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AN INDUCTION THEOREM FOR REARRANGEMENTS

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Introduction. In this paper, an induction theorem for rearrangements involving n-tuples in R^n is proved, showing that a certain proposition regarding a pair of n-tuples related by the weak spectral order $\ll$ is true for any integer $n \geqq 2$ if and only if it is true for $n = 2$. This theorem contains as particular cases a well-known theorem of Hardy–Littlewood–Pólya [**4**, Lemma 2, p. 47], a theorem of Pólya [**8**], a theorem of Rado [**9**, pp. 1–2], two theorems of Mirsky [**6**, Theorem 2, p. 232; **7**, Theorem 4, p. 90], a result given in [**1**, Corollary 2.4, p. 1333] and also [**2**, Proposition 2.1, p. 439]. With this theorem, we give conditions for equality to hold in Mirsky's [**6**, Theorem 2, p. 232], as well as Pólya's [**8**] inequality. Moreover, we also obtain some Hardy–Littlewood–Pólya-type, Muirhead-type, Marshall–Proschan-type and Rado-type rearrangement theorems.

1. Preliminaries. For any n-tuple $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in R^n$, we denote by $\mathbf{x}^* = (x_1^*, x_2^*, \ldots, x_n^*)$ the n-tuple whose components are those of $\mathbf{x}$ arranged in non-increasing order of magnitude. If $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in R^n$ and $\mathbf{b} = (b_1, b_2, \ldots, b_n) \in R^n$, then we say that the *weak spectral inequality* $\mathbf{a} \ll \mathbf{b}$ holds whenever

$$(1.1)\qquad \sum_{i=1}^{k} a_i^* \leqq \sum_{i=1}^{k} b_i^*$$

for $1 \leqq k \leqq n$, and the *strong spectral inequality* $\mathbf{a} \prec \mathbf{b}$ holds if $\mathbf{a} \ll \mathbf{b}$ and if there is equality in (1.1) for $k = n$. Moreover, we write $\mathbf{a} \sim \mathbf{b}$ whenever the components of $\mathbf{a}$ form a permutation of those of $\mathbf{b}$ in which case $\mathbf{a}$ is called a *rearrangement* of $\mathbf{b}$.

As in [**1**] and [**2**], the spectral inequality $\mathbf{a} \prec \mathbf{b}$ (respectively $\mathbf{a} \ll \mathbf{b}$) is said to be *strictly strong* (respectively *strictly weak*) if $\mathbf{a} \nsim \mathbf{b}$ (respectively if the inequality (1.1) is strict for $k = n$).

If $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in R^n$ is any n-tuple, and if $x \in R$ is any number, we often let $(x, \mathbf{x}) = (x, x_1, x_2, \ldots, x_n)$ denote the $n + 1$-tuple obtained by adjoining x to $\mathbf{x}$. If $\mathbf{a}, \mathbf{b} \in R^n$, then it is not hard to see that $\mathbf{a} \ll \mathbf{b}$ if and only if

$$(1.2)\qquad (c, \mathbf{a}) \ll (c, \mathbf{b})$$

for any number $c \in R$. More generally, if $\mathbf{x}_i \in R^n$, $i = 1, 2, \ldots, m$, are n-tuples, let $(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m)$ be the mn-tuple whose components are those of $\mathbf{x}_1, \mathbf{x}_2, \ldots,$

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$\mathbf{x}_m$. Then, if $\mathbf{a}_i \ll \mathbf{b}_i$, $i = 1, 2, \ldots, m$, it is easy to see that

(1.3) $\quad (\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m) \ll (\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_m).$

Inequalities between vectors are defined componentwise and such an inequality is said to be *strict* if at least one of the inequalities between the corresponding components is strict.

In the sequel, an n-tuple is sometimes taken to mean a column vector. We refer to [**1**, Sections 1 and 3] for an extension of the above notions to measurable functions.

