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ON RATIONAL SUBDIVISIONS OF POLYHEDRA WITH RATIONAL VERTICES

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Introduction. This short paper is devoted to the proof of a single theorem, which, in its simplest form, asserts that if Q is a polyhedron in $\mathbf{R}^n$ which can be expressed as the union of finitely many convex polytopes whose vertices are at rational points in $\mathbf{R}^n$, and if $\mathscr{S}$ is a simplicial subdivision of Q, then there is an isomorphic simplicial subdivision $\mathscr{S}'$ of Q in which all vertices are at rational points.

In a subsequent paper [**1**], this theorem is used in generalising known results concerning finitely generated vector lattices to the context of finitely generated lattice-ordered Abelian groups.

Preliminaries. For basic definitions relating to polyhedra and simplicial presentations of polyhedra see Stallings [**3**] or Glaser [**2**].

Let P be a polyhedron, $\mathscr{S}$ a simplicial presentation of P, and x a vertex of $\mathscr{S}$. The *star* of x in $\mathscr{S}$ is the subcomplex of $\mathscr{S}$ consisting of all simplices which contain x as a vertex, together with all their faces. The *link* of x in $\mathscr{S}$ is the subcomplex of $\mathscr{S}$ consisting of all simplices which belong to the star of x in $\mathscr{S}$, but do not contain x as a vertex. Following common convention, the terms 'star' and 'link' will also be used to refer to the polyhedral realisations of these subcomplexes.

A polyhedron P is *rational* if P can be expressed as a finite union of convex polytopes $Q_1, Q_2, \ldots, Q_k$ such that for each $i \leqq k$ the vertices of Q_i are at rational points.

If S is a subset of $\mathbf{R}^n$, the notation conv S is used to denote the convex hull of S (the intersection of all convex subsets of $\mathbf{R}^n$ which contain S), and the notation aff S is used to denote the affine hull of S (the intersection of all affine subspaces of $\mathbf{R}^n$ which contain S).

If C is a convex set in $\mathbf{R}^n$ and x is a point of C, then x is said to be in the *relative interior* of C if there is an open neighbourhood U of x in $\mathbf{R}^n$ such that $U \cap \operatorname{aff} C \subseteq C$.

THEOREM 1. *Let Q be a rational polyhedron in $\mathbf{R}^n$, and let $\mathscr{S}$ be a simplicial presentation of Q with vertices at the points $v_1, v_2, \ldots, v_k$. For any $\epsilon > 0$, there is a set of rational points $v_1', v_2', \ldots, v_k'$ and a simplicial presentation $\mathscr{S}'$ of Q with vertices $v_1', v_2', \ldots, v_k'$ such that*

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(i) $|v_i - v_i'| < \epsilon$ *for* $i = 1, 2, \ldots, k$, *and*

(ii) *the map sending the vertex* v_i *of* $\mathscr{S}$ *to the vertex* v_i' *of* $\mathscr{S}'$ *defines an isomorphism between the simplicial presentations* $\mathscr{S}$ *and* $\mathscr{S}'$ *of* Q.

Proof. Let $\mathscr{T}$ be an abstract simplicial complex, and V the set of vertices of $\mathscr{T}$. A map $f : V \to \mathbf{R}^n$ defines a realisation of $\mathscr{T}$ in $\mathbf{R}^n$ if for any subsets A and B of V in $\mathscr{T}$ the relation

$$\operatorname{conv} f(A) \cap \operatorname{conv} f(B) = \operatorname{conv} f(A \cap B)$$

holds in $\mathbf{R}^n$. It is easy to show that the map $f : V \to \mathbf{R}^n$ defines a realisation of $\mathscr{T}$ in $\mathbf{R}^n$ if and only if, for any two disjoint subsets A and B of V in $\mathscr{T}$, the subsets $\operatorname{conv} f(A)$ and $\operatorname{conv} f(B)$ in $\mathbf{R}^n$ have empty intersection. In particular, if the set of functions $V \to \mathbf{R}^n$ is considered as an Euclidean space in the natural way, then the set of functions defining a realisation of $\mathscr{T}$ in $\mathbf{R}^n$ is an open subset of this space. By virtue of this fact, if $\mathscr{S}$ is a (geometrical) simplicial complex in $\mathbf{R}^n$, with vertices $v = v_0, v_1, \ldots, v_k$ there is an open neighbourhood U of v such that if $v' \in U$, then $v', v_1, \ldots, v_k$ are the vertices of a simplicial complex $\mathscr{S}'$ isomorphic with $\mathscr{S}$ via the map which sends v to v' and v_i to v_i for $i = 1, 2, \ldots, k$. The simplicial complex $\mathscr{S}'$ is then said to be *obtained* from $\mathscr{S}'$ by moving the vertex v to v'.

