17

Anti-de Sitter-like spacetimes

This chapter discusses the construction of ***anti-de Sitter-like spacetimes***, that is, solutions to the vacuum Einstein field equations with an anti-de Sitter-like value of the cosmological constant λ. Following the general discussion in Chapter 10, an anti-de Sitter-like value of the cosmological constant implies a timelike conformal boundary. This feature of anti-de Sitter-like spacetimes marks the essential difference between the analysis contained in this chapter and the ones given in Chapters 15 and 16 for de Sitter-like and Minkowski-like spacetimes, respectively.

While the de Sitter and Minkowski spacetimes are both globally hyperbolic, and, accordingly, perturbations thereof can be constructed by means of suitable initial value problems, *the anti-de Sitter spacetime is not-globally hyperbolic*; see the discussion in Section 14.5. Consequently, anti-de Sitter-like spacetimes cannot be solely reconstructed from initial data. One needs to prescribe some boundary data on the conformal boundary. Thus, the proper setting for the construction of anti-de Sitter-like spacetimes is that of an *initial boundary value problem*. In this spirit, one of the key objectives of this chapter is to identify suitable boundary data for the conformal Einstein field equations.

For both the de Sitter and Minkowski spacetimes it is possible to obtain conformal representations which are compact in time so that global existence of perturbations can be analysed in terms of problems which are local in time. However, the conformal representations of the anti-de Sitter spacetime discussed in Chapter 6 involve an infinite range of time. As a consequence, the main result of this chapter is local in time and makes no assertions about the stability of the anti-de Sitter spacetime. The main result of this chapter can be formulated as follows:

Theorem ***(local existence of anti-de Sitter-like spacetimes).*** *Given smooth anti-de Sitter-like initial data for the Einstein field equations on a three-dimensional manifold $\mathcal{S}$ with boundary and a smooth three-dimensional Lorentzian metric $\boldsymbol{\ell}$ on a cylinder $[0,\tau_\bullet)\times\partial\mathcal{S}$ for some $\tau_\bullet > 0$, and assuming that* *these data satisfy certain corner conditions, there exists a local-in-time solution to the Einstein field equations with an anti-de Sitter-like cosmological constant such that on* $\{0\}\times\mathcal{S}$ *it implies the given anti-de Sitter-like initial data. Moreover, this solution to the Einstein field equations admits a conformal completion such that the intrinsic metric of the resulting (timelike) conformal boundary belongs to the conformal class* $[\boldsymbol{\ell}]$*.*

Thus, the conformal class of the intrinsic metric of the conformal boundary constitutes suitable boundary data for the construction of anti-de Sitter spacetimes. This insight was first obtained in Friedrich (1995).

17.1 General properties of anti-de Sitter-like spacetimes

In what follows, by an ***anti-de Sitter-like spacetime*** it will be understood an asymptotically simple spacetime $(\tilde{\mathcal{M}},\tilde{\boldsymbol{g}})$ with *positive* (i.e. anti-de Sitter-like) cosmological constant. The basic intuition on this type of spacetimes is obtained from the paradigmatic example discussed in Section 6.4. In particular, it has been shown that making use of the conformal factor

$$\Xi_{adS} = a\cos\psi, \qquad a \text{ a constant},$$

the anti-de Sitter spacetime $(\mathbb{R}^4,\tilde{\boldsymbol{g}}_{adS})$ is conformal to the region

$$\tilde{\mathcal{M}}_{adS} \equiv \left\{ p\in\mathbb{R}\times\mathbb{S}^3 \;\middle|\; 0\leq\psi(p)<\frac{\pi}{2}\right\}$$

of the Einstein cylinder $\mathbb{R}\times\mathbb{S}^3$ described in *standard coordinates* (T,ψ,θ,φ). Moreover, the conformal boundary of the spacetime is given by

$$\mathscr{I} \equiv \left\{ p\in\mathbb{R}\times\mathbb{S}^3 \;\middle|\; \psi(p)=\frac{\pi}{2}\right\},$$

which can be verified to be timelike.

17.1.1 General setting for the construction of anti-de Sitter-like spacetimes

Let $(\mathcal{M},\boldsymbol{g},\Xi)$ denote a conformal extension of an anti-de Sitter-like spacetime $(\tilde{\mathcal{M}},\tilde{\boldsymbol{g}})$ with $\boldsymbol{g}=\Xi^2\tilde{\boldsymbol{g}}$. It will be assumed that the spacetime is causal (i.e. it contains no closed timelike curves) and that it contains a smooth, oriented and compact spacelike hypersurface $\mathcal{S}_\star$ with boundary $\partial\mathcal{S}_\star$ which intersects the conformal boundary $\mathscr{I}$ in such a way that $\mathcal{S}_\star\cap\mathscr{I}=\partial\mathcal{S}_\star$. It is convenient to define $\tilde{\mathcal{S}}_\star\equiv\mathcal{S}_\star\setminus\partial\mathcal{S}_\star$. The portion of $\mathscr{I}$ in the future of $\mathcal{S}_\star$ will be denoted by $\mathscr{I}^+$. Furthermore, it will be assumed that the causal future $J^+(\mathcal{S}_\star)$ coincides with the future domain of dependence[1] $D^+(\mathcal{S}_\star\cup\mathscr{I}^+)$ and that $\mathcal{S}_\star\cup\mathscr{I}^+\approx[0,1)\times\mathcal{S}_\star$

[1] In Chapter 14 the domain of dependence has been defined for achronal sets. However, that $\mathcal{S}_\star\cup\mathscr{I}^+$ is not achronal. This feature will not play a role in the subsequent discussion.

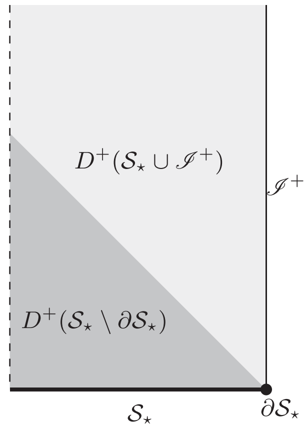

Figure 17.1 Penrose diagram of the set up for the construction of anti-de Sitter-like spacetimes as described in the main text. Initial data prescribed on $\mathcal{S}_\star \setminus \partial\mathcal{S}_\star$ allow one to recover the dark shaded region $D^+(\mathcal{S}_\star \setminus \partial\mathcal{S}_\star)$. In order to recover $D^+(\mathcal{S}_\star \cup \mathscr{I}^+)$ it is necessary to prescribe boundary data on $\mathscr{I}^+$. Notice that $D^+(\mathcal{S}_\star \cup \mathscr{I}^+) = J^+(\mathcal{S}_\star)$.

so that, in particular, $\mathscr{I}^+ \approx [0,1) \times \partial\mathcal{S}_\star$. A schematic depiction of the above setting is given in Figure 17.1. One of the key objectives of the present chapter is to address the question: *what data on $\mathcal{S}_\star \cup \mathscr{I}^+$ are needed to reconstruct the anti-de Sitter-like spacetime $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ in a neighbourhood $\mathcal{U} \subset J^+(\mathcal{S}_\star)$ of $\mathcal{S}_\star$?*

As a consequence of the properties of the standard Cauchy problem and the *localisation property of hyperbolic equations*, the solutions to the conformal Einstein field equations on $D^+(\tilde{\mathcal{S}}_\star)$ are determined, up to diffeomorphisms, in a unique manner by solutions to the constraint equations on $\mathcal{S}_\star$. To recover $J^+(\mathcal{S}_\star) \setminus D^+(\tilde{\mathcal{S}}_\star)$ one needs to prescribe suitable data on the conformal boundary $\mathscr{I}$. The analysis of the suitable boundary data requires the prescription of some appropriate gauge near $\mathscr{I}$. As will be seen, conformal geodesics are ideally suited to provide such a gauge.

The conformal constraints at the conformal boundary

Because for anti-de Sitter-like spacetimes the conformal boundary is a $\boldsymbol{g}$-timelike hypersurface, it follows that the metric $\boldsymbol{g}$ induces on $\mathscr{I}$ a three-dimensional Lorentzian metric $\boldsymbol{\ell}$. As discussed in Section 11.4.4, the conformal Einstein field equations satisfied by the (unphysical) spacetime $(\mathcal{M}, \boldsymbol{g})$ imply on $\mathscr{I}$ a *simplified* set of interior (constraint) equations. It is recalled that a solution to these conformal constraints at the conformal boundary can be computed from the metric $\boldsymbol{\ell}$, a smooth scalar function $\varkappa$ and a symmetric $\boldsymbol{\ell}$-tracefree three-dimensional tensor on $\mathscr{I}$; see Proposition 11.1. The scalar function is, in particular, a conformal gauge-dependent quantity which can be set to zero by considering a different metric in $[\boldsymbol{\ell}]$.

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17.1.2 Conformal geodesics at the conformal boundary

In Section 6.4.2 it has been shown that the anti-de Sitter spacetime can be covered by a congruence of (non-intersecting) conformal geodesics. In this congruence, curves which for some value of their affine parameter $\bar{\tau}$ are tangent to $\mathscr{I}$ remain on $\mathscr{I}$ for all values of $\bar{\tau}$. It will be shown that this observation is, in fact, a generic property of anti-de Sitter-like spacetimes.

