Bull. Aust. Math. Soc. (First published online 2023), page 1 of 3*
doi:10.1017/S0004972723000813
*Provisional—final page numbers to be inserted when paper edition is published

A COUNTEREXAMPLE TO A RESULT OF JABERI AND MAHMOODI

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(Received 25 June 2023; accepted 9 July 2023)

Abstract

We show that $\ell^1(\mathbb{N}_\wedge)$ is φ-amenable for each multiplicative linear functional $\varphi : \ell^1(\mathbb{N}_\wedge) \to \mathbb{C}$. This is a counterexample to the final corollary of Jaberi and Mahmoodi ['On φ-amenability of dual Banach algebras', *Bull. Aust. Math. Soc.* **105** (2022), 303–313] and shows that the final theorem in that paper is not valid.

2020 *Mathematics subject classification*: primary 46H25; secondary 43A07.

Keywords and phrases: φ-amenability, Banach algebra.

1. Introduction and preliminaries

The cohomological notion of amenability was introduced and studied in the pioneering work of Johnson [5]. A Banach algebra $\mathcal{A}$ is amenable if every continuous derivation from $\mathcal{A}$ into a dual Banach $\mathcal{A}$-bimodule E^* is inner. A modification of amenability depending on multiplicative linear functionals was introduced and studied by Kaniuth *et al.* [6]. A Banach algebra $\mathcal{A}$ is called φ-amenable if there exists an element m in $\mathcal{A}^{**}$ such that $m(\varphi) = 1$ and $m(f \cdot a) = \varphi(a)m(f)$ for every $a \in \mathcal{A}$ and $f \in \mathcal{A}^*$, where φ is a multiplicative linear functional on $\mathcal{A}$. For a locally compact group G, the Fourier algebra $\mathcal{A}(G)$ is always φ-amenable. Moreover, the Segal algebra $S^1(G)$ is φ-amenable if and only if G is amenable (see [1, 6]).

Jaberi and Mahmoodi [4] introduced the new concept of φ-injectivity for the category of dual Banach algebras, where φ is a wk^*-continuous multiplicative linear functional on $\mathcal{A}$. A dual Banach algebra $\mathcal{A}$ is φ-injective if whenever $\pi : \mathcal{A} \to \mathcal{L}(E)$ is a wk^*-continuous unital representation on a reflexive Banach space E, then there is a projection $Q : \mathcal{L}(E) \to \pi(\mathcal{A})^\varphi$ such that $Q(STU) = SQ(T)U$ for $S, U \in \pi(\mathcal{A})^c$ and $T \in \mathcal{L}(E)$, where $\pi(\mathcal{A})^\varphi = \{T \in \mathcal{L}(E) : \pi(a)T = \varphi(a)T \quad (a \in \mathcal{A})\}$. They proved that φ-injectivity is equivalent to φ-amenability [4, Theorem 3.6].

There is an important category of dual Banach algebras, called enveloping dual Banach algebras. Let $\mathcal{A}$ be a Banach algebra and let E be a Banach $\mathcal{A}$-bimodule.

An element $x \in E$ is called weakly almost periodic if the module maps $\mathcal{A} \to E$ given by $a \mapsto a \cdot x$ and $a \mapsto x \cdot a$ are weakly compact. The set of all weakly almost periodic elements of E is denoted by $WAP(E)$ [7, Definition 4.1]. Runde observed that $WAP(\mathcal{A}^*)^*$ is a canonical dual Banach algebra associated to an arbitrary Banach algebra $\mathcal{A}$ [7, Theorem 4.10]. By means of the new notion of φ-injectivity, Jaberi and Mahmoodi investigated φ-amenability of the enveloping dual Banach algebra $WAP(\mathcal{A}^*)^*$ [4, Theorem 4.8]. In a short final section of the paper, they claimed that $WAP(\ell^1(\mathbb{N}_\wedge)^*)^*$ is not $\tilde{\varphi}$-amenable, where $\tilde{\varphi}$ is the unique extension of the augmentation character φ on the semigroup algebra $\ell^1(\mathbb{N}_\wedge)$ [4, Theorem 5.4]. From this result, they concluded that $\ell^1(\mathbb{N}_\wedge)$ is not φ-amenable, where φ is the augmentation character [4, Corollary 5.5].