2. An induction theorem. The following theorem gives the main result of this paper.

THEOREM 2.1. *For any integer* $n \geqq 2$, *let* $P[\mathbf{a}, \mathbf{b}, n]$ *be a proposition concerning a pair of* n-*tuples* $\mathbf{a}, \mathbf{b} \in R^n$ *satisfying* $\mathbf{a} \ll \mathbf{b}$ *such that*

(i) $P[\mathbf{a}, \mathbf{b}, n]$ *is compatible with the* (*vector*) *lattice structure of* R^n *with respect to the ordinary partial order* $\leqq$, *i.e.*, $P[\mathbf{a}, \mathbf{b}, n]$ *is true whenever* $\mathbf{a} \leqq \mathbf{b}$;

(ii) $P[\mathbf{a}, \mathbf{b}, n]$ *is compatible with vector extensions, i.e.*, $P[(c, \mathbf{a}), (c, \mathbf{b}), n + 1]$ *is true for any* $c \in R$ *whenever* $P[\mathbf{a}, \mathbf{b}, n]$ *is true;*

(iii) $P[\mathbf{a}, \mathbf{b}, n]$ *is rearrangement invariant, i.e.*, $P[\mathbf{a}', \mathbf{b}', n]$ *is true for all rearrangements* $\mathbf{a}'$ *of* $\mathbf{a}$ *and* $\mathbf{b}'$ *of* $\mathbf{b}$ *whenever* $P[\mathbf{a}, \mathbf{b}, n]$ *is true;*

(iv) $P[\mathbf{a}, \mathbf{b}, n]$ *is transitive, i.e.*, $P[\mathbf{a}, \mathbf{b}, n)$ *is true whenever both* $P[\mathbf{a}, \mathbf{c}, n]$ *and* $P[\mathbf{c}, \mathbf{b}, n]$ *are true, where* $\mathbf{c}$ *is any* n-*tuple satisfying* $\mathbf{a} \ll \mathbf{c} \prec \mathbf{b}$ *or, simply* $\mathbf{a} \ll \mathbf{c} \ll \mathbf{b}$.

Then $P[\mathbf{a}, \mathbf{b}, n]$ *is true for any* $n \geqq 2$ *if and only if* $P[\mathbf{a}, \mathbf{b}, 2]$ *is true.*

Proof. The necessity of the condition is clear.

To prove its sufficiency, we first note that $\mathbf{a}^* \leqq \mathbf{b}^*$ if $a_1^* \leqq b_n^*$, in which case $P[\mathbf{a}^*, \mathbf{b}^*, n]$ or $P[\mathbf{a}, \mathbf{b}, n]$ is true, by virtue of hypotheses (i) and (iii). Thus, we assume that $a_1^* > b_n^*$. Next, suppose that $P[\mathbf{a}, \mathbf{b}, 2]$ is true, that $n > 2$ and that $P[\mathbf{a}, \mathbf{b}, m]$ is true for any $m \leqq n - 1$. Let $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ and $\mathbf{b} = (b_1, b_2, \ldots, b_n)$.

If $a_1^* = b_1^*$, then $\mathbf{a} \ll \mathbf{b}$ implies that $(a_2^*, a_3^*, \ldots, a_n^*) \ll (b_2^*, b_3^*, \ldots, b_n^*)$, by (1.2). But $P[(a_2^*, a_3^*, \ldots, a_n^*), (b_2^*, b_3^*, \ldots, b_n^*), n - 1]$ is true, by the induction hypotheses, and so $P[(a_1^*, a_2^*, \ldots, a_n^*), (b_1^*, b_2^*, \ldots, b_n^*), n]$ is true, by hypothesis (ii), that is, $P[\mathbf{a}, \mathbf{b}, n]$ is true, by hypothesis (iii).

If $b_n^* < a_1^* < b_1^*$, then it is easily seen that there exists a smallest integer i, $1 < i \leqq n$, and an n-tuple $\mathbf{c} = (a_1^*, b_1^*, b_2^*, \ldots, b_{i-2}^*, b_{i-1}^* + b_i^* - a_1^*, b_{i+1}^*, \ldots, b_n^*)$ such that $b_i^* \leqq a_1^* < b_{i-1}^*$ and $\mathbf{a} \ll \mathbf{c} \prec \mathbf{b}$. But, by (1.2), $\mathbf{a} \ll \mathbf{c}$ and $\mathbf{c} \prec \mathbf{b}$ are respectively equivalent to $(a_2, a_3, \ldots, a_n) \ll (b_1, b_2, \ldots, b_{i-2}, b_{i-1} + b_i - a_1, b_{i+1}, \ldots, b_n)$ and $(a_1, b_{i-1} + b_i - a_1) \prec (b_{i-1}, b_i)$, provided that $a_1 \geqq a_2 \geqq \ldots \geqq a_n$ and $b_1 \geqq b_2 \geqq \ldots \geqq b_n$, it is, therefore, clear that $P[\mathbf{a}, \mathbf{c}, n]$ is true in view of hypotheses (ii) and (iii). Moreover, by virtue of the induction hypotheses and hypotheses (ii) and (iii), $P[\mathbf{c}, \mathbf{b}, n]$ is also true. Thus $P[\mathbf{a}, \mathbf{b}, n]$ is true, by hypothesis (iv).