Let P be a polyhedron in $\mathbf{R}^n$, the union of convex polytopes $P_1, \ldots, P_t$, and let X denote the union of the sets of vertices of $P_1, \ldots, P_t$. The convex polytope S will be called a *slice of* P (relative to X) if S is contained in P and the vertices of S belong to X. For each point x of P the symbol $\mathrm{Sl}\,(x)$ will denote the (non-empty) set of slices of P which contain x, and $\bigcap \mathrm{Sl}(x)$ the intersection of all slices in $\mathrm{Sl}(x)$. For each x in P, the set of points y of P such that $\mathrm{Sl}(x) = \mathrm{Sl}(y)$ will be called the *scope of* x (relative to X).

LEMMA 1. *The scope of* x *is a relatively open subset of* $\operatorname{aff}\,(\bigcap \mathrm{Sl}(x))$. *That is, if* y *lies in the scope of* x *there is an open neighbourhood* U *of* y *in* $\mathbf{R}^n$ *such that* $U \cap \operatorname{aff}\,(\bigcap \mathrm{Sl}(x))$ *is contained in the scope of* x. *In particular, the scope of* x *is* $\{x\}$ *if and only if* $\{x\} = \bigcap \mathrm{Sl}(x)$.

Proof. Note first that x (and hence any point in the scope of x) is relatively interior to $\bigcap \mathrm{Sl}(x)$. To see this, let $S_1, \ldots, S_m$ be the slices of P which contain x as a relatively interior point. If $S \in \mathrm{Sl}(x)$ then S contains S_i for some $i \leqq m$, hence $\bigcap \mathrm{Sl}(x) = \bigcap_{i=1} S_i$. Thus

$$x \in \bigcap_{i=1}^{m} \operatorname{rel\,int} S_i \subseteq \operatorname{rel\,int} \bigcap_{i=1}^{m} S_i = \operatorname{rel\,int} \bigcap \mathrm{Sl}\,(x).$$

Now let y be in the scope of x, let V be an open neighbourhood of y in $\mathbf{R}^n$ such that $V \cap \operatorname{aff}\,(\bigcap \mathrm{Sl}(x))$ is contained in $\bigcap \mathrm{Sl}(x)$, and let $T_1, \ldots, T_s$ be the slices of P which do not contain y. Since each T_i is closed, there is an open neighbourhood W of y such that $W \cap T_i$ is empty for $i = 1, 2, \ldots, s$.

Let $U = V \cap W$. It is easy to verify that $U \cap \text{aff}\ (\bigcap \mathrm{Sl}(x))$ is contained in the scope of x, as required.

If $\bigcap \mathrm{Sl}(x) = \{x\}$ then the scope of x is $\{x\}$ trivially, since the scope of x is a subset of $\bigcap \mathrm{Sl}(x)$. Conversely, suppose that $\bigcap \mathrm{Sl}(x) \neq \{x\}$. Then there is a point y distinct from x in $\bigcap \mathrm{Sl}(x)$, and all points of the line segment $[x, y]$ sufficiently close to x lie in the scope of x, since the scope of x is a relatively open subset of aff $\bigcap \mathrm{Sl}(x)$. This completes the proof of Lemma 1.

Now let $\mathscr{S}$ be a simplicial presentation of the polyhedron P introduced above, and let x be a vertex of $\mathscr{S}$ such that the scope of x is not $\{x\}$. Then aff $(\bigcap \mathrm{Sl}(x))$ is an affine subspace of $\mathbf{R}^n$ of dimension at least one, and by Lemma 1, there is an open n-ball W containing x, such that $W \cap \text{aff}\ (\bigcap \mathrm{Sl}(x))$ is contained in the scope of x.

Let V be an open n-ball containing x in $\mathbf{R}^n$ with the property that it $x' \in V$, then a simplicial complex $\mathscr{S}'$, isomorphic with $\mathscr{S}$, is obtained from $\mathscr{S}$ by moving the vertex x to x'. Note that, since x' and the link of x in $\mathscr{S}$ are joinable, V does not meet the link of x in $\mathscr{S}$.

Let $U = V \cap W \cap \text{aff}\ (\bigcap \mathrm{Sl}(x))$.