On the conformal boundary of an anti-de Sitter-like spacetime consider an *adapted* $\boldsymbol{g}$-orthonormal frame $\{\boldsymbol{e_a}\}$ such that $\boldsymbol{e_3}$ is inward pointing and orthogonal to $\mathscr{I}$. This frame can then be extended to a neighbourhood $\mathcal{U}$ of $\mathscr{I}$ by requiring the frame to be parallely propagated in the direction of $\boldsymbol{e}_3$. It follows that the connection coefficients of $\boldsymbol{\nabla}$ associated to this frame satisfy

$$\Gamma_{\boldsymbol{3}}{}^{\boldsymbol{a}}{}_{\boldsymbol{b}} = 0 \qquad \text{on } \mathcal{U}.$$

If one uses ***Gaussian coordinates*** $x = (x^\mu)$ based on $\mathscr{I}$ such that

$$\mathscr{I} = \big\{ p \in \mathcal{U} \,\big|\, x^3(p) = 0 \big\},$$

it follows from writing $\boldsymbol{e_a} = e_{\boldsymbol{a}}{}^{\mu}\boldsymbol{\partial}_\mu$ that

$$e_{\boldsymbol{3}}{}^{\mu} = \delta_3{}^{\mu}, \qquad e_{\boldsymbol{a}}{}^{3} = 0.$$

To analyse the behaviour of conformal geodesics at the conformal boundary it is convenient to consider the equations for these curves expressed in terms of the connection $\boldsymbol{\nabla}$. These equations can be decomposed in components using the adapted frame discussed in the previous paragraph. One writes

$$\dot{\boldsymbol{x}} = z^{\boldsymbol{a}}\boldsymbol{e_a}, \qquad \boldsymbol{\beta} = \beta_{\boldsymbol{a}}\boldsymbol{\omega}^{\boldsymbol{a}}.$$

The conformal curve equations split into two groups. Firstly, one has the ***normal equations***:

$$\begin{aligned}
\dot{x}^3 &= z^{\boldsymbol{a}} e_{\boldsymbol{a}}{}^3 = z^{\boldsymbol{3}},\\
\dot{z}^{\boldsymbol{3}} &= -\Gamma_{\boldsymbol{a}}{}^{\boldsymbol{3}}{}_{\boldsymbol{b}} z^{\boldsymbol{a}} z^{\boldsymbol{b}} - 2(\beta_{\boldsymbol{c}} z^{\boldsymbol{c}}) z^{\boldsymbol{3}} + (z_{\boldsymbol{c}} z^{\boldsymbol{c}})\beta^{\boldsymbol{3}},\\
\dot{\beta}_{\boldsymbol{3}} &= \Gamma_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{3}} z^{\boldsymbol{a}} \beta_{\boldsymbol{c}} + (\beta_{\boldsymbol{c}} z^{\boldsymbol{c}})\beta_{\boldsymbol{3}} - \frac{1}{2}(\beta_{\boldsymbol{c}}\beta^{\boldsymbol{c}}) z_{\boldsymbol{3}} + L_{\boldsymbol{33}} z^{\boldsymbol{3}} + L_{\boldsymbol{i3}} z^{\boldsymbol{i}}.
\end{aligned}$$

Secondly, for $\boldsymbol{i}$, $\alpha = 0, 1, 2$ one has the ***intrinsic equations***:

$$\begin{aligned}
\dot{x}^\alpha &= e_{\boldsymbol{a}}{}^{\alpha} z^{\boldsymbol{a}},\\
\dot{z}^{\boldsymbol{i}} &= -\Gamma_{\boldsymbol{c}}{}^{\boldsymbol{i}}{}_{\boldsymbol{b}} z^{\boldsymbol{c}} z^{\boldsymbol{b}} - 2(\beta_{\boldsymbol{c}} z^{\boldsymbol{c}}) z^{\boldsymbol{i}} + (z_{\boldsymbol{c}} z^{\boldsymbol{c}})\beta^{\boldsymbol{i}},\\
\dot{\beta}_{\boldsymbol{i}} &= \Gamma_{\boldsymbol{b}}{}^{\boldsymbol{c}}{}_{\boldsymbol{i}} \beta_{\boldsymbol{c}} z^{\boldsymbol{b}} + (\beta_{\boldsymbol{c}} z^{\boldsymbol{c}})\beta_{\boldsymbol{i}} - \frac{1}{2}(\beta_{\boldsymbol{c}}\beta^{\boldsymbol{c}}) z_{\boldsymbol{i}} + L_{\boldsymbol{3i}} z^{\boldsymbol{3}} + L_{\boldsymbol{ci}} z^{\boldsymbol{c}}.
\end{aligned}$$

To simplify the analysis of the above equations one can exploit the conformal freedom and choose an element of the conformal class of the intrinsic 3-metric $\boldsymbol{\ell}$ of $\mathscr{I}$ for which

$$s = \frac{1}{4}\nabla^c\nabla_c\Xi + \frac{1}{24}R\Xi = 0.$$

Following the discussion of Section 11.4.4, this can always be done locally. Under this choice of conformal gauge, the solution of the conformal constraint equations on $\mathscr{I}$ implies that

$$\Gamma_{\boldsymbol{a}}{}^{\boldsymbol{3}}{}_{\boldsymbol{b}} = 0, \qquad \Gamma_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{3}} = 0, \qquad L_{\boldsymbol{3a}} = 0.$$

Moreover, one has

$$L_{\boldsymbol{3a}} = 0, \qquad L_{\boldsymbol{ij}} = l_{\boldsymbol{ij}}.$$

That is, the spacetime (unphysical) Schouten tensor on $\mathscr{I}$ is determined by the Schouten tensor of the intrinsic metric $\boldsymbol{\ell}$.

From the previous discussion it follows that the normal subset of the conformal geodesic equations reduces to:

$$\begin{aligned}
\dot{x}^{\boldsymbol{3}} &= z^{\boldsymbol{3}}, \\
\dot{z}^{\boldsymbol{3}} &= -2(\beta_{\boldsymbol{c}}\beta^{\boldsymbol{c}})z^{\boldsymbol{3}} + (z_{\boldsymbol{c}}z^{\boldsymbol{c}})\beta_{\boldsymbol{3}}, \\
\dot{\beta}_{\boldsymbol{3}} &= (\beta_{\boldsymbol{c}}z^{\boldsymbol{c}})\beta_{\boldsymbol{3}} - \frac{1}{2}(\beta_{\boldsymbol{c}}\beta^{\boldsymbol{c}})z^{\boldsymbol{3}} + l_{\boldsymbol{33}}z^{\boldsymbol{3}}.
\end{aligned}$$

These equations are homogeneous in the unknowns $(x^3, z^{\boldsymbol{3}}, \beta_{\boldsymbol{3}})$. Thus, by choosing initial data such that

$$x^3_\star = 0, \qquad \dot{x}^{\boldsymbol{3}}_\star = 0, \qquad \beta_{\boldsymbol{3}\star} = 0, \tag{17.1}$$

one obtains that

$$x^3(\tau) = 0, \qquad z^{\boldsymbol{3}}(\tau) = 0, \qquad \beta_{\boldsymbol{3}}(\tau) = 0$$

for later times. Accordingly, conformal curves with initial data given by (17.1) will remain on $\mathscr{I}$. Looking now at the intrinsic part of the conformal geodesic equations one observes that the equations reduce to

$$\begin{aligned}
\dot{x}^\alpha &= z^{\boldsymbol{i}} e_{\boldsymbol{i}}{}^\alpha, \\
\dot{z}^{\boldsymbol{i}} &= -\Gamma_{\boldsymbol{k}}{}^{\boldsymbol{i}}{}_{\boldsymbol{j}} z^{\boldsymbol{k}} z^{\boldsymbol{j}} - 2(\beta_{\boldsymbol{k}}z^{\boldsymbol{k}})z^{\boldsymbol{i}} + (z_{\boldsymbol{k}}z^{\boldsymbol{k}})\beta^{\boldsymbol{i}}, \\
\dot{\beta}_{\boldsymbol{i}} &= \Gamma_{\boldsymbol{j}}{}^{\boldsymbol{k}}{}_{\boldsymbol{i}} z^{\boldsymbol{j}} \beta_{\boldsymbol{k}} + (\beta_{\boldsymbol{k}}z^{\boldsymbol{k}})\beta_{\boldsymbol{i}} - \frac{1}{2}(\beta_{\boldsymbol{k}}\beta^{\boldsymbol{k}})z_{\boldsymbol{i}} + l_{\boldsymbol{ki}}z^{\boldsymbol{k}}.
\end{aligned}$$

These are the *conformal geodesic equations for the 3-metric* $\boldsymbol{\ell}$ on $\mathscr{I}$.

To verify the consistency between the construction described in the previous paragraphs and the adapted $\boldsymbol{g}$-orthonormal frame $\{\boldsymbol{e_a}\}$, consider a vector $\boldsymbol{v}$ satisfying the Weyl propagation equation

$$\nabla_{\dot{\boldsymbol{x}}}\boldsymbol{v} = -\langle \boldsymbol{\beta}, \boldsymbol{v}\rangle \dot{\boldsymbol{x}} - \langle \boldsymbol{\beta}, \dot{\boldsymbol{x}}\rangle \boldsymbol{v} + \boldsymbol{g}(\boldsymbol{v}, \dot{\boldsymbol{x}})\boldsymbol{\beta}^\sharp,$$

along $\mathscr{I}$. Making the ansatz $\boldsymbol{v} = \alpha \boldsymbol{e}_3$, where α denotes a scalar function on $\mathscr{I}$, one finds the equation $\dot{\alpha} = -\langle \boldsymbol{\beta}, \dot{\boldsymbol{x}}\rangle\alpha$. Thus, if initially one has $\alpha_\star \neq 0$, then $\alpha \neq 0$ at later times. Accordingly, if one prescribes at some point of the conformal geodesic in $\mathscr{I}$ an orthonormal frame $\{\boldsymbol{e_a}\}$ containing a vector which is normal to $\mathscr{I}$, one finds that the solution to the Weyl propagation equations will be a frame

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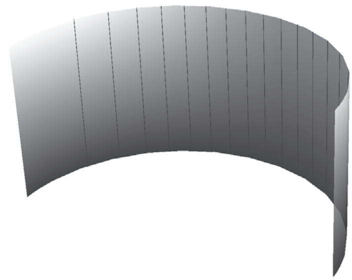

Figure 17.2 Representation of conformal geodesics on the conformal boundary of an anti-de Sitter-like spacetime: those curves that at some point are tangent to $\mathscr{I}$ remain in the conformal boundary and are conformal geodesics for the conformal structure implied by the intrinsic metric $\boldsymbol{\ell}$; see Lemma 17.1. The conformal geodesics are depicted by black lines.

along the conformal geodesic which contains a vector normal to $\mathscr{I}$. Moreover, as the Weyl propagation preserves the orthogonality of vectors, it follows that the elements of the frame which are at some point intrinsic to $\mathscr{I}$ will remain so at later times; see Figure 17.2.

