+d} M_j - v_0 w_0 \prod_{j=j'}^{j'+d} \lambda_0(M_j) \Big\| \leq \Big\| \sum_{\widehat{H}} \lambda_\alpha\Big(\prod_{j=j'}^{j'+d} M_j\Big) v_\alpha w_\alpha - v_0 w_0 \prod_{j=j'}^{j'+d} \lambda_0(M_j) \Big\|$$

$$= \Big\| \sum_{\widehat{H} \smallsetminus \{\chi_0\}} \lambda_\alpha\Big(\prod_{j=j'}^{j'+d} M_j\Big) v_\alpha w_\alpha \Big\|$$

$$\leq \sum_{\widehat{H} \smallsetminus \{\chi_0\}} \frac{\epsilon}{n-1} \|v_\alpha w_\alpha\| = \epsilon. \qquad \blacksquare$$

Now we show that hollowness applies to symmetric and other matrices, with additional constraints. In the symmetric case, the criterion is rather primitive.

The following generalizes [GH, Proposition 5.3]. It is reminiscent of Mineka's criterion (for triviality of boundaries).

If p and q are Laurent polynomials with real coefficients, define their infimum in the obvious way, $p \wedge q := \sum \min\{(p, x^i), (q, x^i)\} x^i$. Obviously, if p and q belong to A^+, then so does $p \wedge q$, and if $p \wedge q = p$, then $(p, x^i) \le (q, x^i)$ for all i.

Lemma 3.2   *Suppose that $(M_j = \sum_{g \in H} q_{gj} m_g)$ is an ergodic sequence of hemicirculant matrices with respect to H. Sufficient for (M_j) to be hollow is that for all maximal subgroups K of H, $\sum_j \sum_{\{(g,h) \mid gh^{-1} \notin K\}} (q_{gj} \wedge q_{hj})(1) = \infty$.*

Remark   If $H = \mathbf{Z}_n$ is cyclic (so that the matrices are circulant) of order n, it follows that sufficient (but not necessary, unless n is a power of a prime) is divergence of $\sum_j \sum_{i \not\equiv i' \mod n} (q_{ij} \wedge q_{i'j})(1)$.

Proof   Let α be a nontrivial element of $\widehat{H}$, and let K be a maximal subgroup containing the kernel of α. Set $a_j = \sum_{\{(g,h) \mid gh^{-1} \notin K\}} (q_{gj} \wedge q_{hj})(1)$. Since there are fewer than n^2 terms in the sum, there exists (g_j, h_j) such that $g_j h_j^{-1} \notin K$ and $(q_{g_j j} \wedge q_{h_j j})(1) \ge a_j / n^2$. Set $f_j = q_{g_j j} \wedge q_{h_j j}$ and expand:

$$\lambda_\alpha(M_j) = \sum_{g \in H} q_{gj} \alpha(g^{-1})$$
$$= (\alpha(g_j^{-1}) + \alpha(h_j^{-1})) f_j + (g_j - f_j) \alpha(g_j^{-1}) + (h_j - f_j) \alpha(h_j^{-1})$$
$$+ \sum_{g \notin \{g_j, h_j\}} q_{gj} \alpha(g^{-1}),$$

since $\sum_H q_{gj}(1) = 1$, and

$$\|\lambda_\alpha(M_j)\| \le 1 - (2 - |\alpha(g_j^{-1}) + \alpha(h_j^{-1})|) f_j(1),$$

since $g_j h_j^{-1} \notin K$, and thus

$$\alpha(g_j h_j^{-1}) \ne 1, \quad \le 1 - 2 \sin \frac{\pi}{N} \cdot f_j(1) \le 1 - a_j \frac{2 \sin \frac{\pi}{N}}{n^2}.$$

(Recall that N is the exponent of H; we could just as well have replaced it with n.) Since $\sum a_j = \infty$, we have $\prod \lambda_\alpha(M_j) \to 0$, verifying hollowness. ∎

The matrix $M = \sum_{g \in H} q_g m_g$ is symmetric if and only if for all g, $q_g = q_{g^{-1}}$.

Proposition 3.3   *Let $(M_j = \sum_{g \in H} q_{gj} m_g)$ be an ergodic sequence of hemicirculant matrices with respect to H. Suppose, in addition, that each M_j is symmetric. Then (M_j) is hollow if for all maximal proper subgroups K of H,*

$$(3.1) \qquad \sum_j \sum_{\{g \in H \mid g^2 \notin K\}} q_{gj}(1) = \infty.$$

In particular, if $|H|$ is odd and the M_j are symmetric, then (M_j) is hollow.

Proof   By symmetry, $q_{gj} \wedge q_{g^{-1}j} = q_{qj}$ and $g(g^{-1})^{-1} = g^2$. Thus

$$\sum_j \sum_{\{g,h|gh^{-1} \notin K\}} (q_{gj} \wedge g_{hj})(1) \geq \sum_j \sum_{\{g|g^2 \notin K\}} q_{gj}(1),$$

and the previous lemma applies.

Let f_j in H be such that $q_{f_j j}(1)$ is maximal among $\{q_{gj}(1)\}$. Then

$$f_j M_j = \sum_{g \in H} q_{f^{-1}g} m_g,$$

and if $f_j^2 = 1$, then $f_j M_j$ is still symmetric but now the maximal coefficient in

$$f_j M_j(1) = \sum q_{f_j g}(1) m_g$$

occurs at the identity coefficient. If $f_j^2 \neq 1$ (as will be the case when $|H|$ is odd and f_j is not the identity), then the coefficients of $M_j(1)$ have maxima at two distinct points. This together with ergodicity is more than enough to guarantee that the condition in Lemma 2.1 (iv),

(3.2)
$$\sum_j \sum_{g \notin K} q_{gj}(1) = \infty$$

for all maximal subgroups K (in place of the maximum coefficient occurring at the identity, which is assumed there).

Now suppose $|H|$ is odd. Then H/K has odd order, so $g^2 \in K$ entails $g \in K$. Thus (3.1) is equivalent to (3.2) (for all maximal subgroups K), hence (M_j) is hollow. ∎

The condition $g^2 \notin K$ (as opposed to $g \notin K$) really is significant. The standard example of a non-AT but AT(2) ergodic action [GH] is given by size two (necessarily symmetric) circulant matrices (over $H = \mathbf{Z}_2$): $M_j = \frac{1}{2}\begin{pmatrix} 1 & x^{5^j} \\ x^{5^j} & 1 \end{pmatrix}$. The sequence (M_j) is not hollow, as it is not even AT.

In many cases, (M_j^2) is hollow. This occurs if, for example, $H = \mathbf{Z}_n$ (cyclic) and $M_j = \frac{1}{2}(1 + x^{2^j} m_{[1]})$ where [1] is the generating element of Z_n (so $m_{[1]}$ is just the usual cyclic permutation matrix of size n). Here (M_j) is not hollow, as is easy to check. Less easy to check is that $\|\prod_{j=0}^d (1 + \xi^l x^{2^j})^2\| = o(4^d)$ if l is not divisible by n (in fact, convergence is faster than $o(s^d)$ for some number $s < 4$, possibly as low as $s = 2$), so (M_j^2) is hollow. However, there exist sequences (M_j) for which hollowness of powers does not occur.

For example, suppose $H = \mathbf{Z}_n$ again, let k be a positive even integer exceeding 2. Set $M_j = \frac{1}{2}(1 + x^{k^j} m_{[1]})$. Then (M_j^s) is not hollow for $s < k$, but is hollow for $s \geq k$. More drastically, if $M_j = \frac{1}{2}(1 + x^{j!} m_{[1]})$ (this example has extravagantly large gaps in the exponents), then (M_j^s) is not hollow for any s. More drastically still, if $M = \frac{1}{2}\begin{pmatrix} 1+x^2 & x \\ x & 1+x^2 \end{pmatrix}$ and $M_j = M(x^{3^j})$, and $f: \mathbf{N} \to \mathbf{N}$ is any function whatsoever, then $(M_j^{f(j)})$ is not hollow. (This is easily deduced from the signs of the coefficients in the products $\lambda_1(\prod M_j^{f(j)}) = \prod \frac{1}{2}(1 - x^{3^j})^{f(j)}$: the coefficient of x^k is $(-1)^k$ times the coefficient of x^k in $\lambda_0(\prod M_j^{f(j)}) = \prod \frac{1}{2}(1 + x^{3^j})^{f(j)}$.)

The following is practically tautological.

Proposition 3.4   *Let* (M_j) *be an ergodic H-hemicirculant sequence. The following are equivalent.*

(i) (M_j) *is hollow;*
(ii) *the kernel of the natural map from the dimension space of* (M_j) *to that of* $(\lambda_0(M_j))$ *is zero;*
(iii) *for all* g *in* H, *the automorphism of the dimension space* (M_j) *given by* $[f, k] \mapsto [m_g f, k]$, *is the identity.*

Suppose $g \colon \mathbf{N} \to \mathbf{N}$ is a function, and set $M_j = \frac{1}{2}\begin{pmatrix} 1 & x^{g(j)} \\ x^{g(j)} & 1 \end{pmatrix}$. For example, $g(j)$ could be the j-th Fibonacci number. Under some conditions (including this example), the sequence (M_j) is hollow. Under somewhat weaker conditions, (M_j) is AT but not necessarily hollow. This requires several lemmas. First, we have a special case in which the method used to show (M_j^2) is AT sometimes works to show (M_j) is AT.

Lemma 3.5   *Let* (M_j) *be an ergodic sequence of H-hemicirculant matrices. Suppose that for all* $\epsilon > 0$, *and all sufficiently large integers* k, *there exists a positive integer* $d \equiv d(\epsilon, k)$ *together with a subset* $R \subseteq T := \{k, k+1, \dots, k+d\}$ *such that for all* $\alpha \neq \beta$ *in* $\widehat{H}$, $\| \prod_{j \in R} \lambda_\alpha(M_j) \prod_{j \in T \smallsetminus R} \lambda_\beta(M_j) \| < \epsilon$. *Then* (M_j) *is AT.*

Proof   Set $M = \prod_T M_j$. Set

$$V = \sum_{\alpha \in \widehat{H}} v_\alpha \prod_{j \in R} \lambda_\alpha(M_j), \quad W = \sum_{\alpha \in \widehat{H}} w_\alpha \prod_{j \in T \smallsetminus R} \lambda_\alpha(M_j).$$

As in the argument of Theorem 2.3, the entries of both V and W are in A^+ (explicitly, $V = \sum q_g e_g$, where $\prod_R M_j = \sum q_g m_g$ and $W = \sum q_g' e_g$, where $\prod_{T \smallsetminus R} M_j = \sum q_g' m_g$). Also as in that argument,

$$VW = \sum_{\alpha \in \widehat{H}} v_\alpha w_\alpha \lambda_\alpha(M) + \sum_{\alpha \neq \beta} v_\alpha w_\beta \prod_{j \in R} \lambda_\alpha(M_j) \prod_{j \in T \smallsetminus R} \lambda_\beta(M_j).$$

Thus, $\| VW - M \| \leq |H|(|H| - 1)\epsilon$. This yields suitable approximate factorizations of the products of the M_j, so (M_j) is AT. ∎

Lemma 3.6   *Let* $g \colon \mathbf{N} \to \mathbf{N}$ *be a function satisfying the following conditions. There exists* $S \subseteq \mathbf{N}$ *together with* $\epsilon_i(s) \in \{0, \pm 1\}$ *(for* $s \in S$*), such that*

(i) *for all* $s \in S$, *we may write* $g(s) = \sum_{i=1}^{N-1} \epsilon_i(s) g(s-i)$, *where* $\sum_i \epsilon_i(s)$ *is even;*
(ii) *on setting* $b(s) = 1 + \sum |\epsilon_i(s)|$, *we have* $\sum 2^{-b(s)} = \infty$;
(iii) *on defining* $supp\, g(s) = \{s\} \cup \{s - i \mid \epsilon_i(s) \neq 0\}$, *we have* $s \neq s' \in S$ *entails* $supp\, g(s) \cap supp\, g(s') = \varnothing$.