The following lemma will be used repeatedly in what is to follow.

LEMMA 2.2. *If* $\mathbf{a} = (a_1, a_2) \ll (b_1, b_2) = \mathbf{b}$, *then there exists a number* r *such that* $0 \leqq r \leqq 1$ and

$$(2.1)\quad \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \leqq \left[r\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + (1-r)\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right] \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}.$$

Moreover, the spectral inequality $\mathbf{a} \ll \mathbf{b}$ *is strictly weak if and only if the inequality* (2.1) *is strict.*

Proof. The result is easily seen to hold by considering the cases that $a_1^* \leqq b_2^*$ and that $a_1^* > b_2^*$ separately.

As a corollary to Theorem 2.1, the following is an extension of a theorem of Hardy–Littlewood–Pólya [**4**, Lemma 2, p. 47].

THEOREM 2.3. *If* $\mathbf{a}$ *and* $\mathbf{b}$ *are (column) vectors in* R^n, *then* $\mathbf{a} \ll \mathbf{b}$ *if and only if there exist a finite number of transformations, say* $T_1, T_2, \ldots, T_m$, *of the form* $rI + (1-r)P$, *where* $0 \leqq r \leqq 1$, I *is the* $n \times n$ *identity matrix and* P *is an* $n \times n$ *"transposition" matrix (i.e., a permutation matrix obtained by permuting two rows of* I*) such that*

$$(2.2)\quad \mathbf{a} \leqq T_1 T_2 \ldots T_m \mathbf{b}.$$

Moreover, the spectral inequality $\mathbf{a} \ll \mathbf{b}$ *is strictly weak if and only if the inequality* (2.2) *is strict.*

Proof. For the first part of the theorem, let $P[\mathbf{a}, \mathbf{b}, n]$ be the proposition as asserted. Then (i) and (iv) of Theorem 2.1 are obvious, whereas (iii) follows from the fact that a permutation matrix is a product of transposition matrices. To prove (ii), suppose that $P[\mathbf{a}, \mathbf{b}, n]$ is true, then

$$\mathbf{a} \leqq \prod_{i=1}^{m} [r_i I + (1 - r_i)P_i]\mathbf{b},$$

where $m \geqq 1$ is some integer, $0 \leqq r_i \leqq 1$, $i = 1, 2, \ldots, m$, and $P_1, P_2, \ldots, P_m$ are transposition matrices. As it is easy to see that

$$\begin{pmatrix} c \\ \mathbf{a} \end{pmatrix} \leqq \prod_{i=1}^{m} \left(r_i \begin{bmatrix} 1 & 0 & \ldots & 0 \\ 0 & & & \\ \cdot & & I & \\ \cdot & & & \\ \cdot & & & \\ 0 & & & \end{bmatrix} + (1 - r_i) \begin{bmatrix} 1 & 0 & \ldots & 0 \\ 0 & & & \\ \cdot & & P_i & \\ \cdot & & & \\ \cdot & & & \\ 0 & & & \end{bmatrix} \right) \begin{pmatrix} c \\ \mathbf{b} \end{pmatrix}$$

for any $c \in R$, we infer that $P[(c, \mathbf{a}), (c, \mathbf{b}), n+1]$ is true. Since $P[\mathbf{a}, \mathbf{b}, 2]$ is precisely Lemma 2.2, we conclude that $P[\mathbf{a}, \mathbf{b}, n]$ is true.

The last assertion is easy.