A more general result

The results obtained in the previous paragraphs make use of a particular metric in the conformal class $[\boldsymbol{\ell}]$. Thus, it is of interest to reformulate them in an arbitrary conformal gauge. As in Chapter 10, the symbol $\simeq$ denotes equality on $\mathscr{I}$. Now, consider on $\mathcal{M}$ a conformal factor $\vartheta > 0$ such that $\vartheta \simeq 1$ to perform a rescaling of the form $\boldsymbol{g}' \equiv \vartheta^2 \boldsymbol{g}$. This rescaling leaves the metric $\boldsymbol{\ell}$ unchanged in the sense that $\boldsymbol{\ell}' \simeq \vartheta^2 \boldsymbol{\ell} \simeq \boldsymbol{\ell}$. Furthermore, one finds that

$$s' \simeq (\nabla^a \Xi \nabla_a \vartheta) \simeq \boldsymbol{e_3}(\vartheta),$$

with $\boldsymbol{e_3} = (\mathbf{d}\Xi)^\sharp$ as $\Xi = x^3$ in local coordinates. The comparison of the above expression with the solution to the conformal constraints at the conformal boundary as given in Section 11.4.4 suggests defining

$$\varkappa \equiv \sqrt{3/\lambda}\, \boldsymbol{e_3}(\vartheta)\big|_{\mathscr{I}}.$$

Defining the covector $\boldsymbol{k} \equiv \vartheta^{-1}\mathbf{d}\vartheta$, and taking into account the transformation properties of conformal geodesics as given in Section 5.5.2, it follows that

$$(x(\tau), \boldsymbol{\beta}'(\tau)), \qquad \text{with } \boldsymbol{\beta}' \equiv \boldsymbol{\beta} - \boldsymbol{k},$$

is a solution to the conformal geodesic equations associated to the connection $\boldsymbol{\nabla}' \equiv \boldsymbol{\nabla} + \boldsymbol{S}(\boldsymbol{k})$. From the definition of $\boldsymbol{k}$ it follows that $\boldsymbol{\nabla}'$ is the Levi-Civita connection of the metric $\boldsymbol{g}' = \vartheta^2 \boldsymbol{g}$. Observe, in particular, that

$$\beta'_{\boldsymbol{3}}(\tau) \simeq -k_{\boldsymbol{3}}(\tau) \simeq -\boldsymbol{e_3}(\vartheta) \simeq -s'.$$

The discussion of this section can be summarised as follows:

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Lemma 17.1 *A conformal geodesic in an anti-de Sitter-like spacetime which passes through a point $p \in \mathscr{I}$, is tangent to $\mathscr{I}$ at p and which satisfies*

$$\langle \boldsymbol{\beta}, \boldsymbol{\nu} \rangle|_p = -s,$$

with $\boldsymbol{\nu}$ the unit normal to $\mathscr{I}$, remains in $\mathscr{I}$ and defines a conformal geodesic for the conformal structure of $\mathscr{I}$. Furthermore, the Weyl propagation equations admit a solution containing a vector field normal to $\mathscr{I}$.

17.2 The formulation of an initial boundary value problem

The properties of conformal geodesics in anti-de Sitter-like spacetimes will now be exploited to construct a conformal Gaussian system for the extended conformal Einstein field equations. As will be seen, the hyperbolic reduction associated to this gauge leads to an initial boundary value problem for the conformal evolution equations.

17.2.1 Construction of a boundary adapted gauge

Following the discussion of Chapter 14, the solution to the Einstein field equations on the domain of dependence $D^+(\tilde{\mathcal{S}}_\star) = D^+(\mathcal{S}_\star \setminus \partial\mathcal{S}_\star)$ is determined in a unique manner, up to diffeomorphisms, by a pair of tensors $(\tilde{\boldsymbol{h}}, \tilde{\boldsymbol{K}})$ satisfying the Einstein constraint equations on $\tilde{\mathcal{S}}_\star$. On $\mathcal{S}_\star$, let

$$\Omega \equiv \Theta\big|_{\tilde{\mathcal{S}}_\star}, \qquad \tilde{\Sigma}_\star \equiv \tilde{\boldsymbol{\nu}}(\Theta)|_{\tilde{\mathcal{S}}_\star},$$

with $\tilde{\boldsymbol{\nu}}$ the future-directed unit normal field of $\tilde{\mathcal{S}}_\star$ with respect to $\tilde{\boldsymbol{g}}$. In addition to the usual smoothness and positivity assumptions, the fields Ω and $\tilde{\Sigma}_\star$ are restricted by their behaviour near $\partial\mathcal{S}_\star$ where one requires that $\Sigma_\star \equiv \boldsymbol{\nu}(\Theta)\big|_{\mathcal{S}_\star} = \Omega^{-1}\tilde{\Sigma}_\star$, with $\boldsymbol{\nu}$ the future-directed $\boldsymbol{g}$-unit normal, to be smooth. Using the above fields one can use Equations (11.1a) and (11.1b) to compute the unphysical fields $(\boldsymbol{h}, \boldsymbol{K})$.

To simplify the subsequent discussion, it is assumed that the initial hypersurface $\mathcal{S}_\star$ is such that the unit normal $\boldsymbol{\nu}$ is tangent to $\mathscr{I}$ on $\partial\mathcal{S}_\star$. Accordingly, one has

$$\Sigma_\star \equiv \boldsymbol{\nu}(\Theta)\big|_{\mathcal{S}_\star} = 0 \qquad \text{on } \partial\mathcal{S}_\star.$$

Moreover, recalling that at the conformal boundary s can be made to vanish by a convenient choice of conformal gauge, it is assumed that

$$s = 0, \qquad \text{on } \partial\mathcal{S}_\star.$$

In what follows, each $p \in \mathcal{S}_\star$ will be considered as the starting point of a future-directed conformal geodesic $(x(\tau), \boldsymbol{\beta}(\tau))$ and an associated Weyl propagated frame $\{\boldsymbol{e_a}\}$. The parametrisation of the curves is naturally chosen so that $\tau = 0$ on $\mathcal{S}_\star$. For points $p \in \tilde{\mathcal{S}}_\star$, the data for these curves are set in terms of $\tilde{\boldsymbol{g}}$ and its Levi-Civita connection $\tilde{\boldsymbol{\nabla}}$ by the conditions:

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(i) $\dot{\boldsymbol{x}}$ is future directed, orthogonal to $\tilde{\mathcal{S}}_\star$ and satisfies the normalisation condition

$$\tilde{\boldsymbol{g}}(\dot{\boldsymbol{x}}, \dot{\boldsymbol{x}})_\star = \Theta_\star^{-2}.$$

(ii) $\boldsymbol{\beta}_\star = \Omega^{-1}\mathbf{d}\Omega$ so that $\langle \boldsymbol{\beta}_\star, \dot{\boldsymbol{x}}_\star \rangle = 0$ —as $\Sigma_\star = 0$ by assumption.
(iii) $\boldsymbol{e}_{\boldsymbol{0}\star} = \dot{\boldsymbol{x}}_\star$ and $\tilde{\boldsymbol{g}}(\boldsymbol{e_a}, \boldsymbol{e_b})_\star = \Theta_\star^{-2}\eta_{\boldsymbol{ab}}$.

On suitable neighbourhoods $\mathcal{W} \subset J^+(\mathcal{S}_\star)$ of $\mathcal{S}_\star$, the conformal geodesics $x(\tau)$ define a smooth timelike congruence in $\mathcal{W}$, $\{\boldsymbol{e_a}\}$ a smooth frame field and $\boldsymbol{\beta}$, a smooth covector. The conformal geodesics can be used to fix a conformal Gaussian coordinate system on $\mathcal{W}$ by setting $x^0 = \tau$ and then extending local coordinates $\underline{x} = (x^\alpha)$ on $\mathcal{S}_\star$ by requiring them to remain constant along conformal geodesics. The coefficients $e_{\boldsymbol{a}}{}^\mu = \langle \mathbf{d}x^\mu, \boldsymbol{e_a} \rangle$ of the frame $\{\boldsymbol{e_a}\}$ with respect to the Gaussian coordinates satisfy on $\mathcal{W}$ the condition $e_{\boldsymbol{0}}{}^\mu = \delta_0{}^\mu$. Observe, however, that in general $e_{\boldsymbol{a}}{}^0 = 0$ only on $\mathcal{S}_\star$. The conformal factor Θ is then fixed on $\mathcal{W}$ by requiring

$$\boldsymbol{g}(\boldsymbol{e_a}, \boldsymbol{e_b}) = \eta_{\boldsymbol{ab}}.$$

The discussion of the conformal geodesics in the conformal boundary needs to be done in terms of the metric $\boldsymbol{g}$ and its Levi-Civita connection $\boldsymbol{\nabla}$. In terms of these, the conformal geodesics are represented by a pair $(x(\tau), \boldsymbol{f}(\tau))$ with $\boldsymbol{f} \equiv \boldsymbol{\beta} - \Theta^{-1}\mathbf{d}\Theta$. Accordingly, one has

$$\boldsymbol{f} = 0, \qquad \text{on } \mathcal{S}_\star.$$

As a result of Lemma 17.1, conformal geodesics which start on $\partial\mathcal{S}_\star$ remain on $\mathscr{I}$. As $s = 0$ on $\partial\mathcal{S}_\star$ one can write

$$s_\star = \Omega\varsigma_\star, \tag{17.2}$$

with $\varsigma_\star$ a smooth function on $\partial\mathcal{S}_\star$. It follows from Proposition 5.1 that

$$\Theta = \Omega\left(1 - \frac{1}{2}\varsigma_\star\tau^2\right), \tag{17.3}$$

while for $d_{\boldsymbol{a}} \equiv \langle \boldsymbol{d}, \boldsymbol{e_a} \rangle$ one obtains the explicit expression

$$d_{\boldsymbol{a}} = \big(\dot{\Theta}, \boldsymbol{e_i}(\Omega)_\star\big), \qquad \boldsymbol{e_i}(\Omega)_\star \equiv (e_{\boldsymbol{i}}{}^\alpha\partial_\alpha\Omega)_\star, \tag{17.4}$$

where the functions Ω, $\varsigma_\star$ and $\boldsymbol{e_i}(\Omega)_\star$ defined initially on $\mathcal{S}_\star$ are extended to $\mathcal{W}$ so that they are constant along conformal geodesics.

Remark. Insight into the behaviour of the conformal factor (17.3) can be obtained from the constraint Equation (11.35c). Using Equation (17.2), exploiting that in an adapted gauge $(\mathbf{d}\Omega)^\sharp = -\boldsymbol{e_3}$ and evaluating at $\partial\mathcal{S}_\star$ one concludes

$$\varsigma_\star \simeq -L_{\boldsymbol{03}}\Sigma - L_{\boldsymbol{33}}.$$

Finally, from Equations (11.40) and (11.41) it follows that in a conformal gauge for which $s \simeq 0$ one also necessarily has $L_{\mathbf{03}} \simeq 0$. Thus, one obtains the simple expression

$$\varsigma_\star \simeq -L_{\mathbf{33}}.$$

In particular, if $L_{\mathbf{33}} > 0$, then from Equation (17.3) the conformal factor Θ vanishes only if Ω vanishes. This observation is consistent with the discussion of Section 17.1.2 – conformal geodesics which start normal to $\mathcal{S}_\star$ and away from $\partial\mathcal{S}_\star$ cannot enter the conformal boundary. Ideally, one would like to deduce the property $L_{\mathbf{33}} > 0$ from an analysis of the conformal constraint equations. For data for the exact de Sitter spacetime, Equation (6.8b) implies $L_{\mathbf{33}} = \frac{1}{2}$ on $\partial\mathcal{S}_\star$. Suitable perturbations of data for the anti-de Sitter spacetime should preserve this property.