Then for all integers k,

$$\lim_{N \to \infty} \left\| \prod_{j=k+1}^{N+k} \frac{1 - x^{g(j)}}{2} \right\| = 0.$$

Proof   Define $S_k = \{s \mid supp\, g(s) \cap \{1, 2, 3, \dots, k\} = \varnothing\}$. Obviously $\sum_{s \in S_k} 2^{-b(s)} = \infty$. For $\epsilon > 0$, there exists $N \equiv N(\epsilon)$ sufficiently large that if

$$A_N = \{s \in S_k \mid supp\, g(s) \subseteq \{k+1, \dots, N+k\}\},$$

then $\prod_{s\in A_N}\left(1-\frac{1}{2^{b(s)}}\right) < \epsilon$. Define $V_N = \prod_{k+1}^{N+k}\{0,1\}$. Say that v in V_N *runs through* supp $g(s)$ if

$$v(i) = \begin{cases} 1 & \epsilon_i(s) = 1, \\ 0 & \epsilon_i(s) = -1. \end{cases}$$

Let $\mathcal{S} = \{v \in V_N \mid v \text{ runs through at least one } g(s) \text{ with } s \in A_N.\}$ Then $|V_N \smallsetminus \mathcal{S}| < \epsilon|V_N| = 2^N\epsilon$, so $|\mathcal{S}| \geq 2^N(1-\epsilon)$.

Define $\phi\colon \mathcal{S} \to V_N$ as follows. For $v \in \mathcal{S}$, find the smallest s, denoted s_0, such that $s \in A_N$ and v runs through supp $g(s)$. Define

$$\phi(v)(i) = \begin{cases} 1 - v(i) & \text{if } i \in \text{supp } g(s_0), \\ v(i) & \text{else.} \end{cases}$$

Then ϕ is one-to-one, and

$$\sum_{k+1}^{N+k} \phi(v)(i) \equiv 1 + \sum v(i) \mod 2, \qquad \sum_{k+1}^{N+k} \phi(v)(i)g(i) = \sum v(i)g(i).$$

Now

$$\left\| \prod(1 - x^{g(j)})\right\| = \sum_{j=k+1}^{k+N} \left| \sum_{\substack{\sum v(i)g(i)=j \\ v\in A_N}} (-1)^{\sum v(i)} \right|.$$

If $v \in \mathcal{S}$, then $\phi(v)$ and v have opposite parities, i.e., $(-1)^{\sum v(i)} + (-1)^{\sum \phi(v)(i)} = 0)$. This means that the contribution of such a v to the total mass is zero. Hence

$$\left\| \prod(1 - x^{g(j)})\right\| \leq |V_N \smallsetminus \mathcal{S}| < 2^N\epsilon. \qquad \blacksquare$$

Part of the hypotheses above practically require $g(n) = o(2^n)$. Since we only use the hypothesis $g(s) = \sum_{i<s}\epsilon_s(i)g(i)$ for $s \in S$, there is no general requirement about the growth of g on all of $\mathbf{N}$. However, in examples, it is easier to assume the recurrence relations hold for all n. This aspect cannot be much improved, since, for example, if $g(n) > \sum_{i<n}g(i)$ for all n, e.g., g increasing and $g(n) \geq 2^n$, then no cancellation occurs, and thus $\| \prod(1 - x^{g(n)})/2\| = 1$. Put another way, if $m = \sum_n \eta(n)g(n)$ where $\eta(n) \in \{0,1\}$, then η is uniquely determined by m.

The parity hypothesis (condition (i) of Lemma 3.6) is crucial; for example, if each

$$g(n) = g(n-1) + g(n-2) + g(n-3)$$

for $n > 3$ and each of $g(1), g(2), g(3)$ is odd, then all $g(n)$ are odd, hence for any product of distinct terms of the form $(1 - x^{g(j)})/2$, the norm is 1 (evaluate at $x = -1$); in particular, (M_j) is not hollow in this case.

On the other hand, if g satisfies $g(j) = g(j-1) + g(j-2)$ (or anything else with an even number of terms), then we can take S to be $3\mathbf{N}$ (to avoid overlaps in the supports), and the hypotheses are all satisfied.

In the case that $g(j)$ is the j-th Fibonacci number, this yields a proof that the $\mathbf{Z}$-action obtained from the sequence $\left\{\frac{1}{2}\left(\begin{smallmatrix} 1 & x^{g(i)} \\ x^{g(i)} & 1 \end{smallmatrix}\right)\right\}$ is hollow. Additionally, (M_j) is isomorphic to the AT system $((1 + x^{g(j)})/2)$. We now drop the parity condition on $\sum_i \epsilon_i(s)$.

Proposition 3.7 *Let $g \colon \mathbf{N} \to \mathbf{N}$ be a function such that there exists $S \subseteq \mathbf{N}$ together with $\epsilon_i(s) \in \{0, \pm 1\}$ (for $s \in S$), satisfying*

- *for all $s \in S$, we may write $g(s) = \sum_{i=1}^{N-1} \epsilon_i(s) g(s-i)$;*
- *on setting $b(s) = 1 + \sum |\epsilon_i(s)|$, we have $\sum 2^{-b(s)} = \infty$;*
- *on defining $\operatorname{supp} g(s) = \{s\} \cup \{s - i \mid \epsilon_i(s) \neq 0\}$, we have $s \neq s' \in S$ entails $\operatorname{supp} g(s) \cap \operatorname{supp} g(s') = \varnothing$.*

Then

(i) $\left(M_j := \frac{1}{2} \begin{pmatrix} 1 & x^{g(i)} \\ x^{g(i)} & 1 \end{pmatrix} \right)$ *is AT.*

(ii) $\left(M_j \right)$ *is hollow if additionally, $\sum_i \epsilon_i(s)$ is even for all $s \in S$. In this case, the dimension spaces corresponding to $\left(M_j \right)$ and to $\left((1 + x^{g(j)})/2 \right)$ are isomorphic.*

Proof (i) Suppose s belongs to S. We have a relation of the form

$$g(s) + \sum_{i \in E_-(s)} g(s-i) = \sum_{i \in E_+(s)} g(s-i),$$

where $E_+(s) = \{i \in \mathbf{N} \mid \epsilon_i(s) = 1\}$ and $E_-(s) = \{i \in \mathbf{N} \mid \epsilon_i(s) = -1\}$. Since $\epsilon_i(s) \neq 0$ entails $i < s$, each set is finite. Since g is positive-valued, $E_+(s)$ is nonempty.

This equality yields that at least two terms cancel in each of

$$(1 + x^{g(s)}) \prod_{E_-(s)} (1 + x^{g(s-i)}) \prod_{E_+(s)} (1 - x^{g(s-i)}),$$

$$(1 - x^{g(s)}) \prod_{E_-(s)} (1 - x^{g(s-i)}) \prod_{E_+(s)} (1 + x^{g(s-i)}).$$

Hence

$$\left\| \frac{(1 \pm x^{g(s)})}{2} \prod_{i \in E_-(s)} \frac{(1 \pm x^{g(s-i)})}{2} \prod_{i \in E_+(s)} \frac{(1 \mp x^{g(s-i)})}{2} \right\| \leq 1 - \frac{2}{2^{b(s)}}.$$

Since $\lambda_0(M_j) = (1 + x^{g(j)})/2$ and $\lambda_1(M_j) = (1 - x^{g(j)})/2$, we will verify the conditions of Lemma 3.5 if we select d large enough that we can choose R inside T so that both R and $T \setminus R$ contain enough terms of S to guarantee that the two products are arbitrarily small. This is possible since $\prod (1 - 2^{-b(s)}) = 0$, as a consequence of $\sum 2^{-b(s)} = \infty$.

(ii) Since the matrices are 2×2 and circulant, the only relevant eigenvalue of M_j is $(1 - x^{g(j)})/2$. So hollowness follows from Lemma 3.6, and does not depend on the argument in (i). ∎

For example, if $g(j)$ is the j-th Fibonacci or Lucas number, then $\left(M_j \right)$ is isomorphic to the AT system $\left((1 + x^{g(j)})/2 \right)$, since in either case, $g(j) = g(j-1) + g(j-2)$, and we can set $S = 3\mathbf{N}$. On the other hand, if $g(j) = g(j-1) + g(j-2) + g(j-3)$ for all sufficiently large j, set $S = 4\mathbf{N}$. Proposition 3.7 (i) yields that the resulting sequence $\left(M_j \right)$ is at least AT, but need not be hollow, as we observed above.

Roughly speaking, if $g(j)$ is $o(2^j)$ (and satisfies a recurrence relation as above), then the resulting $\left(M_j = \frac{1}{2} \begin{pmatrix} 1 & x^{g(j)} \\ x^{g(j)} & 1 \end{pmatrix} \right)$ tends to be at least AT, and could be hollow. A limiting case that is not covered by the hypotheses occurs with $g(j) = 2^j$, for which AT-ness of the resulting sequence is due to [DQ]. It is known that if $g(j) = 5^j$, then the sequence is not AT [GH].

Proposition 3.8   *If (M_j) is an ergodic sequence of hemicirculant matrices, then*

$$(\lambda_0(M_j)M_j) = (\lambda_0(M_j)) \otimes (M_j)$$

is hollow, and thus $(\lambda_0(M_j) \cdot M_j) \cong (\lambda_0(M_j)^2)$.

Proof   We note for α in $\widehat{H}$, that $\lambda_\alpha(\lambda_0(M_j)M_j) = \lambda_\alpha(M_j)\lambda_0(M_j)$. Thus with $\beta = \chi_0$ and $\alpha \neq \chi_0$, setting $N_j = \lambda_0(M_j)M_j$, we have

$$\lambda_\alpha\Big(\prod_{j_0}^{j_0+d} N_j\Big) = \prod_{j_0}^{j_0+d} \lambda_\alpha(M_j)\lambda_\beta(M_j),$$

which goes to zero as $d \to \infty$, by Lemma 2.2. ∎

   If instead, we ask about multiplication by suitable values of the trace, an extra condition is required. For size two matrices, there is an analogous result even in the case that the matrices in the sequences do not commute. From $N^2 = N \operatorname{tr} N - I \det N$, by suitably telescoping, we can arrange that the determinant terms are as small (in norm) as we like. As a consequence, if (N_j) is an ergodic sequence of 2×2 matrices, then there exists a telescoping so that $((N^{(i)})^2) \cong (\operatorname{tr}(N^{(i)}) \cdot N^{(i)})$. This is not as useful as it might appear: there is no assumption that the N_j commute, so the isomorphism class of $((N^{(i)})^2)$ might depend on the choice of telescoping.