On the contrary, we show that $\ell^1(\mathbb{N}_\wedge)$ is φ-amenable for each multiplicative linear functional $\varphi : \ell^1(\mathbb{N}_\wedge) \to \mathbb{C}$ and comment on the reason for this counterexample to the result stated in [4].

2. φ-amenability of $\ell^1(\mathbb{N}_\wedge)$

Let $S = \mathbb{N}$. With the semigroup product $m \wedge n = \min\{m, n\}$, for $m, n \in S$, the set S becomes a semigroup. It is known that $\Delta(\ell^1(S))$ consists of all the functions $\varphi_n : \ell^1(S) \to \mathbb{C}$ given by $\varphi_n(\sum_{i=1}^{\infty} \alpha_i \delta_i) = \sum_{i=n}^{\infty} \alpha_i$, for $n \in S$ (see [2, page 32]). Suppose that $m = \delta_1$. Then $\varphi_1(m) = \varphi_1(\delta_1) = 1$ and

$$am = a\delta_1 = \Big(\sum_{i=1}^{\infty} a_i\Big)\delta_1 = \varphi_1(a)\delta_1 = \varphi_1(a)m, \quad \text{where } a = \sum_{i=1}^{\infty} a_i\delta_i \in \ell^1(S).$$

It follows that $\ell^1(S)$ is φ_1-amenable. For $n > 1$, define $m_n = \delta_n - \delta_{n-1}$. Then,

$$\varphi_n(m_n) = \varphi_n(\delta_n - \delta_{n-1}) = 1 - 0 = 1$$

and

$$am_n = a(\delta_n - \delta_{n-1}) = \sum_{i=n}^{\infty} a_i(\delta_n - \delta_{n-1}) = \varphi_n(a)(\delta_n - \delta_{n-1}) = \varphi_n(a)m_n,$$

where $a = \sum_{i=1}^{\infty} a_i\delta_i \in \ell^1(S)$. It follows that $\ell^1(S)$ is φ-amenable with respect to each multiplicative linear functional $\varphi : \ell^1(S) \to \mathbb{C}$. Thus, [4, Corollary 5.5] is not true.

This counterexample to [4, Corollary 5.5] shows that [4, Theorem 5.4] is also not true. The mistake is the assertion in the second sentence of the proof of Theorem 5.4 that 'there is an isometric isomorphism Θ from $\rho(\ell^1(\mathbb{N}_\wedge)^c$ onto $\rho(\ell^1(\mathbb{N}_\wedge))^{\varphi}$'. An example showing that Θ cannot be isometric can be constructed using [3, Theorem 7.6]. Take $\| \sum_{n=1}^{\infty} a_n\delta_n\| = \sup_F \| \sum_{n\in F} a_n\delta_n\|$, where F is a finite subset of $\mathbb{N}$. Take indices 1 and $2n+1$ so that the corresponding basis elements belong to distinct summands. Set A to be the diagonal matrix having ones at indices 1 and $2n+1$ and zero otherwise. Set B to have ones at indices 1 and $2n+1$ in the first row and zeros otherwise. Then $B = \Theta(A)$ and

https://doi.org/10.1017/S0004972723000813 Published online by Cambridge University Press

$$\begin{aligned}\|A\| &= \sup\Big\{\|a_1\delta_n + a_{2n+1}\delta_{2n+1}\| : \Big\|\sum a_n\delta_n\Big\| = 1\Big\}\\ &= \sum\{(a_1^2 + a_{2n+1}^2)^{1/2} : (a_1^2 + a_{2n+1}^2)^{1/2} = 1\} = 1,\end{aligned}$$

while

$$\begin{aligned}\|B\| &= \sup\Big\{|a_1 + a_{2n+1}| : \Big\|\sum a_n\delta_n\Big\| = 1\Big\}\\ &= \sum\{|a_1 + a_{2n+1}| : (a_1^2 + a_{2n+1}^2)^{1/2} = 1\} = \sqrt{2}.\end{aligned}$$

Consequently, Θ is not isometric. By taking k summands in a similar way, it can be shown that Θ is unbounded on diagonal elements of finite support.

Acknowledgements

The authors would like to thank the anonymous referee for very careful reading and valuable comments that improved the presentation of the manuscript and pinpointed the error in [4]. Also the first author thanks Ilam university for its support.

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https://doi.org/10.1017/S0004972723000813 Published online by Cambridge University Press