Remark. Theorem 2.3 extends [**4**, Lemma 2, p. 47] since it is easily seen that $T_1T_2 \ldots T_m\mathbf{b} \prec \mathbf{b}$ and that $\mathbf{a} \prec \mathbf{b}$ and $\mathbf{a} \leqq \mathbf{x} \prec \mathbf{b}$ imply $\mathbf{x} = \mathbf{a}$.

The following Muirhead-type theorem was noted earlier in [**2**, p. 443]. It can now be obtained as a consequence of Theorem 2.1.

THEOREM 2.4. *If* $\mathbf{a} = (a_1, a_2, \ldots, a_n)$, $\mathbf{b} = (b_1, b_2, \ldots, b_n)$ *and* $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ *are such that* $\mathbf{a} \ll \mathbf{b}$ *and* $x_i > 1$, $i = 1, 2, \ldots, n$, *then*

$$(2.3) \quad \sum! x_1^{a_1} x_2^{a_2} \ldots x_n^{a_n} \leqq \sum! x_1^{b_1} x_2^{b_2} \ldots x_n^{b_n}$$

where $\sum!$ *denotes summation over all the (distinct) permutations of* $(x_1, x_2, \ldots, x_n)$. *Moreover, equality holds in* (2.3) *if and only if either* $\mathbf{a} \sim \mathbf{b}$ *or both* $\mathbf{a} \prec \mathbf{b}$ *and* $x_1 = x_2 = \ldots = x_n$.

Proof. Clearly the hypotheses of Theorem 2.1 are satisfied.

THEOREM 2.5 (Mirsky [**7**, Theorem 4, p. 90]). *If* $\mathbf{a}, \mathbf{b} \in R^n$ *are (column) vectors, then* $\mathbf{a} \ll \mathbf{b}$ *if and only if there exists a doubly stochastic matrix* A *such that* $\mathbf{a} \leqq A\mathbf{b}$.

Proof. The sufficiency of the condition follows from the fact that $A\mathbf{b} \prec \mathbf{b}$, which can be proved as in [**4**, p. 49]. Conversely, its necessity is a direct consequence of Theorem 2.1 and Lemma 2.2.

Remark. In [**7**, Theorem 4, p. 90], the above theorem of Mirsky is obtained as a direct consequence of a theorem of Hardy–Littlewood–Pólya [**4**, Theorem 46, p. 49]. On the other hand, the Hardy–Littlewood–Pólya theorem can also be derived from Mirsky's theorem as a particular case, since it is easily seen that $\mathbf{a} \prec \mathbf{b}$ and $\mathbf{a} \leqq A\mathbf{b} \prec \mathbf{b}$ imply $\mathbf{a} = A\mathbf{b}$.

The following theorem of Pólya also fits nicely into the language of Theorem 2.1.

THEOREM 2.6 (Pólya [**8**]). *If* $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in R^n$ *and* $\mathbf{b} = (b_1, b_2, \ldots, b_n) \in R^n$, *then* $\mathbf{a} \ll \mathbf{b}$ *if and only if*

$$(2.4) \quad \sum_{i=1}^{n} \Phi(a_i) \leqq \sum_{i=1}^{n} \Phi(b_i)$$

for all increasing convex functions $\Phi : R \to R$.

If $\mathbf{a} \ll \mathbf{b}$ *and if* Φ *is strictly increasing and convex, then* $\mathbf{a} \prec \mathbf{b}$ *whenever equality holds in* (2.4).

If $\mathbf{a} \ll \mathbf{b}$ *and if* Φ *is strictly convex and increasing, then equality holds in* (2.4) *if and only if* $\mathbf{a} \sim \mathbf{b}$.

Proof. The proof is obvious by virtue of Theorem 2.1 and Lemma 2.2.

Remark. The conditions concerning equalities in Theorem 2.6 were not given in [**8**].

The following theorem is equivalent to Theorem 2.6 (in the sense that one can be obtained from the other and that each can be established independently of the other).

THEOREM 2.7 [**1**, Corollary 2.4, p. 1333]. *If* $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ *and* $\mathbf{b} = (b_1, b_2, \ldots, b_n)$ *are in* R^n, *then* $\mathbf{a} \ll \mathbf{b}$ *if and only if*

$$(2.5) \quad (\Phi(a_1), \ldots, \Phi(a_n)) \ll (\Phi(b_1), \ldots, \Phi(b_n))$$

for all increasing convex functions $\Phi : R \to R$.