Proposition 3.9   *Suppose that (M_j) is an ergodic sequence of hemicirculant matrices. Then (M_j^2) is hollow if and only if there exists a telescoping $0 = n(1) < n(2) < \cdots$ with $M^{(i)} = \prod_{j=n(i)}^{n(i+1)-1} M_j$ such that the sequence $(\operatorname{Tr}(M^{(i)}) \cdot M^{(i)})$ is hollow.*

Proof   For α in $\widehat{H} \setminus \chi_0$ and J a finite subset of $\mathbf{N}$, we observe that $\operatorname{Tr}(\prod_{j \in J} M_j) = \sum_{\gamma \in \widehat{H}} \prod_J \lambda_\gamma(M_j)$. Thus

$$\lambda_\alpha\big(\operatorname{Tr}(\prod_{j \in J} M_j) \cdot \prod_{j \in J} M_j\big) = \sum_{\gamma \in \widehat{H}} \prod_J \lambda_\gamma(M_j)\lambda_\alpha(M_j)$$

$$= \prod_J \lambda_\alpha(M_j^2) + \sum_{\alpha \neq \gamma} \prod_{j \in J} \lambda_\gamma(M_j)\lambda_\alpha(M_j).$$

   Assume (M_j^2) is hollow. Given any $j = j_0$ we can make the first term on the right side as small as we like by increasing the d in $J = \{j_0, j_1, \ldots, j_0 + d\}$; by Lemma 2.2, and similarly increasing the d (further if necessary), the second summand can be made as small as we like, hence the desired telescoping exists.

   Conversely, assume that $(\operatorname{Tr}(M^{(i)}) \cdot M^{(i)})$ is hollow. By a further telescoping (which amounts to deleting some of the $n(i)$s and re-indexing), we may assume that the second summand and the entire left side are arbitrarily small for $J = \{n(i), n(i) + 1, \ldots, n(i+1) - 1\}$ (and sufficiently large i). Thus the first summand must be small, which yields hollowness of (M_j^2). ∎

4   Tensor Products

Let M and N be hemicirculant matrices with respect to the same group H. We may form $M \otimes N$, which is now hemicirculant with respect to $H \times H$. If (M_j) and (N_j)

are ergodic sequences of hemicirculant matrices, we may form $(M_j \otimes N_j)$, which corresponds to the tensor product of the corresponding dimension spaces.

There are natural A-module maps in this situation. Define $\theta \colon AH \otimes AH \to AH$ via $\theta(e_g \otimes e_h) = e_{gh}$. This is a positive, onto A-module map such that $\theta \circ (M \otimes N) = MN \circ \theta$ (here M and N are H-hemicirculant matrices) requiring a routine verification. We may complexify $A_{\mathbf{C}} = A \otimes \mathbf{C} = \mathbf{C}[x, x^{-1}]$ as usual, and extend $\theta_{\mathbf{C}} \colon A_{\mathbf{C}}H \otimes A_{\mathbf{C}}H \to A_{\mathbf{C}}H$. It is easy to verify from the definitions that for α and β in $\widehat{H}$,

$$\theta_{\mathbf{C}}(v_\alpha \otimes v_\beta) = \begin{cases} v_\alpha & \text{if } \alpha = \beta, \\ 0 & \text{if } \alpha \neq \beta. \end{cases}$$

Since $\theta_{\mathbf{C}}$ is actually defined over the complexes, it easily follows (since $\{v_\alpha\}_{\alpha \in \widehat{H}}$ is a basis for $\mathbf{C}H$), that $\ker \theta_{\mathbf{C}}$ is the free $A_{\mathbf{C}}$-module on $\{v_\alpha \otimes v_\beta\}_{\alpha \neq \beta}$.

The following is a fragment of the commuting diagram obtained from the map θ. The splitting map ψ is given by $\psi(e_g) = \frac{1}{|H|} \sum_{h \in H} e_h \otimes e_{gh^{-1}}$ (the corresponding complexification of ψ sends v_α to $v_\alpha \otimes v_\alpha$).

Now θ induces an A-module map from the dimension space of $(M_j \otimes N_j)$ to that of $(M_j N_j)$, obviously of the form $\Theta[z, k] \mapsto [\theta(z), k]$. This defines Θ on the algebraic direct limits; it clearly extends to the l^1 completions, that is, the dimension spaces themselves. It is clearly of norm 1, positive, onto on positive cones, and sends the unique invariant measure on the domain to that on the image; it also has a splitting, which we will use shortly.

Under some circumstances, Θ will be an isomorphism. Suppose z in AH is sent to zero by θ. We wish to determine whether for all k, and for all $\epsilon > 0$, there exists d such that $\| \prod_{j=k}^{k+d} (M_j \otimes N_j) z \| < \epsilon$. We can write z as an $(A_{\mathbf{C}}$-$)$ linear combination of terms of the form $v_\alpha \otimes v_\beta$ with $\alpha \neq \beta$. Next, we note that $(\prod_{j=k}^{k+d} (M_j \otimes N_j)) v_\alpha \otimes v_\beta = (\prod_{j=k}^{k+d} \lambda_\alpha(M_j) \lambda_\beta(N_j)) v_\alpha \otimes v_\beta$. Define the following property,

$$(4.1) \qquad \text{For all } \alpha \neq \beta \in \widehat{H}, \text{ for all } k, \qquad \lim_{d \to \infty} \prod_{j=k}^{k+d} \lambda_\alpha(M_j) \lambda_\beta(N_j) = 0.$$

When (4.1) holds, it follows that the elements $[z, k]$ in the dimension space of $(M_j \otimes N_j)$ are all zero.

Now assume (1) and suppose $[y, k]$ is such that $\| \theta y \| < \epsilon$. In $A_{\mathbf{C}}H$, we can write $y = z + x$ where $y \mapsto z$ is the projection of $A_{\mathbf{C}}H$ onto $\ker \theta$ (this has norm 1, but all we need is that it has finite norm, which is trivial here), and there exists a constant K, independent of y, such that $\|x\| \leq K\epsilon$. Then it easily follows that the image of $[y, k]$ in the dimension space of $(M_j \otimes N_j)$ has norm at most $K\epsilon$. We can avoid this last argument if we employ the splitting map, $\psi \colon AH \to AH \otimes AH$. Then ψ induces a map Ψ from the dimension space of $(M_j N_j)$ to that of $(M_j \otimes N_j)$ and $\Theta \circ \Psi$ is the identity on the dimension space of $(M_j N_j)$. Hence using Lemma 2.2, we obtain the following.

Lemma 4.1 *Let (M_j) and (N_j) be ergodic sequences of H-hemicirculant matrices. The map Θ from the dimension space of $(M_j \otimes N_j)$ to that of $(M_j N_j)$ is an isomorphism of dimension spaces if (4.1) holds. In particular, this occurs if $M_j = N_j$, i.e., (M_j^2) and $(M_j \otimes M_j)$ yield isomorphic dimension spaces, necessarily AT.*

Another consequence of the arguments used before: if (4.1) holds, then $(M_j N_j)$ is AT (simply use the argument of Lemma 3.6, constructing V_j from M_j and W from N_j). However, when M_j and N_j are quite different from each other, it may be difficult to decide whether (4.1) holds. For example, in general it will fail if $N_j = M_j^T$. This does not imply $(M_j \otimes M_j^T) \not\simeq (M_j M_j^T)$, simply that the method of proof fails. However, after we remind the reader of an invariant developed in [H, §6], we will give a large class of examples for which $(M_j \otimes M_j^T) \not\simeq (M_j M_j^T)$.

5 Isomorphism and Non-isomorphism

Let $(M_j = \sum_{g \in H} q_{gj} m_g)$ be an (ergodic) sequence of H-hemicirculant matrices, with $v_0 = (|H|)^{-1}(1\,1\cdots1)^T$ and $w_0 = |H| v_0^T$ as usual. The row w_0 induces (after a relabelling) the map $\rho \colon AH \to A$ sending $\sum_{g \in H} q_g m_g$ to $\sum q_g$. Then we obtain a commuting diagram (not just approximately commuting)

$$
(5.1) \qquad
\begin{array}{ccccccc}
A^n & \xrightarrow{\;M_{j-1}\;} & A^n & \xrightarrow{\;M_j\;} & A^n & \xrightarrow{\;M_{j+1}\;} & \\
\rho\downarrow\uparrow\sigma & & \rho\downarrow\uparrow\sigma & & \rho\downarrow\uparrow\sigma & & \\
A & \xrightarrow{\;\times\lambda_0(M_{j-1})\;} & A & \xrightarrow{\;\times\lambda_0(M_j)\;} & A & \xrightarrow{\;\times\lambda_0(M_{j+1})\;} &
\end{array}
$$

where the horizontal maps on the second row are multiplications by the indicated elements of A^+, in this case, the large eigenvalue of the M_j, $\lambda_0(M_j) = \sum q_{gj}$. Then ρ extends to P (capital ρ), a positive A-module map from the dimension space of (M_j) to the AT dimension space of $(\lambda_0(M_j))$. With the same elementary techniques as those of the earlier tensor product constructions, it follows that P is an isomorphism if and only if (M_j) is hollow. This is made easier by the presence of the splitting map for P, induced by $\sigma \colon A \to AH$ given by $\sigma(c) = \frac{1}{|H|} c \sum_{g \in H} e_g$.

When (M_j) is hollow, it is also immediate that for all $j_0 \geq 0$,

$$
\lim_{d \to \infty} \left\| \mathrm{tr}\Big(\prod_{j=j_0}^{j_0+d} M_j \Big) - \lambda_0\Big(\prod_{j=j_0}^{j_0+d} M_j \Big) \right\| = 0.
$$

(The norm is the usual l^1 norm, i.e., the total variation norm, on the coefficients.) Since λ_0 is multiplicative on H-hemicirculant matrices, this yields that the dimension space associated with (M_j) is isomorphic (in all senses) to that of a telescoped sequence $(\mathrm{tr}(\prod_{j=n(i)}^{n(i+1)-1} M_j))_i$, where $n(i)$ is a suitable strictly increasing sequence of positive integers.

We can ask, if (M_j) is not hollow, then can it be isomorphic to the AT system $(\lambda_o(M_j))$? In many cases (and possibly all), it is relatively easy to give a negative answer. This uses an invariant (actually, a class of invariants), introduced in [H], related to mass cancellation.