LEMMA 2. *If P, x, $\mathscr{S}$ and U are as defined above, and if x' $(\neq x)$ is a point of U, the simplicial complex obtained from $\mathscr{S}$ by moving the vertex x to x' is also a simplicial presentation of P.*

Proof. It will suffice to prove that the star of x in $\mathscr{S}$ and the star of x' in $\mathscr{S}'$ coincide as polyhedra in $\mathbf{R}^n$, for the vertices of $\mathscr{S}$ other than x are common to $\mathscr{S}$ and $\mathscr{S}'$. Moreover, it suffices to prove that the star of x in $\mathscr{S}$ contains the star of x' in $\mathscr{S}'$, for x and x' stand in symmetric relation.

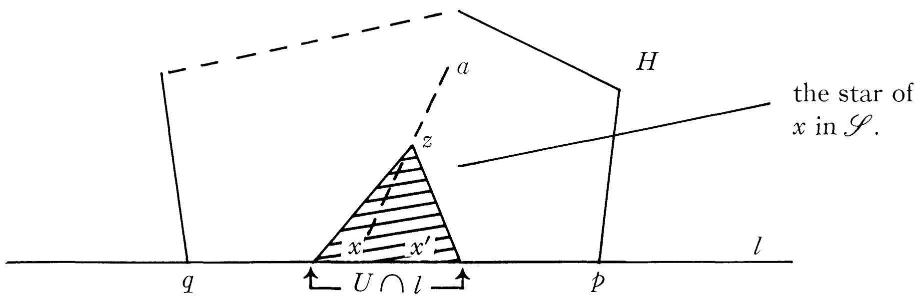
The ray xy with vertex x in P is said to be *locally in P* if there is a point q distinct from x on xy such that the line segment $[x, q]$ is contained in P. The point z on xy is said to be *beyond* y on xy if y is on the line segment $[x, z]$.

Because x and the link of x in $\mathscr{S}$ are joinable, it follows that a ray xy with vertex x can meet the link of $\mathscr{S}$ in at most one point, and meets the link of x if and only if xy is locally in P. Since the join of x and the link of x in $\mathscr{S}$ is the star of x in $\mathscr{S}$, the point $y(\neq x)$ of P is in the star of x in $\mathscr{S}$ if and only if the ray xy meets the link of x in $\mathscr{S}$ in a point beyond y.

Now let l be the affine hull of x and x'. Then l is a 1-dimensional affine subspace of aff $(\bigcap \mathrm{Sl}(x))$, and $U \cap l$ is an open interval of l containing x and x' and contained in the scope of x. In particular, the two rays with vertex x which are contained in l are locally in P, and meet the link of x in $\mathscr{S}$. Suppose that the line segment $[x, x']$ meets the link of x in $\mathscr{S}$ in p if produced beyond x' and in q if produced beyond x. Both p and q are outside the line segment $[x, x']$, for U is disjoint from the link of x in $\mathscr{S}$. Thus, if y is any point of the line segment $[p, q]$, then the ray $x'y$ meets the link of x' in $\mathscr{S}'$ (which is also the link of x in $\mathscr{S}$) in a point beyond y, and y belongs to the star of x' in $\mathscr{S}'$. Hence all points of l which belong to the star of x in $\mathscr{S}$ also belong to the star of x' in $\mathscr{S}'$.

Let a lie in the star of x in $\mathscr{S}$, but not on l. There is a unique plane Π containing x, x' and a, and l divides Π into two open half planes. Let H denote that open half plane, bounded by l, which contains a.

Let $T_1, \ldots, T_m$ be the slices of P which do not contain x, and let N be an open neighbourhood of x in $\mathbf{R}^n$ such that N and T_i have empty intersection for $i = 1, 2, \ldots, m$. Let z be a point common to N and the line segment $[x, a]$, other than x itself. Since a is in the star of x in $\mathscr{S}$, z is in P, and there is a slice S of P which contains z. Since $z \in N$, S necessarily contains x; hence S contains the scope of x in P. In particular, S contains $U \cap l$, an open interval of l containing x and x', and, being convex, must contain the convex hull of z and $U \cap l$. It follows that if b is a point of H, the ray xb with vertex x is locally in P, and meets the link of x in $\mathscr{S}$. Thus, the intersection of H with the link of x in $\mathscr{S}$, together with the line segment $[p, q]$ form a polyhedral 1-sphere in Π.



Let K be the polyhedral 2-cell bounded by this 1-sphere. Since a lies in the star of x in $\mathscr{S}$, a necessarily lies in K.