17.2.2 The conformal evolution system

Combining the gauge construction with the hyperbolic reduction for the extended conformal field equations discussed in Section 13.4 one obtains an evolution system of the form

$$\partial_\tau \hat{\boldsymbol{\upsilon}} = \mathbf{K}\hat{\boldsymbol{\upsilon}} + \mathbf{Q}(\hat{\boldsymbol{\Gamma}})\hat{\boldsymbol{\upsilon}} + \mathbf{L}(x)\boldsymbol{\phi}, \tag{17.5a}$$

$$\big(\mathbf{I} + \mathbf{A}^0(\boldsymbol{e})\big)\partial_\tau \boldsymbol{\phi} + \mathbf{A}^\alpha(\boldsymbol{e})\partial_\alpha \boldsymbol{\phi} = \mathbf{B}(\hat{\boldsymbol{\Gamma}})\boldsymbol{\phi}, \tag{17.5b}$$

where the notation of Proposition 13.3 is retained and the matrix-valued function $\boldsymbol{L}(x)$ is given explicitly in terms of the conformal gauge fields Θ and $d_{\boldsymbol{a}}$ as given by Equations (17.3) and (17.4). In the above system, Equation (17.5b) is understood to correspond to the boundary-adapted Bianchi evolution system (13.60a) and (13.60b) in Chapter 13. The evolution system (17.5a) and (17.5b) is ideally suited to the formulation of a boundary value problem, as the equations described by the subsystem (17.5a) are mere ***transport equations*** along the conformal boundary which do not need to be supplemented by boundary conditions. Hence, all the boundary conditions arise from the subsystem (17.5b) associated to the evolution of the Weyl tensor.

Following the discussion of initial boundary value problems for symmetric hyperbolic equations as described in Section 12.4, the identification of suitable boundary conditions for Equation (17.5b) stems from an analysis of the ***normal matrix*** $\mathbf{A}^3$ at the conformal boundary. Making use of the explicit expression for the principal part of the boundary-adapted Bianchi system given in Equation (13.61) and taking into account that, in the *boundary adapted conformal Gaussian gauge*, one has

$$e_{\mathbf{00}}{}^3 \simeq 0, \qquad e_{\mathbf{11}}{}^3 \simeq 0,$$

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it follows that

$$\mathbf{A}^3 \simeq 2e_{\boldsymbol{01}}{}^3\big|_{\mathscr{I}} \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

This normal matrix is almost in the form required by the theory of Chapter 12. It needs only to be verified that the evolution of the frame coefficient $e_{\boldsymbol{01}}{}^3$ on $\mathscr{I}$ can be decoupled from that of the components of the Weyl tensor. An inspection of the conformal evolution Equations (13.59a)–(13.59g) – of which Equation (17.5a) above is a schematic representation – shows that whenever $\Theta = 0$, the evolution equations for *certain components* of the fields $e_{\boldsymbol{AB}}{}^{\alpha}$, $\chi_{(\boldsymbol{AB})\boldsymbol{CD}}$, $\Theta_{\boldsymbol{CD}(\boldsymbol{AB})}$ decouple from the evolution of $\phi_{\boldsymbol{ABCD}}$. Thus, it is possible to determine the frame coefficient $e_{\boldsymbol{01}}{}^3$ directly from the initial data at $\partial\mathcal{S}_\star$ – hence, it is independent of any boundary value prescriptions on $\mathscr{I}$. This observation will be discussed in some detail in the following subsection.

Remark. The normal matrix for the *standard Bianchi system* is given by

$$\mathbf{A}^3 \simeq 2e_{\boldsymbol{01}}{}^3\big|_{\mathscr{I}} \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

so that this normal matrix leads to a much more complicated analysis of boundary conditions.

17.2.3 Behaviour of the frame at the conformal boundary

In this section, the discussion is restricted to a suitable open neighbourhood $\mathcal{W}$ of a point on $\partial\mathcal{S}_\star$ such that the intersection with conformal geodesics is connected. Consistent with the discussion in Section 17.2.1, one introduces on $\mathcal{S}_\star \cap \mathcal{W}$ an adapted three-dimensional spatial frame $\{\boldsymbol{e_i}\}$ such that $\boldsymbol{e_3}$ is orthogonal and inward directed at $\partial\mathcal{S}_\star$ and such that $\nabla_{\boldsymbol{3}}\boldsymbol{e_a} = 0$ on $\mathcal{S}_\star \cap \mathcal{W}$. One introduces coordinates $\underline{x} = (x^\alpha)$ on $\mathcal{S}_\star \cap \mathcal{W}$ so that x^3 vanishes on $\partial\mathcal{S}_\star$ and $\langle \mathbf{d}x^\alpha, \boldsymbol{e_3}\rangle = \delta_{\boldsymbol{3}}{}^\alpha$ on $\mathcal{S}_\star \cap \mathcal{W}$. A conformal Gaussian gauge system satisfying the above assumptions near $\partial\mathcal{S}_\star$ will be called a ***boundary adapted gauge***.

For future reference it is observed that the conformal evolution Equations (13.59b), (13.59e) and (13.59f) reduce, on the conformal boundary, to

$$\partial_\tau e_{\boldsymbol{AB}}{}^\alpha \simeq -\chi_{(\boldsymbol{AB})}{}^{\boldsymbol{PQ}} e_{\boldsymbol{PQ}}{}^\alpha, \tag{17.6a}$$

$$\partial_\tau \chi_{(\boldsymbol{AB})\boldsymbol{CD}} \simeq -\chi_{(\boldsymbol{AB})}{}^{\boldsymbol{PQ}} \chi_{\boldsymbol{PQCD}} - \Theta_{\boldsymbol{AB}(\boldsymbol{CD})}, \tag{17.6b}$$

$$\partial_\tau \Theta_{\boldsymbol{CD}(\boldsymbol{AB})} \simeq -\chi_{(\boldsymbol{CD})}{}^{\boldsymbol{PQ}} \Theta_{\boldsymbol{PQ}(\boldsymbol{AB})} + \mathrm{i}\sqrt{2} d^{\boldsymbol{P}}{}_{(\boldsymbol{A}}\mu_{\boldsymbol{B})\boldsymbol{CDP}}. \tag{17.6c}$$

The above evolution equations at the conformal boundary are conveniently analysed in terms of a ***1+1+2 spinorial formalism***. Given a spinorial basis $\{\epsilon_{\boldsymbol{A}}{}^{A}\}$ such that

$$\tau^{\boldsymbol{AA}'} = \delta_{\boldsymbol{0}}{}^{\boldsymbol{A}}\delta_{\boldsymbol{0}'}{}^{\boldsymbol{A}'} + \delta_{\boldsymbol{1}}{}^{\boldsymbol{A}}\delta_{\boldsymbol{1}'}{}^{\boldsymbol{A}'},$$

it is convenient to introduce a spatial spinor $\rho^{AA'}$ with components with respect to the basis $\{\epsilon_{\boldsymbol{A}}{}^{A}\}$ given by

$$\rho^{\boldsymbol{AA}'} \equiv \delta_{\boldsymbol{0}}{}^{\boldsymbol{A}}\delta_{\boldsymbol{0}'}{}^{\boldsymbol{A}'} - \delta_{\boldsymbol{1}}{}^{\boldsymbol{A}}\delta_{\boldsymbol{1}'}{}^{\boldsymbol{A}'}.$$

The space spinor counterpart of $\rho^{\boldsymbol{AA}'}$ is given by

$$\rho_{\boldsymbol{AB}} \equiv \tau_{\boldsymbol{B}}{}^{\boldsymbol{A}'}\rho_{\boldsymbol{AA}'} = -2\delta_{(\boldsymbol{A}}{}^{\boldsymbol{0}}\delta_{\boldsymbol{B})}{}^{\boldsymbol{1}}.$$

It can be verified that, in addition to the condition $\sqrt{2}\boldsymbol{e_0} = \tau^{\boldsymbol{AA}'}\boldsymbol{e_{AA'}}$, one has

$$\sqrt{2}\boldsymbol{e_3} = \rho^{\boldsymbol{AA}'}\boldsymbol{e_{AA'}} = \rho^{\boldsymbol{AB}}\boldsymbol{e_{AB}} = 2\boldsymbol{e_{01}}, \qquad \text{on } \mathcal{S}_\star \cap \mathcal{W}. \tag{17.7}$$

In particular, one has

$$\boldsymbol{e_{AB}}(\Theta) = d_{\boldsymbol{AB}} = -\sqrt{\lambda/6}\rho_{\boldsymbol{AB}} \qquad \text{on } \partial\mathcal{S}_\star.$$

The spinor $\rho^{\boldsymbol{AB}}$ will be used to split space spinor fields into parts orthogonal and tangent to $\mathscr{I}$. Accordingly, one defines

$$\begin{aligned}
e^{3\perp} &\equiv \rho^{\boldsymbol{AB}}e_{\boldsymbol{AB}}{}^3, & e_{\boldsymbol{AB}}{}^{3\parallel} &\equiv \rho_{(\boldsymbol{A}}{}^{\boldsymbol{C}}e_{\boldsymbol{B})\boldsymbol{C}}{}^3\\
\chi^{\perp\perp} &\equiv \rho^{\boldsymbol{AB}}\rho^{\boldsymbol{CD}}\hat{\chi}_{\boldsymbol{ABCD}}, & \chi^{\parallel\perp}{}_{\boldsymbol{AB}} &\equiv \rho_{(\boldsymbol{A}}{}^{\boldsymbol{E}}\hat{\chi}_{\boldsymbol{B})\boldsymbol{ECD}}\rho^{\boldsymbol{CD}},\\
\chi^{\perp\parallel}{}_{\boldsymbol{CD}} &\equiv \rho^{\boldsymbol{AB}}\hat{\chi}_{\boldsymbol{ABE}(\boldsymbol{C}}\rho^{\boldsymbol{E}}{}_{\boldsymbol{D})}, & \chi^{\parallel\parallel}{}_{\boldsymbol{ABCD}} &\equiv \rho_{(\boldsymbol{A}}{}^{\boldsymbol{E}}\hat{\chi}_{\boldsymbol{B})\boldsymbol{EF}(\boldsymbol{C}}\rho^{\boldsymbol{F}}{}_{\boldsymbol{D})},\\
\Theta^{\perp\perp} &\equiv \rho^{\boldsymbol{AB}}\rho^{\boldsymbol{CD}}\hat{\Theta}_{\boldsymbol{ABCD}}, & \Theta^{\parallel\perp}{}_{\boldsymbol{AB}} &\equiv \rho_{(\boldsymbol{A}}{}^{\boldsymbol{E}}\hat{\Theta}_{\boldsymbol{B})\boldsymbol{ECD}}\rho^{\boldsymbol{CD}},
\end{aligned}$$

where

$$\hat{\chi}_{\boldsymbol{ABCD}} \equiv \chi_{(\boldsymbol{AB})\boldsymbol{CD}}, \qquad \hat{\Theta}_{\boldsymbol{ABCD}} \equiv \Theta_{\boldsymbol{AB}(\boldsymbol{CD})}.$$