In [H], it was intended to be used to distinguish pairs of AT systems, but in fact, can be extended to any dimension space. Let $N_j \colon A^{n(j)} \to A^{n(j+1)}$ be a sequence of matrices with entries from A^+ such that all column sums of $N_j(1)$ are 1. Let $\{p_i\}_{i \in I}$ (in examples, $I = \mathbf{N}$, but could be any infinite set) be a collection of elements of $A_{\mathbf{C}} = \mathbf{C}[x, x^{-1}]$ (not A^+) each with norm 1. Associate with the sequences $\{p_i\}, (N_j)$ a real number $s \equiv s(\{p_i\}, (N_j))$ in $[0, 1]$ as follows.

Fix positive integers l and d, and $p \in \{p_i\}$, and define the operator l^1 norm on the products $pN_{l+d} \cdot N_{l+d-1} \cdots N_l$ (using the norm inherited from the limiting-dimension space). Define $s_{l,p} = \lim_{d \to \infty} \|pN_{l+d} \cdot N_{l+d-1} \cdots N_l\|$. Since each N_j has norm 1, it easily follows that the limit exists and is a number in the unit interval. Now set $s_l = \inf_{p \in \{p_i\}} s_{l,p}$. As $l \mapsto s_l$ is increasing, $s(\{p_i\}, (N_j))) := \lim_{l \to \infty} s_l$ is well defined and in the unit interval. The invariant is $s(\{p_i\}, \cdot)$. We must check that it is an invariant. In fact, it is even an invariant for neutral isomorphisms (see Section 2) as well.

First, we note that it is invariant under telescoping and deletion of a finite set of N_j. Now suppose we have an approximately commuting diagram in (1.1) (here the M_j need not be hemicirculant, or even square), meaning, with respect to the l^1 norms induced by the invariant linear functionals, we have

$$\sum \|S_j R_j - M_j\| < \infty \quad \text{and} \quad \sum \|R_{j+1} S_j - N_j\| < \infty.$$

(By repeatedly telescoping, we may arrange this out of any approximately commuting isomorphism.)

Given $\epsilon > 0$, there exists l such that both sums

$$\sum_{j \geq l} \|S_j R_j - M_j\| \quad \text{and} \quad \sum_{j \geq l} \|R_{j+1} S_j - N_j\|$$

are less than ϵ (the norms are on different spaces, but the notation would become cumbersome if we distinguished them). Then to within ϵ, for any polynomial p,

$$pM_{l+d} M_{l+d-1} \cdots M_d \sim pS_{l+d} N_{l+d-1} \cdots N_l R_l.$$

Since we can assume R_j and S_j have respective norms at most 1, it follows that

$$s_{l,p}(\{M_j\}) \leq \epsilon + s_{l,p}(\{N_j\}).$$

Hence the corresponding result holds dropping the l and taking infima over all the elements p of $\{p_i\}$, and, of course, then we interchange the roles of M_j and N_j.

As an aside, we almost always need an infinite choice of $\{p_i\}$. For example, the AT system given by $\left(\frac{1}{2}(1 + x^{2^j})\right)$ represents the dyadic odometer. If $\{p_i\}$ is just the singleton $\{(1 - x)/2\}$, then $s_0 = 0$ (almost complete mass cancellation) but $s_l = 1$ (no mass cancellation) for $l \geq 1$, hence $s = 1$, which in this case is uninformative. On the other hand, if $p_i = (1 - x^{2^{i-1}})/2$, then $s_l = 0$ for all l and thus $s = 0$, which is frequently useful.

For example, it allows us to distinguish the sequence $\left(M_j = \frac{1}{2}(I + x^{2^j} P)\right)$ (where P is the cyclic permutation matrix of size n, corresponding to $H = \mathbf{Z}_n$) from the dyadic odometer. We note that $\lambda_0(M_j) = \frac{1}{2}(1 + x^{2^j})$, so the sequence in the bottom line of (5.1) is that of the dyadic odometer. As above, let $p_i = (1 - x^{2^{i-1}})/2$, so that $s(\{\frac{1}{2}(1 - x^{2^j})\}, (\lambda_0(M_j)) = 0$. On the other hand, a simple computation reveals that the $(1,1)$ entry of $\prod_0^d M_j$ is $2^{-d} \sum_{\eta(a) \equiv 0 \mod n} x^a$ where a varies over the integers from

1 to 2^{d+1}, and $\eta(a)$ is the number of 1 in the binary expansion of a. It easily follows that $\|((1-x)/2)\prod_0^d M_j\|$ is at least $1-1/n-2^{-d}$ (it can be computed exactly), so that s_0 of the sequence is at least $1-1/n$. However, $M_j = M_0(x^{2^j})$, so $s_0 = s_j$ (since the sequence $\{p_i\}$ is invariant under $x \mapsto x^2$), so the s value of (M_j) is at least $1-1/n$. In particular, (M_j) is not isomorphic to the dyadic (or any) odometer.

In fact, we obtain different s values for different values of n (this requires more precise, but elementary computations), so distinct choices of n yield nonisomorphic systems.

Conjecture 5.1 *If (M_j) is ergodic, hemicirculant, and not hollow, then $(M_j) \not\cong (\lambda_0(M_j))$.*

We can *almost* prove this. A slightly stronger assumption, that (M_j^2) be not hollow, is sufficient. There are examples where non-isomorphism occurs (so that (M_j) is not hollow), but with (M_j^2) hollow.

Lemma 5.2 *Suppose that (M_j) is an ergodic sequence of hemicirculant matrices such that (M_j^2) is not hollow. Then $(M_j) \not\cong (\lambda_0(M_j))$. More generally, if (M_j^{k+1}) is not hollow, then $(M_j^k) \not\cong (\lambda_0(M_j^k))$.*

Proof If $k=1$, (M_j^2) is not hollow and there thus exists $\alpha \in \widehat{H} \smallsetminus \{\chi_0\}$ such that for some positive integer N_0, $\lim_{l\to\infty} \|\prod_{j=N_0}^{N_0+l} \lambda_\alpha(M_j)^2\| > 0$. It follows that given $\epsilon > 0$, there exists $N \equiv N(\epsilon)$ such that $\lim_{l\to\infty} \|\prod_{j=N}^{N+l} \lambda_\alpha(M_j)^2\| > 1-\epsilon$.

Set $p_i' = \prod_{j=N}^{N+i} \lambda_\alpha(M_j)$ (in $A_{\mathbf{C}} = \mathbf{C}[x, x^{-1}]$), so that $1 \geq \|p_i'\| > 1-\epsilon$, and define the normalized versions $p_i = p_i'/\|p_i'\| \in A_{\mathbf{C}}$. Then for each i and all k,

$$\Big\| p_i \prod_{j=N}^{N+k} \lambda_\alpha(M_j) \Big\| > 1-\epsilon.$$

Thus $s(\{p_i\}, (M_j)) \geq s(\{p_i\}, (\lambda_\alpha(M_j))) \geq 1-\epsilon$, so that $s(\{p_i\}, (M_j)) \geq 1-\epsilon$ (because the choice of $\{p_i\}$ depended on ϵ, we cannot obtain 1 exactly).

One the other hand, for $N' \geq N$,

$$p_i \prod_{j=N'}^{N'+k} \lambda_\alpha(M_j) = \prod_{i=N}^{N+i} \lambda_0(M_j) \prod_{j=N'}^{N'+k} \lambda_\alpha(M_j).$$

If we choose i large enough that $N+i > N'+k$, then Lemma 2.2 applies, so the norm of the product can be made arbitrarily small. Thus $s(\{p_i\}, (\lambda_0(M_j)) = 0$, and therefore nonisomorphism occurs.

If $k > 1$, we find N such that for all i, $\|\prod_{j=N}^{N+i} \lambda_\alpha(M_j)^{k+1}\| > 1-\epsilon$, set $p_i = \prod \lambda_\alpha(M_j)$, and proceed as above. $\blacksquare$

There do exist non-hollow (M_j) with (M_j^2) hollow, such that the same nonisomorphism holds, so the question remains whether the additional hypothesis really is necessary.

Suppose $M_j = \frac{1}{2}\left(\begin{smallmatrix} 1 & x^{2^j} \\ x^{2^j} & 1 \end{smallmatrix}\right)$. Radu Monteanu has shown (unpublished) that this represents the Morse–Thue transformation. It was shown by Dooley and Quas [DQ] to be AT. It is easy to verify that (M_j^2) is hollow: $\|\prod_{j=N}^{N+k-1}(1-x^{2^j})^2\| = \boldsymbol{o}(4^k)$ (in fact, it is $\boldsymbol{o}(\beta^k)$ for β substantially less than 4). However, $(M_j) \not\cong (\frac{1}{2}(1+x^{2^j})) = (\lambda_0(M_j))$ as the Morse is not an odometer; we can use $p_i = \frac{1}{2}(1-x^{2^j})$.

The map $AH \to AH$ given by $e_g \mapsto e_{g^{-1}}$ induces an isomorphism between the dimension spaces of (M_j) and of (M_j^T). This allows us to show that not only is the map in Lemma 4.1 not generally an isomorphism between $(M_j \otimes M_j^T)$ and $(M_j M_j^T)$ (which we already knew), but in fact, the two dimension spaces are not generally isomorphic.

Corollary 5.3   *Suppose that $|H|$ is odd and (M_j) is an ergodic hemicirculant sequence such that (M_j^2) is not hollow. Then $(M_j \otimes M_j^T) \not\cong (M_j M_j^T)$.*

Proof   First, $(M_j^T) \cong (M_j)$. Next, $(M_j M_j^T)$ is symmetric and of odd size, hence is hollow (Proposition 3.3). If $(M_j \otimes M_j^T) \cong (M_j M_j^T)$, then we would have the following chain of isomorphisms,

$$(M_j^2) \cong (M_j \otimes M_j) \cong (M_j \otimes M_j^T) \cong (M_j M_j^T) \cong (\lambda_0(M_j M_j^T)) = (\lambda_0(M_j^2)).$$

This contradicts $(M_j^2) \not\cong (\lambda_0(M_j^2))$. ∎

Example   *Another computation.* Let $H = \mathbf{Z}_n$ be the cyclic group of order n, let P be the corresponding cyclic permutation matrix of size n (corresponding to the generator, $[1]$, of $\mathbf{Z}_n$), and for $j = 0, 1, 2, \ldots$, set $M_j = \frac{1}{2}(I + x^{3^j}P)$. Then (M_j) and (M_j^2) are ergodic but not hollow, while (M_j^3) is hollow. It is reasonable to conjecture that (M_j) is not AT, or even not AT(2); but by Theorem 2.3 (i), (M_j^2) is AT. We wish to determine the corresponding sequence of polynomials (Q_j) (each $Q_j \in A^+$ and $Q_j(1) = 1$) that is obtained by the argument therein such that $(M_j^2) \cong (Q_j)$.

Indexing $\widehat{H}$ by $0, 1, 2, \ldots, n-1$ (instead of α), we see immediately that $\lambda_k(M_j) = \frac{1}{2}(1 + x^{3^j}\xi^k)$, where $\xi = \exp(2\pi i/k)$, for $k = 0, 1, \ldots, n-1$. For $k \neq l$ (corresponding to $\alpha \neq \beta$), we calculate the norm of $\lambda_k(M_j)\lambda_l(M_j)$. In all cases, we see that the maximum of the norm over all pairs is some number $\gamma \equiv \gamma(n) < 1$. As a sample computation $\gamma(3) = \frac{3}{4}$ comes from

$$(1 + \xi x)(1 + \xi^2 x) = 1 - x + x^2 \quad (1 + \xi x)(1 + x) = 1 - \xi^2 x + \xi x^2$$

(because $M_j = M_0(x^{3^j})$, the dimension space is stable under $x \mapsto x^3$ and we need only consider the case that $j = 0$; also, when $n = 3$, $1 + \xi + \xi^2 = 0$).