Consider the ray $x'a$ with vertex x'. Since a lies in K, and there is a point b beyond a on $x'a$ not in K, there is necessarily a point y, beyond a on the ray $x'a$, common to $x'a$ and the boundary of K. The point y belongs to the link of x in $\mathscr{S}$, for a does not lie on $[p, q]$. On the other hand, x' and the link of x in $\mathscr{S}$ are joinable, so that y is the unique point of intersection of the ray $x'a$ and the link of x in $\mathscr{S}$. Since the link of x in $\mathscr{S}$ is the link of x' and $\mathscr{S}'$ it follows that a belongs to the star of x' in $\mathscr{S}'$. This completes the proof of Lemma 2.

To complete the proof of Theorem 1, let the polyhedron Q referred to in the statement of the theorem be expressed as a union of convex polytopes $Q_1, Q_2, \ldots, Q_t$ having vertices at rational points. Let X be the union of the sets of vertices of $Q_1, Q_2, \ldots, Q_t$.

If x is an irrational vertex of the simplicial presentation $\mathscr{S}$ of Q, then the scope of x cannot consist solely of the point x, for if $\{x\} = \bigcap \mathrm{Sl}(x)$, then x is necessarily a rational point. Hence, by Lemma 2, a new simplicial presentation $\mathscr{S}'$ of Q isomorphic with $\mathscr{S}$, is obtained from $\mathscr{S}$ by moving x to any point x' in a relative neighbourhood U of x in the affine hull of $\bigcap \mathrm{Sl}(x)$. Since, in this instance, the affine hull of $\bigcap \mathrm{Sl}(x)$ is a rational subspace of $\mathbf{R}^n$, rational points are dense in the neighbourhood U of x. Thus, given $\epsilon > 0$, there is a simplicial presentation $\mathscr{S}'$ of Q, with vertices at rational points $v_1', \ldots, v_k'$ such that $|v_i - v_i'| < \epsilon$ for $i = 1, 2, \ldots, k$, and such that the map sending v_i to v_i' for $i = 1, 2, \ldots, k$ defines an isomorphism between the simplicial presentations $\mathscr{S}$ and $\mathscr{S}'$ of Q.

COROLLARY. *Suppose that P and Q are rational polyhedra in $\mathbf{R}^n$, and that $\mathscr{S}$ and $\mathscr{T}$ are simplicial presentations of P and Q respectively such that $\mathscr{S}$ is a subcomplex of $\mathscr{T}$.*

Then there are simplicial presentations $\mathscr{S}'$ and $\mathscr{T}'$ of P and Q respectively, with vertices at rational points, such that $\mathscr{S}$ and $\mathscr{S}'$ are isomorphic, $\mathscr{T}$ and $\mathscr{T}'$ are isomorphic, and $\mathscr{S}'$ is a subcomplex of $\mathscr{T}'$.

Proof. Let $Q_1, Q_2, \ldots, Q_t$ be a finite set of convex polytopes with rational vertices such that $Q = \bigcup_{i=1}^{t} Q_i$ and $P = \bigcup_{i=1}^{r} Q_i$ for some $r \leqq t$. Let X be the union of the sets of vertices of $Q_1, Q_2, \ldots, Q_t$ and Y the union of the sets of vertices of $Q_1, Q_2, \ldots, Q_r$, so that $Y \subseteq X$.

Suppose that $v_1, v_2, \ldots, v_m$ are the vertices of $\mathscr{S}$, and $v_{m+1}, v_{m+2}, \ldots, v_n$ are the remaining vertices of $\mathscr{T}$. If $i \leqq m$, then $v_i \in Q$, and the scope of v_i is defined relative to X and relative to Y. Moreover, it is clear that the scope of v_i relative to X is a subset of the scope of v_i relative to Y in this case.

Let $v_1', v_2', \ldots, v_m'$ be vertices of a simplicial presentation $\mathscr{T}'$ of Q, isomorphic with $\mathscr{T}$, having rational vertices, constructed as in the proof of Theorem 1. For each i, the rational point v_i' lies in the scope of v_i relative to X. In particular v_i' lies in the scope of v_i relative to Y for $i = 1, 2, \ldots, m$. Thus, by Lemma 2, the subcomplex $\mathscr{S}'$ of $\mathscr{T}'$ corresponding to $\mathscr{S}$, having vertices $v_1', v_2', \ldots, v_m'$ is itself a simplicial presentation of Q, giving the required result.

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