Observing that $\partial_\tau\rho_{\boldsymbol{AB}} = 0$, it follows from Equations (17.6a)–(17.6c) that

$$\begin{aligned}
\partial_\tau e_{\boldsymbol{AB}}{}^3 &\simeq -\hat{\chi}_{\boldsymbol{AB}}{}^{\boldsymbol{PQ}}e_{\boldsymbol{PQ}}{}^3,\\
\partial_\tau\big(\hat{\chi}_{\boldsymbol{ABCD}}\rho^{\boldsymbol{CD}}\big) &\simeq -\hat{\chi}_{\boldsymbol{AB}}{}^{\boldsymbol{PQ}}\hat{\chi}_{\boldsymbol{PQCD}}\rho^{\boldsymbol{CD}} - \hat{\Theta}_{\boldsymbol{ABCD}}\rho^{\boldsymbol{CD}},\\
\partial_\tau\big(\hat{\Theta}_{\boldsymbol{CDAB}}\rho^{\boldsymbol{AB}}\big) &\simeq -\hat{\chi}_{\boldsymbol{AB}}{}^{\boldsymbol{PQ}}\hat{\Theta}_{\boldsymbol{PQCD}}\rho^{\boldsymbol{AB}},
\end{aligned}$$

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where it has been used that $d^{\boldsymbol{P}}{}_{(\boldsymbol{A}\mu_{\boldsymbol{B}})\boldsymbol{CDP}}\rho^{\boldsymbol{AB}} = 0$ as $d_{\boldsymbol{AB}}$ and $\rho_{\boldsymbol{AB}}$ are proportional to each other. By further contractions with $\rho^{\boldsymbol{AB}}$ one finds that the above equations split into the subsystems

$$\partial_\tau e_{\boldsymbol{AB}}{}^{3\|} \simeq \frac{1}{2}\chi^{\|\perp}{}_{\boldsymbol{AB}}e^{3\perp} + \chi^{\|\|}e_{\boldsymbol{AB}}{}^{3\|}, \tag{17.8a}$$

$$\partial_\tau \chi^{\|\perp}{}_{\boldsymbol{AB}} \simeq \frac{1}{2}\chi^{\|\perp}{}_{\boldsymbol{AB}}\chi^{\perp\perp} + \chi^{\|\|}{}_{\boldsymbol{ABPQ}}\chi^{\|\perp\boldsymbol{PQ}} - \Theta^{\|\perp}{}_{\boldsymbol{AB}}, \tag{17.8b}$$

$$\partial_\tau \Theta^{\|\perp}{}_{\boldsymbol{AB}} \simeq \frac{1}{2}\chi^{\|\perp}{}_{\boldsymbol{AB}}\Theta^{\perp\perp} + \chi^{\|\|}{}_{\boldsymbol{ABPQ}}\Theta^{\|\perp\boldsymbol{PQ}}, \tag{17.8c}$$

and

$$\partial_\tau e^{3\perp} \simeq \frac{1}{2}\chi^{\perp\perp}e^{3\perp} + \chi^{\perp\|}{}_{\boldsymbol{PQ}}e^{3\|\boldsymbol{PQ}}, \tag{17.9a}$$

$$\partial_\tau \chi^{\perp\perp} \simeq \frac{1}{2}\left(\chi^{\perp\perp}\right)^2 + \chi^{\perp\|}{}_{\boldsymbol{PQ}}\chi^{\|\perp\boldsymbol{PQ}} - \Theta^{\perp\perp}, \tag{17.9b}$$

$$\partial_\tau \Theta^{\perp\perp} \simeq \frac{1}{2}\Theta^{\perp\perp}\chi^{\perp\perp} + \chi^{\perp\|}{}_{\boldsymbol{PQ}}\Theta^{\|\perp\boldsymbol{PQ}}. \tag{17.9c}$$

Initial data for $e^{3\perp}$ and $e_{\boldsymbol{AB}}{}^{3\|}$ at $\partial\mathcal{S}_\star$ follow directly from (17.7). Namely, one has

$$e^{3\perp}\big|_{\partial\mathcal{S}_\star} = \sqrt{2}, \qquad e_{\boldsymbol{AB}}{}^{3\|}\big|_{\partial\mathcal{S}_\star} = 0. \tag{17.10}$$

For $\chi^{\|\perp}{}_{\boldsymbol{AB}}$ and $\chi^{\perp\perp}$, initial data can be extracted from the conformal constraint Equation (11.35b) which, taking into account that by assumption $\Sigma = 0$ and $L_{\boldsymbol{a}} = 0$ on $\mathcal{S}_\star$, takes the form $\chi_{\boldsymbol{a}}{}^{\boldsymbol{c}}D_{\boldsymbol{c}}\Omega = 0$ on $\partial\mathcal{S}_\star$. It follows then that

$$\chi^{\perp\perp} = 0, \qquad \chi^{\|\perp}{}_{\boldsymbol{AB}} = 0, \qquad \text{on } \partial\mathcal{S}_\star. \tag{17.11}$$

Finally, to compute the data for $\Theta^{\|\perp}{}_{\boldsymbol{AB}}$ and $\Theta^{\perp\perp}$ one considers the conformal constraint (11.35c) which, in the present context, takes the form

$$D_{\boldsymbol{3}}s = -D^{\boldsymbol{b}}\Omega L_{\boldsymbol{b3}}.$$

Recalling that $s = \Omega\varsigma_\star$ and that, in local Gaussian coordinates, $\Omega = x^3$ one concludes that

$$\Theta^{\perp\perp} = 2\varsigma_\star, \qquad \Theta^{\|\perp}{}_{\boldsymbol{AB}} = 0, \qquad \text{on } \partial\mathcal{S}_\star. \tag{17.12}$$

Using the initial conditions (17.10), (17.11) and (17.12) together with the homogeneity of the subsystem (17.8a)–(17.8c), it follows directly that

$$e_{\boldsymbol{AB}}{}^{3\|} \simeq 0, \qquad \chi^{\|\perp}{}_{\boldsymbol{AB}} \simeq 0, \qquad \Theta^{\|\perp}{}_{\boldsymbol{AB}} \simeq 0.$$

The solution to the subsystem (17.9a)–(17.9c) is given by

$$e^{3\perp} = -\frac{2\sqrt{2}}{2+\tau^2\varsigma_\star}, \qquad \chi^{\perp\perp} = -\frac{4\tau\varsigma_\star}{2+\tau^2\varsigma_\star}, \qquad \Theta^{\perp\perp} = \frac{4\varsigma_\star}{2+\tau^2\varsigma_\star}.$$

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The discussion in this section is summarised in the following:

Lemma 17.2 *For any solution to the conformal evolution Equations* (17.5a) *and* (17.5b) *satisfying on $\partial\mathcal{S}_\star$ the conditions* (17.10), (17.11) *and* (17.12), *one has that the normal matrix $\mathbf{A}^3\big|_{\mathscr{I}}$ of the boundary adapted Bianchi system is given by*

$$\mathbf{A}^3 \simeq \frac{2\sqrt{2}}{2+\tau^2\varsigma_\star}\begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

irrespectively of the value of $\phi_{\boldsymbol{ABCD}}$ on $\mathcal{W}\cap\mathscr{I}$.

17.2.4 Identification of boundary conditions

The results of the previous paragraphs allow the identification of maximally dissipative boundary conditions for the conformal evolution equations. Following the discussion in Section 12.4, the basic condition to be satisfied by the normal matrix is the inequality

$$\langle \boldsymbol{\phi}, \mathbf{A}^3\big|_{\mathscr{I}}\boldsymbol{\phi}\rangle \leq 0,$$

which, assuming that $2+\tau^2\varsigma_\star > 0$, implies that

$$|\phi_4|^2 - |\phi_0|^2 \leq 0. \tag{17.13}$$

To characterise the subspaces of $\mathbb{C}^5$ satisfying the above condition consider two smooth complex-valued functions c_1 and c_2 on $\mathscr{I}$ and let

$$\phi_4 = c_1\phi_0 + c_2\bar{\phi}_0.$$

Exploiting that $(c_1\phi_0 - c_2\bar{\phi}_0)(\bar{c}_1\bar{\phi}_0 - \bar{c}_2\phi_0) \geq 0$ one finds that

$$|\phi_4|^2 - |\phi_0|^2 \leq (|c_1|^2 + |c_2|^2 - 1)|\phi_0|^2.$$

Thus, condition (17.13) is satisfied if one requires

$$|c_1|^2 + |c_2|^2 \leq 1.$$

The above discussion shows that suitable ***inhomogeneous maximally dissipative*** boundary conditions for the conformal evolution equations are given by

$$\phi_4 - c_1\phi_0 - c_2\bar{\phi}_0 = q, \qquad |c_1|^2 + |c_2|^2 \leq 1, \tag{17.14}$$

with c_1, c_2, q smooth complex-valued functions on $\mathscr{I}$.

Corner conditions

As seen in Section 12.4, the smoothness of a solution to an initial boundary value problem requires certain compatibility conditions between the initial data and the boundary conditions at the edge $\partial\mathcal{S}_\star$ – so-called ***corner conditions***. Following the general discussion given in Section 12.4, one can use the boundary-adapted Bianchi system (17.5b) to determine a formal expansion in terms of τ of the vector $\boldsymbol{\phi}$ on $\mathscr{I}$ near $\partial\mathcal{S}_\star$. This expansion implies, in turn, an expansion for $\phi_4 - c_1\phi_0 - c_2\bar{\phi}_0$ and must be consistent with the prescription of the freely specifiable function q. The explicit form of these corner conditions is rather cumbersome. In what follows, it will be assumed that these corner conditions are satisfied to any order.