Now we find a telescoping $0 = n(1) < n(2) < n(3) < \cdots$ so that for each $k \neq l$,

$$\left\| \lambda_k\left(\prod_{n(t)}^{n(t+1)-1} M_j\right) \lambda_l\left(\prod_{n(t)}^{n(t+1)-1} M_j\right) \right\| < \infty;$$

sufficient for this is simply $\sum \gamma^{n(i+1)-n(i)} < \infty$. We can set $n(i+1) = i(i+1)/2$, so that $n(i+1) - n(i) = i$. (This is a lot bigger than necessary.) Now define $M^{(i)} =$

$\prod_{n(i)}^{n(i+1)-1} M_j$. We find the coefficients in the expansion of $M^{(i)} = \sum_{u \in H} c_u m_u$, as a circulant matrix.

For an integer t, let $t = \sum \eta_i(t) 3^i$ be its ternary representation ($\eta_i(t) \in \{0, 1, 2\}$). Let $V_m = \{t < 3^{m+1} \mid \eta_i(t) \in \{0, 1\} \text{ for all } i\}$, and for t in V_m, define $\eta(t) = \sum_i \eta_i(t)$, *i.e.*, the number of 1s in the ternary expansion of t. For $j \in \widehat{H}$ and $0 < a < b$, define

$$P_{j,a,b} := \sum_{\{t \in V_{b-a} \mid \eta(t) \equiv j \bmod n\}} x^{t3^a}.$$

Then a simple induction argument yields that with $c_{ui} = P_{u,n(i),n(i+1)}$, we have $M^{(i)} = \sum_{u \in H} c_{ui} P^u / 2^i$, using $n(i+1) - n(i) = i$ (this can also be obtained by noting that $\lambda_u(M^{(i)}) = \prod(1 + \xi^u x^{u3^{n(i)}})/2^i$ and applying the finite Fourier transform).

By the method of Theorem 2.3, we have an approximate factorization $(M^{(i)})^2 \sim V^{(i)} W^{(i)}$, where $V^{(i)} = \frac{1}{n} \sum e_u c_{ui}$ and $W^{(i)} = \sum e_u c_{n-u,i}$. This yields an isomorphism between (M_j^2) and $(p_i := W^{(i+1)} V^{(i)})$. The earlier computation yields

$$p_i = \sum_{u \in \mathbf{Z}_n} P_{u,n(i),n(i+1)} P_{n-u,n(i+1),n(i+2)})),$$

and this equals $P_{0,n(i),n(i+2)}$. Thus $(p_i = P_{0,n(i),n(i+2)})$. It is very likely that

$$\left(\left(\frac{1}{|H|} \mathrm{tr}(M^{(i)}) \right)^2 \right)$$

is not isomorphic to (p_i), but at the moment, there are technical difficulties in showing this. (The idea is to use the mass cancellation invariants discussed above.)

In this example, all of the automorphisms on (M_j^2) given by H are nontrivial. This is one way of constructing finite order automorphisms of AT actions that induce nontrivial automorphisms of the dimension space. Of course, this realizes the dual action of the original approximately inner action of H on the underlying ITPFI given by $(\lambda_0(M_j^2))$.

Example An example of a hemicirculant ergodic sequence (M_j) such that $(M_j \otimes M_j^T)$ is not isomorphic to $(M_j M_j^T)$. This is a modification of the previous example. Let $H = \mathbf{Z}_n$ where $n \geq 5$ and is odd. Select $k \geq 4$, and set $M_j = \frac{1}{2}(I + Px^{k^j})$. Since $k \geq 4$, (M_j^3) is not hollow, and the result follows from Corollary 5.3.

If (M_j) is hollow, there exists a telescoping $n(i) < n(i+1)$ so that if $M^{(i)} := M_{n(i+1)-1} \cdot M_{n(i+1)-2} \cdots M_{n(i)}$, then $(M_j) \cong (\mathrm{tr}(M^{(i)}))$. However, we noted that for some sequences of hemicirculant matrices of the form (M_j^2), the isomorphism is only obtained with overlapping sequences; that is $(M_j^2) \cong (\mathrm{tr}(M^{(i+1)} M^{(i)}))$ (and generally not isomorphic to $(\mathrm{tr}((M^{(i)})^2))$). This overlapping is essential (and one might view it as natural, since $(M_j M_{j-1}) \cong (M_j^2)$), and extends to the most general case.

Suppose $(N_i \colon A^{m(i)} \to A^{m(i+1)})$ is a sequence of matrices with entries from A^+ such that the column sums, when evaluated at $x = 1$, are all 1. Suppose it is known that (N_j) (or more accurately, its associated dimension space) is AT. Then, after a possible

telescoping (which we incorporate into the notation, to avoid unnecessarily complicating an already complicated situation), there exist columns V_i (in $(A^{m(i+1) \times 1})^+$) and columns W_i (in $(A^{1 \times m(i)})^+$) such that $\sum \|N_i - V_i W_i\| < \infty$. Then $(N_i) \cong (W_{i+1} V_i)$, the latter, of course, being a sequence of normalized elements of A^+ (we can slightly perturb the sequence so that the values at $x = 1$ are all exactly one, rather than close).

Now take any telescoping of the already telescoped (N_j), say corresponding to $0 = n(1) < n(2) < \cdots$, and this time, take the products including *both* ends of the intervals, that is, define $N_{(i)} = N_{n(i+1)} \cdot N_{n(i+1)-1} \cdots N_{n(i)}$ (the more usual telescoping would not include $N_{n(i+1)}$). Set $P_j = W_{j+1} V_j$ (an element of A^+). Then

$$N_{(i)} \sim V_{n(i+1)} W_{n(i+1)} V_{n(i+1)-1} W_{n(i+1)-1} \cdots V_{n(i)} W_{n(i)}$$
$$= V_{n(i+1)} W_{n(i)} \left(W_{n(i+1)} V_{n(i+1)-1} \cdots W_{n(i)+1} V_{n(i)} \right)$$
$$= \left(V_{n(i+1)} W_{n(i)} \right) \cdot P_{n(i+1)-1} \cdot P_{n(i+1)-2} \cdots P_{n(i)}$$
$$= \left(V_{n(i+1)} W_{n(i)} \right) \prod_{j=n(i)}^{n(i+1)-1} P_j.$$

Thus, if N_j were square matrices,

$$\operatorname{tr} N_{(i)} \sim \left(W_{n(i)} V_{n(i+1)} \right) \prod_{j=n(i)}^{n(i+1)-1} P_j.$$

Since $(N_j) \cong (P_j)$, we obtain the isomorphism of AT systems, $(\operatorname{tr} N_{(i)}) \cong (N_j) \otimes (W_{n(i)} V_{n(i+1)})$. This is true for *every* telescoping of the originally telescoped sequence (N_j) (although in principal, the isomorphism class of $(\operatorname{tr} N_{(i)})$ may depend on the choice of telescoping, $n(i)$). We note in particular, the overlapping phenomenon $N_{(i)}$ and $N_{(i-1)}$ which both contain the term $N_{n(i)}$ (in the former, at the extreme right of the product, in the latter, at the extreme left), although no such overlapping occurs with the P_j.

One might expect that if the telescoping were sufficiently sparse (if $i \mapsto n(i)$ grows really quickly), the extra polynomials $W_{n(i)} V_{n(i+1)}$ should not play much of a role, that is, it is possible that $(\operatorname{tr} N_{(i)}) \cong (N_j)$ for sufficiently fast growing $n(i)$.

The overlapping has significance for actual computations. For example, when N_j is hemicirculant, it is often straightforward to calculate the $(1,1)$ entry of $N_{(i)}$ (this is the trace divided by n). However, it is not so straightforward to calculate even $\operatorname{tr} N_{(i)} N_{(i-1)}$ (and computing the traces of products of larger numbers of matrices is still more tedious), precisely because of the overlap.

It also means that a theorem such as, if (N_j) is AT, then there is a telescoping so that $\operatorname{tr}(N_{n(i)} N_{n(i-1)})$ is almost $\operatorname{tr} N_{n(i)} \operatorname{tr} N_{n(i-1)}$ (that is, the trace is almost multiplicative after suitably telescoping), is probably not true. Had this been true, it would have been really useful for showing that many sequences are not AT.

6 Perspective

In terms of the type III von Neumann algebras that are classified by dimension spaces, families of hemicirculant matrices have a particularly elementary interpretation. Begin with a product type W*-algebra (and corresponding AT action and dimension

space), let H be a (finite abelian) group of product type automorphisms (necessarily approximately inner), assume the action is ergodic, and form the crossed product, another W*-algebra. Its dimension space is given by an ergodic sequence of H-hemicirculant matrices, and the inclusion of the original in the crossed product translates in the dimension space setting to the diagram (5.1).

The original action of H, being approximately inner, is not otherwise visible on the underlying AT dimension space. However, its dual action on the crossed product is represented by the action of H in its regular representation as permutation matrices; these commute the (M_j). Of course, the dual action may also turn out to be approximately inner, in which case these permutation matrices act trivially on the dimension space level. This occurs precisely when the sequence (M_j) is hollow, as is easy to verify.

In terms of $\mathbf{Z}$-actions, *i.e.*, ergodic transformations, sequences of hemicirculant matrices correspond to a class of H-actions on an AT dynamical system (presumably, there are non-product type actions inequivalent to any product action, even for $H = \mathbf{Z}_2$, hence not all H-actions are determined by hemicirculant matrices), resulting in an almost everywhere finite-to-one (at most $|H|$ to one) map between the spaces. A particular consequence is that the resulting systems have entropy zero, but AT(n) implies entropy zero anyway [M]. Hollow sequences yield one-to-one maps, that is, isomorphisms.

We can also relate this to corresponding ideas from topological (rather than measure-theoretic) classification, in this case of actions of compact groups on AF C*-algebras (in particular). From an AF algebra, given as a direct limit of finite-dimensional algebras, $\lim A_k$ dense in C, let G be any compact group (not necessarily abelian), and suppose for each k, there is a unitary representation $\pi_k \colon G \to A_k$ compatible with the maps $A_k \to A_{k+1}$. This yields an action of G on A called *locally representable*.

It was shown in [HR, Theorem III.1], that $K_0(C \times G)$ (viewed as an ordered module over the representation ring) together with some additional data is a complete invariant (up to unitary group actions on C*-algebras) for the action. Moreover, all such actions can be obtained by considering all possible Bratteli diagrams that represent C, and for each one, replacing the integer entry (describing the multiplicity) by a character (not necessarily irreducible) of G whose dimension equals the multiplicity, that is, if χ is a character, the arrow with multiplicity d can be weighted by χ if $\chi(1) = d$. This yields Bratteli diagrams with weights from the positive cone of the representation ring.