If the spectral inequality $\mathbf{a} \ll \mathbf{b}$ *is strictly weak and if* Φ *is strictly increasing and convex, then the spectral inequality* (2.5) *is also strictly weak.*

If $\mathbf{a} \ll \mathbf{b}$ *and if* Φ *is strictly convex and increasing, then the spectral inequality* (2.5) *is strong if and only if* $\mathbf{a} \sim \mathbf{b}$, *in which case* $(\Phi(a_1), \Phi(a_2), \ldots, \Phi(a_n)) \sim (\Phi(b_1), \Phi(b_2), \ldots, \Phi(b_n))$.

Proof. The case that $n = 2$ follows directly from Lemma 2.2 while the general result is an immediate consequence of this and Theorem 2.1.

Theorem 2.1 also contains another theorem of Mirsky as a direct consequence and gives conditions for equality to hold in his inequality.

THEOREM 2.8 (Mirsky [**6**, Theorem 2, p. 232]). *If* $\mathbf{a}, \mathbf{b} \in R^n$, *then* $\mathbf{a} \ll \mathbf{b}$ *if and only if* $\Phi(\mathbf{a}) \leqq \Phi(\mathbf{b})$ *for all increasing, convex and symmetric functions* $\Phi : R^n \to R$.

If $\mathbf{a} \ll \mathbf{b}$ *and if* Φ *is strictly increasing, convex and symmetric, then* $\mathbf{a} \prec \mathbf{b}$ *whenever* $\Phi(\mathbf{a}) = \Phi(\mathbf{b})$.

If $\mathbf{a} \ll \mathbf{b}$ *and if* Φ *is strictly convex, increasing and symmetric, then* $\Phi(\mathbf{a}) = \Phi(\mathbf{b})$ *if and only if* $\mathbf{a} \sim \mathbf{b}$.

Proof. The sufficiency of the condition for the first part of the theorem follows as in [**7**, p. 22].

To prove its necessity, we need only verify hypothesis (ii) of Theorem 2.1. But this is a consequence of the fact a convex increasing function (of $n + 1$ variables) remains convex increasing if one variable is kept fixed.

The rest is easy.

The following theorem which generalizes a theorem of Rado [**9**, pp. 1–2] yields another consequence of Theorem 2.1.

THEOREM 2.9. *For any n-tuple* $\mathbf{b} \in R^n$, *let* $\mathscr{H}(\mathbf{b})$ *denote the convex hull of the set of all rearrangements of* $\mathbf{b}$. *Then an n-tuple* $\mathbf{a} \in R^n$ *satisfies* $\mathbf{a} \ll \mathbf{b}$ *if and only if there exists an n-tuple* $\mathbf{c} \in \mathscr{H}(\mathbf{b})$ *such that* $\mathbf{a} \leqq \mathbf{c}$.

Proof. The sufficiency of the condition follows from the fact that $\mathbf{c} \in \mathscr{H}(\mathbf{b})$ implies $\mathbf{c} \prec \mathbf{b}$, which is proved in [**7**, p. 2].

To establish its necessity, we need only note that hypothesis (iv) of Theorem 2.1 follows from the fact that a convex combination of some convex combinations of rearrangements of $\mathbf{b}$ is again a convex combination of rearrangements of $\mathbf{b}$.

Remark. Theorem 2.9 does generalize Rado's theorem [**9**, pp. 1–2] since $\mathbf{a} \prec \mathbf{b}$ and $\mathbf{a} \leqq \mathbf{c} \prec \mathbf{b}$ imply $\mathbf{a} = \mathbf{c}$.

We shall now glean from Theorem 2.1 a Marshall–Proschan-type rearrangement theorem [**5**, p. 87].