17.2.5 The local existence result

The analysis of the boundary conditions leads to a local existence result for an initial boundary value problem for the conformal evolution system (17.5a) and (17.5b) with boundary conditions of the form (17.14). This result is a direct application of Theorem 12.6. More precisely, one has the following:

Proposition 17.1 (***local existence for the initial boundary value problem***) *Given an initial boundary value problem for Equations* (17.5a) *and* (17.5b) *with smooth initial data*

$$\big(\hat{\boldsymbol{\upsilon}}_\star(\underline{x}), \boldsymbol{\phi}_\star(\underline{x})\big), \qquad \text{on } \mathcal{S}_\star,$$

and inhomogeneous maximally dissipative boundary data

$$\phi_4 - c_1\phi_0 - c_2\bar{\phi}_0 = q, \qquad |c_1|^2 + |c_2|^2 \leq 1, \qquad \text{on } \mathscr{I},$$

with c_1, c_2, q smooth complex-valued functions on $\mathscr{I}$ and assuming that the required corner conditions at $\partial\mathcal{S}_\star$ between initial and boundary data are satisfied to any order, there exists $\tau_\bullet > 0$ such that the initial boundary value problem has a unique smooth solution $(\hat{\boldsymbol{\upsilon}}(\tau,\underline{x}), \boldsymbol{\phi}(\tau,\underline{x}))$ defined on

$$\mathcal{M}_{\tau_\bullet} \equiv [0,\tau_\bullet) \times \mathcal{S}.$$

Remark. Although the above result is local in time, it is nevertheless global in space. As already mentioned, existence on $D^+(\mathcal{S}_\star\setminus\partial\mathcal{S}_\star)$ follows from the standard Cauchy problem. The solutions away from the boundary and those close to the boundary are then patched together to render the full solution.

17.2.6 Propagation of the constraints

In order to transform the existence result given by Proposition 17.1 into an assertion about the Einstein field equations it is necessary to provide an analysis of the propagation of the constraints.

The subsidiary evolution system associated to the conformal evolution Equations (17.5a) and (17.5b) has been discussed in Proposition 13.4. The key structural feature of these subsidiary equations is that they are homogeneous in the zero quantities. A further crucial feature is that the equations for the zero quantities

$$\hat{\Sigma}_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}}, \quad \hat{\Xi}^{\boldsymbol{c}}{}_{\boldsymbol{dab}}, \quad \hat{\Delta}_{\boldsymbol{abc}}, \quad \delta_{\boldsymbol{a}}, \quad \gamma_{\boldsymbol{ab}}, \quad \varsigma_{\boldsymbol{ab}}$$

are all *transport equations*, and, accordingly, they do not give rise to boundary conditions on $\mathscr{I}$. For the zero quantity $\Lambda_{\boldsymbol{abc}}$ associated to the Bianchi identity, the subsidiary system implied by the boundary-adapted system contains no derivatives with respect to the coordinate x^3 and, thus, has a vanishing normal matrix; compare Equations (13.66a)–(13.66c). It follows that the subsidiary evolution equations require no boundary condition on $\mathscr{I}$. From the uniqueness result for initial boundary value problems, Theorem 12.5, if the conformal Einstein equations are satisfied on $\mathcal{S}$ – that is, the zero quantities vanish – then they are also satisfied on $\mathcal{M}_{\tau_\bullet}$. Combining this discussion with Proposition 8.3 one obtains the following existence result for the Einstein field equations:

Theorem 17.1 (***propagation of the constraints for the initial boundary value problem***) *Consider smooth anti-de Sitter-like initial data for the extended conformal Einstein field equations on a three-dimensional manifold $\mathcal{S}$ and boundary initial data of the form* (17.14) *on $\mathscr{I}$. Assume that the above data satisfy the required corner conditions to all orders on $\partial\mathcal{S}_\star = \mathcal{S}_\star \cap \mathscr{I}$. Then the solution of the initial boundary value problem given by Proposition 17.1 implies a solution to the extended conformal Einstein field equations on $\mathcal{M}_{\tau_\bullet}$. This solution, in turn, implies an anti-de Sitter-like solution to the vacuum Einstein field equations on*

$$\tilde{\mathcal{M}}_{\tau_\bullet} \equiv \mathcal{M}_{\tau_\bullet} \setminus \mathscr{I},$$

for which $\mathscr{I}$ represents the conformal boundary.

Remark. For an ***anti-de Sitter-like initial data set*** it is understood a collection of conformal fields satisfying the conformal constraint equations with the required anti-de Sitter asymptotic behaviour; see Section 11.7.

17.3 Covariant formulation of the boundary conditions

From a geometric point of view, the formulation of the boundary conditions in Proposition 17.1 is not satisfactory. The fields appearing in the maximally dissipative boundary conditions (17.14) are expressed with respect to a certain *boundary adapted gauge*. This gauge specification is an integral part of the boundary conditions: changes on the adapted boundary imply changes in the data. It is therefore important to recast the conditions (17.14), or at least

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a subclass thereof, in a covariant manner. In what follows, attention will be restricted to the subclass

$$\phi_4 - c\bar{\phi}_0 = q, \qquad c \quad \text{constant}, \ |c| \leq 1. \tag{17.15}$$

17.3.1 Space spinor split of the boundary data

To recast the boundary condition (17.15) in a covariant manner, it is first necessary to express the fields in terms of objects intrinsic to the conformal boundary $\mathscr{I}$. It is convenient to make use of a ***timelike spinor formalism*** based on the spacelike spinor

$$\rho^{\boldsymbol{AA'}} = \delta_{\boldsymbol{0}}{}^{\boldsymbol{A}}\delta_{\boldsymbol{0'}}{}^{\boldsymbol{A'}} - \delta_{\boldsymbol{1}}{}^{\boldsymbol{A}}\delta_{\boldsymbol{1'}}{}^{\boldsymbol{A'}},$$

as defined in Section 17.2.3, to project spinorial fields into $\mathscr{I}$ in analogy to the space spinor splits with respect to $\tau^{\boldsymbol{AA'}}$. The spinor $\rho^{\boldsymbol{AA'}}$ is the spinorial counterpart of the inward-pointing normal $\boldsymbol{\nu} = \boldsymbol{e_3}$ to $\mathscr{I}$. Notice, however, the normalisation $\rho_{\boldsymbol{AA'}}\rho^{\boldsymbol{AA'}} = -2$. Define the *space spinor version* $\tau_{\boldsymbol{AB}}$ of $\tau_{\boldsymbol{AA'}}$ as

$$\tau_{\boldsymbol{AB}} = \rho_{\boldsymbol{B}}{}^{\boldsymbol{B'}}\tau_{\boldsymbol{AB'}} = 2\delta_{(\boldsymbol{A}}{}^{\boldsymbol{0}}\delta_{\boldsymbol{B})}{}^{\boldsymbol{1}}.$$

Now, taking into account the decomposition of the spinorial counterpart of the Weyl spinor one can compute its ***electric*** and ***magnetic parts*** with respect to $\rho^{\boldsymbol{AA'}}$ as

$$E_{\boldsymbol{ABCD}} \equiv \frac{1}{2}\rho_{\boldsymbol{B}}{}^{\boldsymbol{A'}}\rho^{\boldsymbol{EE'}}\rho_{\boldsymbol{D}}{}^{\boldsymbol{C'}}\rho^{\boldsymbol{FF'}}d_{\boldsymbol{AA'EE'CC'FF'}} = \frac{1}{2}\big(\phi_{\boldsymbol{ABCD}} + \phi^{\ddagger}_{\boldsymbol{ABCD}}\big),$$

$$B_{\boldsymbol{ABCD}} \equiv \frac{1}{2}\rho_{\boldsymbol{B}}{}^{\boldsymbol{A'}}\rho^{\boldsymbol{EE'}}\rho_{\boldsymbol{D}}{}^{\boldsymbol{C'}}\rho^{\boldsymbol{FF'}}d^{*}_{\boldsymbol{AA'EE'CC'FF'}} = -\frac{\mathrm{i}}{2}\big(\phi_{\boldsymbol{ABCD}} - \phi^{\ddagger}_{\boldsymbol{ABCD}}\big),$$

with

$$\phi^{\ddagger}_{\boldsymbol{ABCD}} \equiv \rho_{\boldsymbol{A}}{}^{\boldsymbol{A'}}\rho_{\boldsymbol{B}}{}^{\boldsymbol{B'}}\rho_{\boldsymbol{C}}{}^{\boldsymbol{C'}}\rho_{\boldsymbol{D}}{}^{\boldsymbol{D'}}\bar{\phi}_{\boldsymbol{A'B'C'D'}}.$$

By construction $E_{\boldsymbol{ABCD}} = E_{(\boldsymbol{ABCD})}$ and $B_{\boldsymbol{ABCD}} = B_{(\boldsymbol{ABCD})}$.

The spinors $E_{\boldsymbol{ABCD}}$ and $B_{\boldsymbol{ABCD}}$ can be decomposed in a $1+2$ manner with respect to the spinor $\tau_{\boldsymbol{AB}}$. The subsequent discussion will be restricted to $B_{\boldsymbol{ABCD}}$, but an identical analysis can be carried out for $E_{\boldsymbol{ABCD}}$. This decomposition is best carried out using tensor frame components and then translating the result into spinors. One obtains

$$B_{\boldsymbol{ABCD}} = \mu_{\boldsymbol{ABCD}} + \mu_{\boldsymbol{AB}}\tau_{\boldsymbol{CD}} + \tau_{\boldsymbol{AB}}\mu_{\boldsymbol{CD}} + \frac{1}{4}\mu\big(3\tau_{\boldsymbol{AB}}\tau_{\boldsymbol{CD}} - 2\epsilon_{\boldsymbol{A}(\boldsymbol{C}}\epsilon_{\boldsymbol{D})\boldsymbol{B}}\big), \tag{17.16}$$

with the fields

$$\mu_{\boldsymbol{ABCD}} = \mu_{(\boldsymbol{ABCD})}, \qquad \mu_{\boldsymbol{AB}} = \mu_{(\boldsymbol{AB})}, \qquad \mu = \bar{\mu},$$

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satisfying

$$\tau^{\boldsymbol{AB}}\mu_{\boldsymbol{ABCD}} = 0, \qquad \tau^{\boldsymbol{AB}}\mu_{\boldsymbol{AB}} = 0.$$

The geometric interpretation of the various spinors follows from the above properties. By inspection, it can be shown that the only non-vanishing components of the spinor $\mu_{\boldsymbol{ABCD}}$ are given by $\overline{\mu_{\boldsymbol{1111}}} = \mu_{\boldsymbol{0000}}$. Similarly, for the rank-2 spinor $\mu_{\boldsymbol{AB}}$ one has the non-vanishing components and $\mu_{\boldsymbol{00}} = \overline{\mu_{\boldsymbol{11}}}$. From the definitions of the magnetic parts of $\phi_{\boldsymbol{ABCD}}$ it follows that

$$\mu_{\boldsymbol{1111}} = -\frac{\mathrm{i}}{2}\big(\phi_{\boldsymbol{1111}} - \bar{\phi}_{\boldsymbol{0'0'0'0'}}\big) \qquad \mu_{\boldsymbol{11}} = -\frac{\mathrm{i}}{2}\big(\phi_{\boldsymbol{0111}} - \bar{\phi}_{\boldsymbol{1'0'0'0'}}\big),$$
$$\mu = -\mathrm{i}\big(\phi_{\boldsymbol{0011}} - \bar{\phi}_{\boldsymbol{1'1'0'0'}}\big).$$