If we begin with a Bratteli diagram that is initially weighted with polynomials (in one variable) with positive integer coefficients (necessarily not adding to one), we obtain an action of the circle on the underlying AF algebra. For example, a product type action would correspond to a sequence of polynomials in one variable. We can further elaborate on this diagram by now permitting characters of $T \times H$, where H is a finite abelian group, yielding an H-action on the crossed product by T. We can then convert this into a product type action in the measure theoretic setting selecting an extreme trace on the underlying AF algebra and then dividing by suitable rationals so that things add to one.

Going in reverse, we merely must approximate sufficiently well any real numbers that appear in the coefficients in the matrix entries by rational numbers (since the measure-theoretic classification is immune to tiny perturbations), and then multiply each matrix entry by a suitable positive integer to remove the denominators. Of course, measure-theoretic classification is much coarser than topological, and topological classification is sensitive to the approximation and choice of integer.

7 Powers of Transformations

Let (X, μ, T) be a dynamical system, with T ergodic and μ quasi-invariant with respect to T. The Poisson boundary can be represented via the dimension space of a suitable sequence of maps $N_j \colon A^{m(j)} \to A^{m(j+1)}$ together with a trace-induced factoring through an extremal trace on the direct limit of $N_j(1) \colon \mathbf{R}^{m(j)} \to \mathbf{R}^{m(j+1)}$ where evaluation at 1 means evaluation $x \mapsto 1$ to each coordinate, creating a real column stochastic matrix.

Suppose we want to consider the square or a higher power of T, that is, the dynamical system (X, μ, T^n), for some $n > 0$. A first problem is that this need not be ergodic (the use of dimension spaces is more or less confined to ergodic transformations that are insensitive to sufficiently small perturbations, whereas in the non-ergodic case, arbitrarily small perturbations can add or delete atoms). However, in general there is a process for obtaining a dimension space for T^n (where convenient, we abbreviate the triple (S, μ, X) to its first component) out of the original one for T.

We can read off from the construction whether T^n is ergodic, and if so, the resulting dimension space corresponds to a special $\mathbf{Z}_n$ action. In particular, if T is AT, then the dimension space for T^n comes from a commuting sequence of $n \times n$ matrices that are very close to being circulant, and allow us to read off properties. For example, if T is AT(k) and T^n is ergodic, then T^n is AT(kn) (but it frequently happens that it is AT). We give examples wherein T is AT, T^2 is ergodic, but T^2 is not AT. It also follows from earlier results that if T^n is ergodic and T is AT, then $T^n \otimes T^n$ is AT.

First we give the construction. Let $X = x^n$, and form $B = \mathbf{R}[X, X^{-1}]$ sitting inside $A = \mathbf{R}[x^{\pm 1}]$, each equipped with the usual ordering. Then A, viewed as a B-module, is free on the set $\{1, x, \ldots, x^{n-1}\}$. Write $A = \bigoplus A_i$ (as B-modules) where $A_i = x^i B$. Now let p be a polynomial in A with no negative coefficients. Let Q be the companion matrix of the polynomial (in Z), $Z^n - x^n = Z^n - X$,

$$Q = \begin{pmatrix} 0 & 0 & 0 & \ldots & 0 & X \\ 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 1 & 0 & \ldots & 0 & 0 \\ & & & \ddots & & \\ 0 & 0 & 0 & \ldots & 1 & 0 \end{pmatrix}.$$

Then Q belongs to $\mathrm{M}_n B^+$; let $\mathcal{B}(p) := p(Q)$. Then $p(Q)$ has entries in B^+, its column sums when evaluated at $x = 1$ (and at $X = 1$!) equal $p(1) = 1$, and if we form the diagonal matrix (with entries from A^+), $\Delta = \mathrm{diag}\,(x^{n-1}, x^{n-2}, \ldots, 1)$, we have $\Delta Q \Delta^{-1} = xP$, where P is the cyclic permutation matrix. In particular, $\Delta \mathcal{B}(p) \Delta^{-1} = p(xP)$, a circulant matrix. Of course, this is not implementable over B, so we cannot simply transfer everything in what follows to circulant matrices.

Now the map $\times x\colon A \to A$, where we identify A with the free B module, $A = \bigoplus_{i=0}^{n-1} x^i B$, has Q as its matrix representation with respect to the basis $\{x^i\}$. There is a natural map $\varphi\colon B^n \to A$ sending $(b_0, b_2, \dots, b_{n-1})^T \mapsto \sum x^i b_i$ (which effectively is the identity on A after identifications).

Now let N_j be an $m(j+1) \times m(j)$ matrix with entries from A^+ such that all the column sums of $N_j(1)$ (evaluation at $x = 1$) are 1. Form a matrix $\mathcal{B}(N_j)$ of size $m(j+1)n \times m(j)n$ by replacing each entry $(N_j)_{lm}$ of N_j by the $n \times n$ matrix $\mathcal{B}((N_j)_{lm}) = (N_j)_{lm}(Q)$. The resulting matrix has entries in B^+, and its column sums, when evaluated at $X = 1$, are all 1. The claim is now that the dimension space (now viewed as a B-module, equivalently as a module over $l^1(n\mathbf{Z})$) corresponding to $(\mathcal{B}(N_j))$ implements T^n.

This is practically a tautology. The natural map $\varphi\colon B^n \to A$ extends to $(B^n)^m \to A^m$ in the obvious way, and the diagram

$$
\begin{array}{ccccccc}
(B^n)^{m(j)} & \xrightarrow{\mathcal{B}(N_j)} & (B^n)^{m(j+1)} & \xrightarrow{\mathcal{B}(N_{j+1})} & (B^n)^{m(j+2)} & \xrightarrow{\mathcal{B}(N_{j+2})} & \cdots \\
\varphi\downarrow & & \varphi\downarrow & & \varphi\downarrow & & \\
A^{m(j)} & \xrightarrow{N_j} & A^{M(j+1)} & \xrightarrow{N_{(j+1)}} & A^{M(j+2)} & \xrightarrow{N_{(j+2)}} & \cdots
\end{array}
$$

commutes and the vertical maps are isomorphisms of ordered B-modules. Under this, multiplication by X in the terms of the top row translates to multiplication by x^n on the bottom row. Hence T^n is represented by the B-dimension space arising from $(\mathcal{B}(N_j))$. An obvious consequence is that if T is $\mathrm{AT}(k)$ and T^n is ergodic, then T^n is $\mathrm{AT}(nk)$ (the definition of $\mathrm{AT}(m)$ does not require ergodicity. With ergodicity, $\mathrm{AT}(m)$ is equivalent to the sequence of matrices being at most $m \times m$, whereas without ergodicity, only the reverse implication applies).

At this stage, we may be tempted to use Δ to change this to matrices blown up by circulant matrices. If we can choose $m(j) = 1$ for all j, that is, if T is AT, after applying Δ, we would obtain a sequence of circulant matrices, and so use all the results of the preceding section. Unfortunately, this is not permissible, since we are restricted to B- (or $l^1(n\mathbf{Z})$-) maps between dimension spaces. Fortunately, we can often deduce results from the circulant case.

The test for ergodicity, however, is identical, since after evaluation at $X = 1$, the resulting $\mathcal{B}(p_j)(1)$ is circulant. For example, if $(p_j = (1 + x^{2^{j-1}})/2)$, then T is the dyadic odometer. Obviously T^2 and thus T^{2n} are not ergodic: T^2 is the disjoint union of two copies of the odometer. We can also see this from

$$
\mathcal{B}(p_j) = \frac{1}{2}\mathrm{diag}\left(1 + X^{2^{j-1}}, 1 + X^{2^{j-1}}\right),
$$

so the limiting dimension group is just a direct sum of two copies the original odometer sequence (with X playing the role of x).

On the other hand, if n is odd, T^n is ergodic and $\mathcal{B}(p_j)$ is $(I + Q^{2^{j-1}})/2$, which, when evaluated at $X = 1$, yields the sequence of matrices $\frac{1}{2}(I + P^{2^{j-1}})$. Since n is odd, there exist infinitely many k such that $2^k \equiv 1 \mod n$, and it easily follows that the matrix products converge to the standard rank one matrix. In fact, for odometers, there is a much stronger result available. As Thierry Giordano observed, any ergodic

power of this type of odometer is conjugate to the original, as follows from the (L^1) pure point spectrum property

It can also be derived directly from the dynamical origins, at least in the case that the supernatural number is of the form n^∞ for some n, and likely more generally. Let $Y = \prod_{\mathbf{N}} \mathbf{Z}_n$ with the product topology (a Cantor set); the odometer views Y as $\mathbf{Z}_{(n)}$ (the n-adic completion of the integers; the elements of the sequence space can be viewed as power series), and simply adds 1. Assume $(k, n) = 1$; the k-th power of the odometer adds k. However, since k is relatively prime to n, k is invertible in $\mathbf{Z}_{(n)}$ and thus $\times k$ (multiplication by k) is also a self-homeomorphism of Y, using the multiplicative (rather than the additive) structure of $\mathbf{Z}_{(n)}$, call it p. Then we have the commuting diagram

$$
\begin{array}{ccc}
Y & \xrightarrow{\ +1\ } & Y \\
{\scriptstyle \times k}\big\downarrow & & \big\downarrow{\scriptstyle \times k} \\
Y & \xrightarrow{\ +k\ } & Y
\end{array}
$$

The top row is the odometer, the bottom row is its k-th power, and the vertical maps are invertible maps (exploiting the multiplicative structure) which conjugate the odometer to its k-th power, *i.e.*, $Tp = pT^k$. (It is not clear how to extend this elementary argument to more general odometers for which there is no underlying multiplication, *e.g.*, if the supernatural number has no infinite multiplicities.) This approach has the advantage that it yields a continuous conjugacy between (Y, T^k) and (Y, T) as continuous self-homeomorphisms on the Cantor set Y.

Unexpected isomorphisms. We would expect that the following is either false, or it is true but easy to prove. It is true, but currently is not easy to prove.

Proposition 7.1 *Suppose (M_i) is an ergodic sequence of matrices over B^+.*

(i) *Suppose that (M_i') is an ergodic sequence of matrices over B^+ such that there exists an A^+-module isomorphism between the corresponding dimension modules. Then they are isomorphic as B^+-modules, that is, $(M_i)_B \cong (M_i')_B$.*

(ii) *If there is an A^+-module isomorphism $(M_i) \cong (N_i)$ where (N_i) is an $AT(n)$ ergodic sequence with entries from A, then (M_i) is $AT(n)$ with respect to B.*

In particular, if viewed as an ergodic sequence of matrices with entries from A^+, and $(M_i)_A$ is AT, then $(M_i)_B$ is AT. Moreover, if $(M_i)_A \cong (p_i)_A$ where $p_i \in B^+$ and $p_i(1) = 1$, then $(M_i)_B \cong (p_i)_B$. This justifies, in some cases, using the conjugation argument available over A, but not over B, to determine isomorphism over B.