THEOREM 2.10. *If* $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ *and* $\mathbf{b} = (b_1, b_2, \ldots, b_n)$ *are in* R^n *and if* $X_1, X_2, \ldots, X_n$ *are interchangeable nonnegative random variables, then* $\mathbf{a} \ll \mathbf{b}$ *if and only if*

$$\Phi(a_1X_1, \ldots, a_nX_n) \ll \Phi(b_1X_1, \ldots, b_nX_n) \tag{2.6}$$

for all increasing, convex and symmetric functions $\Phi : R^n \to R$, *provided the composite random variables have finite expectations.*

Suppose $\Phi : R^n \to R$ *is strictly increasing, convex and symmetric. If the spectral inequality* $\mathbf{a} \ll \mathbf{b}$ *is strictly weak and if* $P\{X_i > 0\} > 0$ *for at least one* i, $1 \leqq i \leqq n$, *then the spectral inequality* (2.6) *is strictly weak.*

Moreover, if $\mathbf{a} \ll \mathbf{b}$ *and if* Φ *is increasing, strictly convex and symmetric, then the spectral inequality* (2.6) *is strong if and only if either* $\mathbf{a} \sim \mathbf{b}$ *or* $X_i = 0$ *almost surely,* $i = 1, 2, \ldots, n$.

Proof. For the first part of the theorem, the sufficiency of the condition follows as in [**6**, Theorem 2, p. 232] by putting $X_i = 1$, $i = 1, 2, \ldots, n$.

To prove its necessity, we need only establish the case that $n = 2$, and the general result then follows immediately from Theorem 2.1. Suppose $(a_1, a_2) \ll (b_1, b_2)$. Then, by Lemma 2.2, there exists a number r such that $0 \leqq r \leqq 1$ and $(a_1, a_2) \leqq r(b_1, b_2) + (1 - r)(b_2, b_1)$. Thus, $\Phi(a_1X_1, a_2X_2) \leqq r\Phi(b_1X_1, b_2X_2) + (1 - r)\Phi(b_2X_1, b_1X_2) \prec \Phi(b_1X_1, b_2X_2)$ since the symmetry of Φ and the interchangeability of X_1 and X_2 imply $\Phi(b_2X_1, b_1X_2) \sim \Phi(b_1X_1, b_2X_2)$.

For the second part of the theorem, if $(a_1, a_2) \ll (b_1, b_2)$, where $a_1 + a_2 < b_1 + b_2$, and if either $P\{X_1 > 0\} > 0$ or $P(\{X_2 > 0\} > 0$, then it follows from Lemma 2.2 that

$$(a_1X_{\pi(1)}, a_2X_{\pi(2)}) < r(b_1X_{\pi(1)}, b_2X_{\pi(2)}) + (1 - r)(b_2X_{\pi(1)}, b_2X_{\pi(2)})$$

for some $0 \leqq r \leqq 1$ and for some permutation π of the integers 1, 2. Thus, if $\Phi : R^2 \to R$ is strictly increasing, convex and symmetric, then it is clear that $E[\Phi(a_1X_1, a_2X_2)] < E[\Phi(b_1X_1, b_1X_2)]$.

The rest is similar.

The following theorem gives a further application of Theorem 2.1 to a result established earlier in [**2**].

THEOREM 2.11 [**2**, Proposition 2.1, p. 439]. *Let* A *be a matrix whose rows are the distinct permutations of a nonnegative* n*-tuple* $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in R^n$. *If* $\mathbf{a}, \mathbf{b} \in R^n$ *are (column) vectors such that* $\mathbf{a} \ll \mathbf{b}$, *then* $A\mathbf{a} \ll A\mathbf{b}$.

Proof. Clearly, we need only show that the asserted proposition satisfies (ii) of Theorem 2.1. Let $c \in R$ and $x_{n+1} \geqq 0$ be given. Let B be the matrix

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whose rows are the distinct permutations of $(x_1, x_2, \ldots, x_{n+1})$ and let A_i be the matrix whose rows are the distinct permutations of the n-tuple obtained by deleting x_i from $(x_1, x_2, \ldots, x_{n+1})$, $i = 1, 2, \ldots, n+1$. Then $cx_i + A_i\mathbf{a} \ll cx_i + A_i\mathbf{b}$, $i = 1, 2, \ldots, n+1$, and so $B\begin{pmatrix} c \\ \mathbf{a} \end{pmatrix} \ll B\begin{pmatrix} c \\ \mathbf{b} \end{pmatrix}$, by (1.3).

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