It follows from the above expressions and their analogues for $E_{\boldsymbol{ABCD}}$ that the boundary condition (17.15) can be rewritten in terms of the components of the spinors $E_{\boldsymbol{ABCD}}$ and $B_{\boldsymbol{ABCD}}$. Of particular interest are the cases

$$c = 1: \quad B_{\boldsymbol{1111}} = q, \tag{17.17a}$$

$$c = -1: \quad E_{\boldsymbol{1111}} = q. \tag{17.17b}$$

The Bianchi constraints at the conformal boundary

Now, assume that one is provided with boundary data in the form (17.17a) or (17.17b). A natural question is whether it is possible to recover the full spinor $E_{\boldsymbol{ABCD}}$ and, respectively, $B_{\boldsymbol{ABCD}}$. It is recalled that the conformal field equation

$$\nabla^{\boldsymbol{A}}{}_{\boldsymbol{A}'}\phi_{\boldsymbol{ABCD}} = 0$$

implies on $\mathscr{I}$ the constraint equations

$$\mathcal{D}^{\boldsymbol{PQ}}\eta_{\boldsymbol{PQAB}} = 0, \qquad \mathcal{D}^{\boldsymbol{PQ}}\mu_{\boldsymbol{PQAB}} = 0, \tag{17.18}$$

with $\mathcal{D}_{\boldsymbol{AB}} \equiv \rho_{(\boldsymbol{A}}{}^{\boldsymbol{A}'}\nabla_{\boldsymbol{B})\boldsymbol{A}'}$; see Section 11.4. The above equations are the spinorial versions of the conformal constraints (11.39f) and (11.39g). They can be decomposed by introducing the directional derivatives

$$\mathcal{P} \equiv \tau^{\boldsymbol{AA}'}\nabla_{\boldsymbol{AA}'}, \qquad \delta_{\boldsymbol{AB}} \equiv \tau_{(\boldsymbol{A}}{}^{\boldsymbol{Q}}\mathcal{D}_{\boldsymbol{B})\boldsymbol{Q}},$$

along, respectively, the direction dictated by the conformal geodesics threading the conformal boundary and the direction orthogonal to them. A direct computation gives

$$\mathcal{D}_{\boldsymbol{AB}} = \frac{1}{2}\tau_{\boldsymbol{AB}}\mathcal{P} + \delta_{\boldsymbol{AB}}.$$

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Combining this split with the decomposition (17.16) of the spinor B_{ABCD} one finds that the constraint Equations (17.18) imply the system

$$\begin{aligned} 2\mathcal{P}\mu + 4\delta^{AB}\mu_{AB} &= 2\mu_{AB}\mathcal{P}\tau^{AB} - 3\mu\mathcal{D}^{AB}\tau_{AB} \\ &\quad + 2\tau^{EF}\mathcal{D}^{AB}\mu_{ABEF}, \end{aligned} \tag{17.19a}$$

$$\begin{aligned} 4\mathcal{P}\mu_{CD} + 2\delta_{CD}\mu &= 4(\mu_{CD}\mathcal{D}^{EF}\tau_{EF} + \mu_{EF}\mathcal{D}^{EF}\tau_{CD}) - 3\mu\mathcal{P}\tau_{CD} \\ &\quad + 4(\delta_C{}^E\delta_D{}^F + \tau_{CD}\tau^{EF})\mathcal{D}^{AB}\mu_{ABEF}. \end{aligned} \tag{17.19b}$$

A similar system is satisfied by the components of E_{ABCD}. Direct inspection reveals that the above equations constitute a *linear symmetric hyperbolic system (intrinsic to $\mathscr{I}$)* for the fields μ and μ_{AB} if the field μ_{ABCD} is provided; that is, μ_{ABCD} plays the role of *source terms*. The terms involving derivatives with respect to the spinor field τ_{AB} appearing in the right-hand sides of the above equations can be simplified if one assumes a boundary-adapted gauge on $\mathscr{I}$.

The discussion of the previous paragraphs can be summarised in the following manner: suppose one is given boundary data on $\mathscr{I}$ of the form (17.17a) and suppose one knows the values of the fields μ and μ_{AB} on $\partial\mathcal{S}_\star$; then, at least in a neighbourhood of the edge $\partial\mathcal{S}_\star$, it is possible to determine the components μ and μ_{AB} by solving the hyperbolic system (17.19a) and (17.19b). A similar discussion holds for the electric part.

17.3.2 Prescribing the Cotton tensor of the conformal boundary

Despite the formal symmetry between the boundary conditions (17.17a) and (17.17b), the former condition possesses a much stronger geometric content. As a consequence of Equation (11.42), the magnetic part of the rescaled Weyl tensor corresponds, essentially, to the components of the Cotton tensor y_{ijk} of the intrinsic Lorentzian metric $\boldsymbol{\ell}$ of $\mathscr{I}$. Thus, one can ask whether, given the components y_{ijk} of a tensor on $\mathscr{I}$ with the symmetries of the Cotton tensor, it is possible to find a Lorentzian metric $\boldsymbol{\ell}$ on $\mathscr{I}$ such that y_{ijk} are the components, with respect to a *boundary-adapted frame*, of the Cotton tensor of $\boldsymbol{\ell}$. If this is possible, then, as a consequence of its conformal transformation properties, *one has obtained a way of reexpressing a subset of the general maximally dissipative boundary conditions for the conformal field equations in terms of the conformal structure on $\mathscr{I}$*. One has the following result, adapted from lemma 7.1 in Friedrich (1995):

Proposition 17.2 (***geometric formulation of boundary conditions***) *Suppose one has a solution to the extended conformal field equations with anti-de Sitter-like cosmological constant on $\mathcal{M}_{\tau_\bullet} = [0,\tau_\bullet)\times\mathcal{S}$ for $\tau_\bullet > 0$ for which $\mathscr{I} = [0,\tau_\bullet)\times\partial\mathcal{S}$ represents the conformal boundary. Let $\boldsymbol{g}$ denote the metric on $\mathcal{M}_{\tau_\bullet}$ obtained from the solution to the conformal field equations and let $\boldsymbol{\ell}$ denote the 3-metric induced on $\mathscr{I}$ by $\boldsymbol{g}$. Assume that the boundary-adapted conformal Gaussian gauge system can be extended to all of $\mathcal{M}_{\tau_\bullet}$. One then has:*

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(i) *Given the restriction to* $\partial\mathcal{S}_\star$ *of the data for the conformal Einstein field equations in the boundary-adapted gauge and given the conformal class* $[\boldsymbol{\ell}]$*, it is possible to compute the function* q *appearing in the boundary condition* (17.17a)*.*

(ii) *Conversely, given on* $\partial\mathcal{S}_\star$ *the restriction of the data for the conformal Einstein field equations in the boundary-adapted gauge and the boundary condition* (17.17a)*, it is possible to determine, in a unique manner, the conformal class* $[\boldsymbol{\ell}]$*.*

Proof To prove (i) it is observed that as a consequence of Lemma 17.1, the boundary-adapted conformal Gaussian gauge at the conformal boundary can be constructed by solving the conformal geodesic equations for the metric $\boldsymbol{\ell}$. Once the associated Weyl-propagated frame $\{\boldsymbol{e_i}\}$ has been obtained, one can directly compute the components $y_{\boldsymbol{ijk}}$ of the Cotton tensor. Using the discussion of the previous subsection one can, in turn, compute the function q appearing in the boundary condition (17.17a).

The proof of (ii) is much more involved and only a sketch of the main ideas will be provided. Here, one has to verify whether a given three-dimensional tensor is the Cotton tensor of a three-dimensional Lorentzian metric. In view of the Lorentzian nature of this problem, one can address this question by formulating a suitable initial value problem on $\mathscr{I}$ with data on $\partial\mathcal{S}_\star$ for the evolution equations implied by the structural equations on $\mathscr{I}$. Formulated in this manner one has a situation which is very similar to the Cauchy problem for the extended conformal Einstein field equations.

In what follows, let $\boldsymbol{D}$ denote the Levi-Civita covariant derivative of the metric $\boldsymbol{\ell}$, and let $\hat{\boldsymbol{D}}$ denote a Weyl connection in the conformal class of $\boldsymbol{\ell}$. As in the four-dimensional case, the connections are related to each other via a relation of the form $\hat{\boldsymbol{D}} - \boldsymbol{D} = \boldsymbol{S}(\boldsymbol{f})$, with $\boldsymbol{f}$ representing a three-dimensional covector and $\boldsymbol{S}$ the three-dimensional version of the transition tensor discussed in Section 5.2.1. Let $\{\boldsymbol{e_i}\}$ denote an $\boldsymbol{\ell}$-orthogonal frame on $\mathscr{I}$, and let $\hat{\gamma}_{\boldsymbol{i}}{}^{\boldsymbol{j}}{}_{\boldsymbol{k}}$ be the associated connection coefficients of the connection $\hat{\boldsymbol{D}}$. Moreover, let $\hat{l}_{\boldsymbol{ij}}$ denote the components of the Schouten tensor of the connection $\hat{\boldsymbol{D}}$. In analogy to the discussion of the conformal field equations, it is convenient to introduce a number of zero quantities encoding the ***structure equations*** to be satisfied by the various geometric fields:

$$\hat{\Sigma}_{\boldsymbol{i}}{}^{\boldsymbol{k}}{}_{\boldsymbol{j}}\boldsymbol{e_k} \equiv [\boldsymbol{e_i}, \boldsymbol{e_j}] - (\hat{\gamma}_{\boldsymbol{i}}{}^{\boldsymbol{k}}{}_{\boldsymbol{j}} - \hat{\gamma}_{\boldsymbol{j}}{}^{\boldsymbol{k}}{}_{\boldsymbol{i}})\boldsymbol{e_k},$$

$$\hat{\Xi}^{\boldsymbol{k}}{}_{\boldsymbol{lij}} \equiv \boldsymbol{e_i}(\hat{\gamma}_{\boldsymbol{j}}{}^{\boldsymbol{k}}{}_{\boldsymbol{l}}) - \boldsymbol{e_j}(\hat{\gamma}_{\boldsymbol{i}}{}^{\boldsymbol{k}}{}_{\boldsymbol{l}}) + \hat{\gamma}_{\boldsymbol{m}}{}^{\boldsymbol{k}}{}_{\boldsymbol{l}}(\hat{\gamma}_{\boldsymbol{j}}{}^{\boldsymbol{m}}{}_{\boldsymbol{i}} - \hat{\gamma}_{\boldsymbol{i}}{}^{\boldsymbol{m}}{}_{\boldsymbol{j}})$$
$$+ \hat{\gamma}_{\boldsymbol{j}}{}^{\boldsymbol{m}}{}_{\boldsymbol{l}}\hat{\gamma}_{\boldsymbol{i}}{}^{\boldsymbol{k}}{}_{\boldsymbol{m}} - \hat{\gamma}_{\boldsymbol{i}}{}^{\boldsymbol{m}}{}_{\boldsymbol{l}}\hat{\gamma}_{\boldsymbol{j}}{}^{\boldsymbol{k}}{}_{\boldsymbol{m}} - 2S_{\boldsymbol{l}[\boldsymbol{i}}{}^{\boldsymbol{km}}\hat{l}_{\boldsymbol{j}]\boldsymbol{m}},$$