Elementary properties of $A = \mathbf{R}[x^{\pm 1}]$, $B = \mathbf{R}[x^{\pm k}]$ equipped with the l^1 norms. Write $a = \sum_{i \in \mathbf{Z}_k} x^i a^{(i)}$ in terms of the standard basis of B as an A-module, $\{1, x, \dots, x^{k-1}\}$. Then we have the following.

- $\|a\| = \sum \|a^{(i)}\|$.
- $\text{dist}(a, B) = \sum_{i \in \mathbf{Z}_k \setminus \{0\}} \|a^{(i)}\|$ and is achieved at $b = a^{(0)}$.
- If $a \in A^+$, then $\|a\| = a(1)$ and all $a^{(i)} \in B^+$.
- If $a \in A^+$, then $\text{dist}(a, B) = \text{dist}(a, B^+)$.
- If $a_j \in A^+$, then $\text{dist}(\sum a_j, B) = \sum \text{dist}(a_j, B)$.

Lemma 7.2 *Suppose a, c are in A^+ and there exists b in B^+ such that $\|ac - b\| < \epsilon a(1)c(1)$. Provided $(k^2 - k)\epsilon < 1/2$, there exists j in $\{0, 1, 2, \ldots, k - 1\}$ together with a' and c' in B^+ such that $\|x^j a' - a\| < (k^2 - k)\epsilon a(1)$ and $\|x^{-j} c' - c\| < (k^2 - k)\epsilon c(1)$.*

Proof Write $a = \sum_{i=0}^{k-1} a_i x^i$ and $c = \sum_{i=0}^{k-1} c_i x^i$ where $a_i, c_i \in B^+$. For some real number $r > 0$, suppose that at least two of a_i have norm at least $ra(1)$. Select l such that $c_l(1) \geq c(1)/k$. Because all the terms have coefficients that are in B^+, the norm is just evaluation at $x = 1$ or $X = 1$. Of the, at least, two choices for i such that $a_i(1) > ra(1)$, at least one of them, say i', is not congruent modulo k to $-l$. Hence in the product ac, the term $x^{i'+l} a_{i'} c_l$ appears, and again, as all the coefficients are positive polynomials and $i' + l \not\equiv 0 \mod k$, it follows that $a_{i'}(1)c_l(1) < \epsilon a(1)c(1)$. So $ra(1)c_l(a)/k < \epsilon a(1)c(1)$. This entails $r < \epsilon k$.

If we select $r = \epsilon k$, we obtain a contradiction. Hence at most one of a_i, say a_j, has norm at least $\epsilon k a(1)$, so that the rest of the terms have norm strictly less. Then $\|a - x^j a_j\| < (k-1)k\epsilon a(1)$. Similarly, with c, we obtain j' such that $\|c - x^{j'} c_{j'}\| < k(k-1)\epsilon c(1)$. If $j + j \neq k, 0$, we see that $a_j(1)c_{j'}(1)$ contributes completely to the error (as previously), so $\epsilon a(1)c(1) > a_j(1)c_{j'}(1) > a(1)(1 - k(k-1)\epsilon)c(1)(1 - k(k-1)\epsilon)$. This will lead to a contradiction if $\epsilon < (1 - (k^2 - k)\epsilon)^2$. Then $\epsilon < 1/2(k^2 - k)$ is sufficient. Hence $j + j' = 0$ or k, and we are done. ∎

In the following, all the inequalities boil down to optimization on intervals or rectangles, and it always turns out that the minima are achieved on the boundary. Because the functions being minimized are at worst quadratic, the arithmetic is quite simple.

Lemma 7.3 *Suppose $a, a' \in A^+$, with $\max\{a^{(i)}(1)\} = \eta a(1)$ and $\max\{a'^{(i)}(1)\} = \mu a'(1)$. If $\mu, \eta > 1/2$, then $\mathrm{dist}(aa', B^+)/aa'(1) \geq \eta + \mu - 2\eta\mu > \max\{1 - \mu, 1 - \eta\}$; otherwise, $\mathrm{dist}(aa', B^+) > 1/2$.*

Proof Write $a = \sum x^i a^{(i)}$ and $a' = \sum x^i a'^{(i)}$, so that with $j \neq 0$, $(aa')^{(0)} = \sum a^{(i)} a'^{(-i)}$ and $(aa')^{(j)} = \sum a^{(i)} a'^{(j-i)}$. Hence

$$\mathrm{dist}(aa', B) = \sum_{j \neq 0} \sum_i a^{(i)}(1) a'^{(j-i)}(1) = a(1)a'(1) - \sum a^{(i)}(1) a'^{(-i)}(1)$$

We may assume that $a(1) = a'(1) = 1$ (dividing by the appropriate scalars). Fix $0 \leq \eta, \mu < 1$, and consider the following problem.

Minimize $1 - \sum_{i=0}^{k-1} X_i Y_i$ subject to $X_i, Y_i \geq 0$, $\sum X_i = \sum Y_i = 1$, and $X_j \leq \eta$, $Y_j \leq \mu$ for all j.

[Set $X_i = a^{(i)}(1)$ and $Y_i = a'^{(-i)}(1)$.] In the interior of the domain, the only critical point (after replacing X_0 and Y_0 by $1 - \sum_{i \neq 0} X_i$ and $1 - \sum_{i \neq 0} Y_i$, respectively)—$X_i = Y_j = 1/k$—yields the maximum. So the minimum must occur on the boundary.

Now assume $\eta, \mu > 1/2$. It easily follows that the minimum occurs only when for some i, $X_i = \eta$ and $Y_i = \mu$ (the same i), and then only when there exist some other $j \neq i$ such that $X_j = 1 - \eta$ and $Y_j = 1 - \mu$. So the minimum value is $1 - \eta\mu - (1 - \eta)(1 - \mu) = \eta + \mu - 2\eta\mu$.

Drop the condition $\mu, \eta > 1/2$ and write $1 = r\eta + \psi$ and $1 = s\mu + \phi$, where r, s are positive integers, $0 \leq \psi < \eta$, and $0 \leq \phi < \mu$; the minimum occurs when (up to relabelling), $X_1 = \eta, X_2 = \eta, \ldots, X_r = \eta, X_{r+1} = \psi, Y_1 = \mu = \cdots = Y_s$, and $Y_{s+1} = \phi$.

Hence if $r = s$, the minimum value is

$$1 - r\eta\mu - (1 - r\eta)(1 - r\mu) = r(\eta + \mu) - (r^2 + r)\eta\mu.$$

If $r \geq 2$, then the minimum value (obtained by minimizing with the constraints $1/(r+1) < \mu, \eta \leq 1/r$) is larger than $1 - 1/r$, which exceeds $1/2$.

If we assume $1/2 < \eta, \mu \leq 1$, then $\eta + \mu - 2\eta\mu > \max\{1 - \eta, 1 - \mu\}$.

If $r < s$, the minimum value is $1 - r\eta\mu > 1 - r/(r+1)(s+1) > 1/2$. Similarly, if $s > r$, the minimum value is at least $1/2$. $\blacksquare$

Next we deal with sums of products and elimination of small terms. Suppose a has two components, $a^{(0)}$ and $a^{(j)}$ such that each has mass at least some number $a(1)\delta$. We obtain an estimate for $\text{dist}(aa', B^+)$ when $a', a \in A^+$. The coefficient of x^t in aa' is $\sum a^{(i)} a'^{(t-j)}$. Suppose $t \neq 0$. Then, summing over all $t \neq 0$, the error is at least $a^{(0)}(1) \sum a'^{(t)}(1) + a^{(j)}(1) \sum_{u \neq -j} a'^{(u)}(1)$. In particular, $a'^{(0)}(1)$ appears in the second summand. Hence $\text{dist}(aa', B^+) \geq \delta a(1) a'(1)$. Thus we have the following.

Lemma 7.4   *Suppose $a, a' \in A^+$, and there exists $\delta > 0$ such that at least two components of a have mass at least $\delta a(1)$. Then $\text{dist}(aa', B^+) \geq \delta aa'(1)$.*

Proof of Proposition 7.1   For $a \in A^+$, define $\eta(a) = (a(1) - \max_i\{a^{(i)}\}(1))/a(1)$. If $\eta(a) < 1/2$, then the i_0 for which $a^{(i_0)}(1)$ is maximal is unique, and thus we can define $\pi(a)$ to be $x^{i_0} a^{(i_0)}$. It is helpful to keep track of the index i_0, so we define the affiliated map $\Pi(a) = (a^{(i_0)}, i_0)$ with values in $B^+ \times \mathbf{Z}_k$. The error resulting from replacing a by $\pi(a)$ is $\eta(a)a(1)$.

If the second largest of $\{a^{(i)}\}$ occurs at $i_1 \neq i_0$, and $a^{(i_1)} = \delta$, then the previous result yields $\text{dist}(aa', B^+) \geq \delta aa'(1)$. However, the second largest is at least as large as $(1 - \eta(a))/(k - 1)$. Hence we deduce that in general,

$$d(aa', B^+) \geq (1 - \eta(a))aa'(1)/(k - 1).$$

Fix a column of V, and pick one of its entries a such that $(1 - \eta(a)) > (k - 1)\sqrt{\epsilon}$. Suppose that a is the i-th entry. Let c_t vary over the entries of the j-th column of W. Then the sum $\sum ac_j$ appears as part of the sum of the entries of the i-th column of W. Now sum this over all the possible a in the i-th column (with the same condition on η), obtaining $e := \sum a_s c_j$, where a_s varies over the entries in the i-th column of V such that $(1 - \eta(a_s)) > (k - 1)\sqrt{\epsilon}$. Then $\text{dist}(e, B^+) \geq \sqrt{\epsilon}e(1) \sum c_j(1)$.

If we knew that $\sum c_j(1) = 1$ for all j, we would deduce $e(1) < \sqrt{\epsilon}$. Now do this simultaneously for all the columns of V. Since the norm is the maximum column sum, we deduce that simultaneously replacing all the entries satisfying the condition on their η by zero increases the norm of V by at most $\sqrt{\epsilon}$.

Hence at a cost of increasing ϵ to $\sqrt{\epsilon} + \epsilon$, we may suppose that all the nonzero entries of V satisfy $(1 - \eta(a)) \leq (k - 1)\sqrt{\epsilon}$. The next stage is to replace each entry by $\pi(a)$, the projection onto the X^0-component (out of the $X^0, \ldots, X^{k-1}$ components). The error along each column is just the sum of $a_i(1)(1 - \eta(a_i))$, which is bounded above

by $(k-1)\sqrt{\epsilon}$, since the column sum of $a_i(1)$ is 1. Hence at a cost of an additional $(k-1)\sqrt{\epsilon}$, we can assume that the entries of V are all of the form $x^{f(i,j)}b_{ij}$ where $f: \mathbf{N} \times \mathbf{N} \to \{0,1,2,\dots,k-1\}$. (The cumulative error on V is now $k\sqrt{\epsilon}+\epsilon$.)