$$\hat{\Delta}_{\boldsymbol{ijk}} \equiv \hat{D}_{\boldsymbol{i}}\hat{l}_{\boldsymbol{jk}} - \hat{D}_{\boldsymbol{j}}\hat{l}_{\boldsymbol{ik}} - y_{\boldsymbol{ijk}},$$

$$\Lambda_{\boldsymbol{j}} \equiv D^{\boldsymbol{i}}y_{\boldsymbol{ij}},$$

where

$$y_{ij} \equiv -\frac{1}{2}\epsilon_{j}{}^{kl} y_{ikl}, \qquad y_{i}{}^{i} = 0, \qquad y_{ij} = y_{ji},$$

is the so-called ***Bach tensor***. The zero quantity $\hat{\Sigma}_{i}{}^{k}{}_{j}$ encodes the vanishing of the torsion of the connection $\hat{D}$, $\hat{\Xi}^{k}{}_{lij}$ contains the relation between the geometric and algebraic curvatures (the Ricci identities), $\hat{\Delta}_{ijk}$ describes the second Bianchi identity for $\hat{D}$ while Λ_{j} corresponds to the so-called ***third Bianchi identity*** – the differential identity satisfied by the Bach tensor.

To obtain a hyperbolic reduction of the above equations one considers the conformal Gaussian system implied by the conformal geodesics on $\mathscr{I}$. Using arguments similar to the ones in the four-dimensional case one has

$$e_{i}{}^{\alpha} = \delta_{i}^{\alpha}, \qquad \hat{\gamma}_{0}{}^{k}{}_{j} = 0, \qquad \hat{l}_{0j} = 0, \tag{17.20}$$

and one considers the *evolution equations*

$$\hat{\Sigma}_{0}{}^{k}{}_{j} e_{k} = 0, \qquad \hat{\Xi}^{k}{}_{l0j} = 0, \qquad \hat{\Delta}_{0jk} = 0, \qquad \hat{\Lambda}_{j} = 0. \tag{17.21}$$

Taking into consideration the gauge conditions (17.20), it can be verified that the first three equations in (17.21) are transport equations on $\mathscr{I}$. The fourth equation requires a more careful discussion: using the solution to the conformal constraint equations as given by Equation (11.42) some components of y_{ij} can be expressed in terms of the boundary conditions; for the remaining components one has that Equations (17.19a) and (17.19b) imply a symmetric hyperbolic system. Thus, one has obtained a symmetric hyperbolic system for the fields $e_{i}{}^{\alpha}$, $\hat{\gamma}_{i}{}^{k}{}_{j}$, $\hat{l}_{ij}$ and for the components of y_{ij} not determined by the boundary conditions. Initial data on $\partial\mathcal{S}_{\star}$ for these fields can be computed from the restriction to $\partial\mathcal{S}_{\star}$ of the initial data for the conformal evolution equations. Hence, using the general theory of symmetric hyperbolic systems as discussed in Chapter 12, one obtains a solution to Equations (17.21) in a neighbourhood $\mathcal{U}$ in $\mathscr{I}$ of $\partial\mathcal{S}_{\star}$. To show that this solution implies, in turn, a solution to the equations

$$\hat{\Sigma}_{i}{}^{k}{}_{j} e_{k} = 0, \qquad \hat{\Xi}^{k}{}_{lij} = 0, \qquad \hat{\Delta}_{ijk} = 0, \qquad \Lambda_{j} = 0, \qquad \text{on } \mathcal{U}$$

provided that they are satisfied at $\partial\mathcal{S}$, one needs to discuss the *propagation of the constraints* along the lines of Section 13.4.5. The resulting frame $\{e_{i}\}$ can be used to construct on $\mathcal{U} \subset \mathscr{I}$ a Lorentzian metric $\boldsymbol{\ell}$. This metric characterises the conformal class of the intrinsic metric of the conformal boundary. □

Reflective boundary conditions

An important class of boundary conditions covered by the prescription (17.17a) is that of the so-called ***reflective boundary conditions***. These correspond to the particular choice of $q = 0$ so that one has

$$\phi_{11111} = \bar{\phi}_{0'0'0'0'}, \qquad \text{on } \mathscr{I}.$$

In what follows, this boundary condition will be supplemented by the conditions

$$\phi_{\mathbf{0111}} = \bar{\phi}_{\mathbf{0'0'0'1'}}, \qquad \phi_{\mathbf{0011}} = \bar{\phi}_{\mathbf{0'0'1'1'}}, \qquad \text{on } \partial\mathcal{S}_\star.$$

Accordingly, from the discussion in Section 17.3.1 it follows that $B_{\boldsymbol{ABCD}} = 0$ on $\partial\mathcal{S}_\star$. Furthermore, using the interior evolution system (17.19a) and (17.19b) one has

$$B_{\boldsymbol{ABCD}} = 0, \qquad \text{on } \mathscr{I}.$$

As $B_{\boldsymbol{ABCD}}$ corresponds to the Cotton tensor of $\mathscr{I}$, it follows that *reflective boundary conditions together with some supplementary conditions at the edge imply that the intrinsic metric on* $\mathscr{I}$ *is conformally flat.*

As pointed out in Friedrich (2014a), despite the above neat geometric characterisation of reflective boundary conditions, if one wants to construct a smooth solution to the initial boundary value problem, one still needs to satisfy an infinite hierarchy of corner conditions. Whether this requirement is compatible with the known procedures for constructing anti-de Sitter-like initial data remains an open question.

Comparison with other initial boundary value problems for the Einstein field equations

Initial boundary value problems in general relativity arise in a natural manner in numerical applications. There exists a number of treatments of the well-posedness of this type of partial differential equation problem for the Einstein field equations; see, for example, Friedrich and Nagy (1999) and Kreiss et al. (2009). The approach and formulation of the Einstein equations considered in the former reference are similar to the ones discussed in this book.

The analysis in Friedrich and Nagy (1999) makes use of a frame formulation of the Einstein field equations. The equations employed in this reference can be obtained from the standard conformal Einstein field equations discussed in Section 8.3.1 by setting $\Xi = 1$. Given these equations, the question is what type of boundary data need to be prescribed on a, in principle, arbitrary, timelike hypersurface to obtain a well-posed initial boundary value problem and to ensure the propagation of the constraints. It turns out that the allowed boundary data are essentially expressed as a combination of components of the Weyl tensor (with respect to a boundary adapted frame) of the form given in Equation (17.14).

Despite these parallels, the situation of the initial boundary value problem analysed in Friedrich and Nagy (1999) and the one discussed in this chapter differ in a key aspect: the boundary hypersurface in anti-de Sitter spacetimes has a *canonic* character. As a consequence, it is possible to formulate covariant boundary conditions, and one ends up with a setting where ***geometric uniqueness*** of the solutions can be ensured. In Friedrich and Nagy (1999) it was not possible to obtain a geometric formulation of the boundary conditions on the timelike hypersurface. Thus, they remain tied to the prescription of the boundary-adapted gauge. As geometric uniqueness cannot be asserted, it is, in principle, not possible to determine whether two seemingly different boundary conditions will lead to the same spacetime, modulo diffeomorphisms. A further discussion can be found in Friedrich (2009).

17.4 Other approaches to the construction of anti-de Sitter-like spacetimes

The analysis of this section has been concerned with the construction of four-dimensional anti-de Sitter-like spacetimes by means of an initial boundary value problem for the conformal Einstein field equations. There are, however, other approaches to this problem if, for example, one assumes the existence of a *static Killing vector* on the spacetime. The assumption of staticity is a strong one and renders results of a global nature. As an example of this type of statement one has the following theorem from Anderson et al. (2002):

Theorem 17.2 (***existence of static anti-de Sitter-like spacetimes***) *Let $\boldsymbol{\ell}$ denote a smooth strictly globally static Lorentzian metric of non-negative scalar curvature on $\mathbb{R} \times \mathbb{S}^2$. Then $(\mathbb{R} \times \mathbb{S}^2, \boldsymbol{\ell})$ is the conformal boundary of a complete strictly globally static vacuum Lorentzian metric on $\mathbb{R}^4$ with anti-de Sitter-like cosmological constant.*

A ***strictly globally static spacetime*** is a spacetime containing an everywhere timelike vector which is orthogonal to the level sets of a globally defined time function. The proof of this result relies on the use of the Fefferman-Graham obstruction tensor; see Fefferman and Graham (1985, 2012). Related to the above theorem is the *rigidity result* given in Anderson (2006), in which it is shown that complete non-singular anti-de Sitter-like spacetimes with a globally stationary conformal infinity and an asymptotically stationary *bulk* must be globally stationary. This result seems to suggest the instability of anti-de Sitter-like spacetimes, at least for certain types of boundary conditions. This expectation has been reinforced by the evidence of turbulent instability observed in numerical simulations of spherically symmetric solutions of the Einstein-scalar field system with anti-de Sitter-like boundary conditions reported by Bizon and Rostworowski (2011).

17.5 Further reading

The approach to the construction of anti-de Sitter-like spacetimes discussed in this chapter has been adapted from the seminal analysis in Friedrich (1995). Boundary conditions for a range of test fields in the anti-de Sitter spacetime have been studied in Ishibashi and Wald (2004). General properties of the exact anti-de Sitter spacetime are examined in detail in Griffiths and Podolský (2009), while properties of anti-de Sitter-like spacetimes are discussed in Henneaux and Teitelboim (1985) and Frances (2005). An issue which has not been touched on in this chapter is that of the definition of the mass for anti-de Sitter-like spacetimes. Conformal approaches to this question have been discussed, for example, in Ashtekar and Magnon (1984) and Ashtekar and Das (2000). Readers interested in a discussion of the issue of the stability/instability of the anti-de Sitter spacetime are referred to the reviews by Bizon (2013) and Maliborski and Rostworowski (2013) and references within.

A considerable part of the interest on anti-de Sitter-like spacetime stems from the so-called AdS/CFT correspondence; see, for example, Maldacena (1998), Witten (1998) and Witten and Yau (1999). A good discussion of the issues involved from a mathematician's point of view are presented in Anderson (2005b).