Since WV is close to a matrix N with entries in B^+, we can, of course, do the same thing to W. This allows us to assume that the cumulative error resulting from replacing V and W by their primes is at most $k\sqrt{\epsilon}+\epsilon$). Let

$$\kappa = \max\{\|V'W' - M\|, \|W'V', N\|\},$$

so that $\kappa < 2k\sqrt{\epsilon} + 2\epsilon$. Write the entries of V (generically denoted a), as

$$v_{ij} = x^{f(i,j)}b_{ij}, \quad w_{jl} = x^{g(j,l)})b'_{jl},$$

where the b all lie in B^+ and the ranges of f and g are in $\{0,1,2,\dots,k-1\}$.

Now assume that W_i and V_i are defined, with appropriate summable errors $\|W_iV_i - M_i\| = \epsilon(i)$, etc., and we may assume (by remarks above) that $\sum \sqrt{\epsilon(i)} < \infty$. By the previous construction, we can assume the entries of W_i are all of the form $(W_i)_{jl} = x^{g_i(j,l)}b_{ijl}$ and $(v_i)_{jl} = x^{f_i(j,l)}b'_{ijl}$ where f_i, g_i have ranges in $\{0,1,2,\dots,k-1\}$. We may replace the f_i by f'_i where the range of each f'_i is in $\{0,-1,-2,\dots,-(k-1)\}$.

At this stage, we have a trick. Replace all the entries of all the V_i and W_i by removing the $x^{f_i(j,l)}$ and $x^{g_i(j,l)}$ terms, so the entries of the new W_i and V_i are, respectively, b_{ijl} and b'_{ijl}. Now it is almost trivial that the resulting errors in approximating M_i, respectively, N_i, by products $M_i - W'_iV'_i$ and $N_i - V'_{i+1}W'_i$ do not increase! The one-sided version (where we do not assume the existence of the N_i, but merely define $N_i := V'_{i+1}W'_i$) of this of course applies, and this will yield part (ii).

To see this, we look at an individual term in an individual entry: if $f_i(j,l) + g_i(l,m) \neq 0$ (because of our assumption on the values for f_i and g_j, we cannot obtain $f_i(j,l) + g_i(l,m) = \pm k$), then the corresponding term yields an error (in the original factorization) of $b_{ijl}b_{lm}(1)$ to that term (since it is part of the distance to B). Hence when we remove f_i and g, the error will be at most that, and since the errors in the distance to B are additive, we find that the error resulting from this contribution will be at most that from the original. On the other hand, if $f_i(j,l) + g_i(j,l) = 0$, the corresponding term will be exactly the same as that in the original factorization. So the outcome is that $\|W_iV_i - W'_iV'_i\| < 2k(\sqrt{\epsilon(i)} + 2\epsilon(i)$ and similarly (if necessary) for $\|V_iW_{i+1} - V'_iW'_{i+1}\|$. ∎

Remark In other words, if two B- (or $l^1(k\mathbf{Z})$-) dimension spaces become isomorphic on tensoring over B with A (or $l^1(\mathbf{Z})$), then they were isomorphic to begin with.

As a sample computation (the basis for the general odometer result), with an extra point, consider the sequence of matrices generated by T^2, where T is the 3-odometer. Here T is represented by the sequence of polynomials $(p_j := (1 + x^{3^j} + x^{2\cdot 3^j})/3)$. Then $\mathcal{B}(p_j) = \frac{1}{3}\begin{pmatrix} 1+X^{3^j} & X^{(3^j+1)/2} \\ X^{(3^j-1)/2} & 1+X^{3^j} \end{pmatrix}$. Products of these matrices can be calculated directly, but instead, let us consider the corresponding matrices over A (instead of B).

Replacing X by x^2, and applying conjugation with Δ, we obtain a sequence

$$\left(M_j = \frac{1}{3} \begin{pmatrix} 1 + (x^{3^j})^2 & x^{3^j} \\ x^{3^j} & 1 + (x^{3^j})^2 \end{pmatrix} \right).$$

This is an ergodic circulant sequence, and the eigenvalues of M_j are $(1 \pm x^{3^j} + x^{2 \cdot 3^j})/3$. It easily follows that (M_j) is not hollow (this is always true when the sequence is derived from ergodic T^2 in this fashion). We can see that (M_j) is AT (without referring to $\mathcal{B}(p_j)$) by observing that $\prod_{j=0}^{N-1} M_j = 3^{-N} \begin{pmatrix} P_0 & P_1 \\ P_1 & P_0 \end{pmatrix}$, where $P_0 = \sum_{\text{even } i < 3^N} x^i$ and $P_1 = \sum_{\text{odd } i < 3^N} x^i$, and since $\|x^{\pm 1} P_1 - P_0\| = 2$, we can conclude that (M_j) is AT. Now the obvious approximate factorization yields that $(M_j) \cong ((1 + x^{2 \cdot 3^j} + x^{4 \cdot 3^j})/3)$, exactly as we would have obtained from $(\mathcal{B}(p_j)) \cong ((1 + X^{3^j} + X^{2 \cdot 3^j})/3)$ by replacing X by x^2.

Note however, that (M_j) is not isomorphic to the 3-odometer (we can apply mass cancellation invariants). Its dimension group corresponds to the odometer with supernatural number $2 \cdot 3^\infty$, which although Kakutani equivalent to the 3-odometer, is not conjugate to it. (We must remember that isomorphisms of $(\mathcal{B}(p_j))$ are implemented over $B = \mathbf{R}[X^{\pm 1}]$, while isomorphisms of (M_j) are implemented over A.)

As a slightly different example, consider what happens when T is represented by $(p_j = (1 + x^{g(j)})/2)$ as in Proposition 3.7. Then

$$\mathcal{B}(p_j) = \begin{cases} \frac{1}{2}((1 + X^{g(j)/2})I) & \text{if } g(j) \text{ is even,} \\ \frac{1}{2}(I + X^{(g(j)-1)/2} \begin{pmatrix} 0 & X \\ 1 & 0 \end{pmatrix}) & \text{if } g(j) \text{ is odd.} \end{cases}$$

It follows that T^2 is ergodic if and only if infinitely many $g(j)$ are odd.

If, for example, *all* the $g(j)$ are odd, then $\Delta \mathcal{B}(p_j)\Delta^{-1} = M_j = \frac{1}{2}\begin{pmatrix} 1 & x^{g(j)} \\ x^{g(j)} & 1 \end{pmatrix}$, so that if we assume the conditions on $g(j)$ of Proposition 3.7, then $(M(j))$ is AT, and thus so is $(\mathcal{B}(p_j))$, the latter by Proposition 7.1.

If instead $g(j)$ is the j-th Fibonacci number (counting 2 as the third one), then $g(3j)$ is even. When we telescope in threes, we obtain

$$\left(M_{3j}M_{3j+1}M_{3j+2} = \frac{1}{8}(1 + x^{g(3j)})\begin{pmatrix} 1 & x^{g(3j+1)} \\ x^{g(3j+1)} & 1 \end{pmatrix}\begin{pmatrix} 1 & x^{g(3j+2)} \\ x^{g(3j+2)} & 1 \end{pmatrix} \right).$$

This is not hollow (in contrast to the corresponding sequence considered in Lemma 3.6), but is AT (this is true much more generally). Therefore T^2 is AT.

We can ask whether T and T^2 are conjugate (when the latter is ergodic), that is, whether their corresponding dimension spaces are isomorphic. This is trickier, especially since it happens to be true for odometers, and we expect it will not be for generic AT (and other) actions.

If in the preceding example, we set $g(j) = 5^j$ (so none of the previous results apply), then the corresponding $\left(M_j = \frac{1}{2}\begin{pmatrix} 1 & x^{5^j} \\ x^{5^j} & 1 \end{pmatrix} \right)$ is obviously ergodic, but is known [GH] *not* to be AT. Therefore, by Proposition 7.1, T^2 is not AT, but is ergodic.

Now we have a simple but useful result on non-hollowness of sequences of the form $(p_i(xP))$ that arise from conjugation of $(\mathcal{B}(p_i))$ by Δ.

If S and T are ergodic transformations represented, respectively, by (M_j) and (N_j), then we define the transformation $S \otimes T$ to be the transformation determined by

the dimension space obtained from $(M_j \otimes N_j)$. When we assume (as we have throughout) that there is a unique invariant measure for each of S and T, then it easily follows that the corresponding trace on the dimension space of $(M_j \otimes N_j)$ is ergodic; hence, at least when there is an invariant measure for each, $S \otimes T$ is ergodic. Unfortunately, there does not seem to be a dynamical characterization of $S \otimes T$, even when $S = T$. In the situation of topological dynamics, at least for Vershik's adic transformations, $S \otimes T$ has been given a dynamical meaning [BeH, Appendix A], but it is not entirely satisfactory.

Lemma 7.5 *Suppose that the AT transformation T is represented by the sequence of polynomials (p_i), and suppose that T^k is ergodic. Let P denote the standard cyclic permutation matrix of size k. Then for all positive integers l, the ergodic circulant sequence (of size k matrices) $((p_i(xP))^l)$ is not hollow.*

Proof Since T^k is ergodic, the sequence $(\mathcal{B}(p_i))$ of matrices over B^+ is ergodic. Conjugating each of the terms $\mathcal{B}(p_i) = p_i(Q)$ with Δ and replacing X by x^k, we obtain a new sequence of matrices from A given by $(p_i(xP))$. Ergodicity of the former sequence implies that of the latter (since the criteria for ergodicity just depend on the real matrices obtained by assigning x and X to 1).

With $H = \mathbf{Z}_n$, the dual group runs over the k characters, χ_j, sending the generator to ξ^j where $\xi = \exp 2\pi\sqrt{-1}/k$; index the corresponding eigenvalue functions λ_j, $j \in \mathbf{Z}_k$. Now $\lambda_j(p(xP)) = p(x\xi^j)$, and thus $\lambda_j(\prod p_i(xP)) = p(x\xi^j)$ where $p = \prod p_i$. If we write $p = \sum c_t x^t$ with $\sum c_t = 1$ and $c_t \geq 0$, then $p(x\xi^j) = \sum c_t \xi^{tj} x^t$, so that $\|p(x\xi^j)\| = \sum |c_t \xi^{tj}| = \sum c_t = 1$. Thus $\lambda_j(p(xP)) = \prod \lambda_j(p_i(xP))$ has norm one. Hence $(p_i(xP))$ is not hollow.

Now $T^k \otimes T^k \otimes \cdots \otimes T^k$ (with l factors) is ergodic (tensor products of ergodic transformations are ergodic, unlike the situation for cartesian products), and is the k-th power of $T \otimes T \otimes \cdots \otimes T$, which in turn is represented by (p_i^l). Hence the result of the previous paragraph applies to $(p_i^l(xP) = (p_i(xP))^l)$. ∎

Combined with earlier results, this says that $(p_i(xP))$ arising in this construction is not isomorphic to the original sequence (p_i) (although it can be isomorphic to $(p_i(x^k))$, as occurs for odometers).

The following is an easy consequence of earlier results. However, its significance is reduced by the fact that we really do not know what $T \otimes T$ is dynamically.

Proposition 7.6 *Suppose T is AT and for a positive integer k, T^k is ergodic. Then $T^k \otimes T^k$ is AT.*

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Mathematics Department, University of Ottawa, Ottawa, ON K1N 6N5
Email: dehsg@uottawa.ca