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Polycharacters of Cocommutative Hopf Algebras

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Abstract. In this paper we extend a well-known theorem of M. Scheunert on skew-symmetric bicharacters of groups to the case of skew-symmetric bicharacters on arbitrary cocommutative Hopf algebras over a field of characteristic not 2. We also classify polycharacters on (restricted) enveloping algebras and bicharacters on divided power algebras.

1 Introduction

In the paper [1] the authors looked into the structure of bicharacters of certain Hopf algebras, which are close to coquasitriangular. One of the results proven there states that any skew-symmetric bicharacter $\beta\colon H \otimes H \to R$ where R is a commutative algebra and H is a Hopf algebra over a field $\mathcal{K}$ can be written as a skew-symmetrization of a 2-cocycle $\sigma\colon H \otimes H \to R$ in the following cases (see the precise meaning below):

1. H is a pointed cocommutative Hopf algebra over a field of characteristic 0;
2. H is a Hopf algebra of the form $H = H_0 \# \mathcal{K}[G]$, where H_0 is generated by primitive elements.

One of the main results of this paper is the proof of this result in the case of arbitrary cocommutative Hopf algebras even in the case of positive characteristic p provided that $p \neq 2$. This is our Corollary 4.7 to Theorem 4.1. Among the other results of [1] one can find the description of the group of all (symmetric, skew-symmetric) bicharacters of certain connected Hopf algebras such as enveloping algebras and restricted enveloping algebras over a field of positive characteristic. Here we find the description of this group for yet another example of co-commutative Hopf algebras, the divided power algebras; see Theorem 3.9. It should be mentioned that a complete description of bicharacters on the finite abelian groups is given by A. Zolotykh [7].

We refer the reader to [1] for some applications of the results about bicharacters to certain generalized Lie algebra structures on H-comodule algebras. Actually (see all definitions in [1]), it follows from our results in the present paper that, given such an H-comodule algebra L, with a generalized Lie structure defined by a skew-symmetric bicharacter $\beta\colon H \otimes H \to R$, where H is now an *arbitrary* cocommutative Hopf algebra over a field of characteristic different from 2, there exists a 2-cocycle σ such that L, twisted by σ, is either an ordinary Lie algebra or an ordinary Lie superalgebra.

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2 Definitions and Preliminary Results

Let $\mathcal{K}$ be a field, H a cocommutative Hopf algebra over $\mathcal{K}$, R a commutative algebra over $\mathcal{K}$. The following (Sweedler's or sigma) notation is standard throughout the paper. Given an element h in a Hopf algebra H, it is conventional to denote the value of the coproduct map Δ on h by

$$\Delta h = \sum_{(h)} h_1 \otimes h_2 \text{ or simply } \Delta h = \sum h_1 \otimes h_2 \text{ or even } \Delta h = h_1 \otimes h_2.$$

In the following definition and some other places of the paper we write the components of the tensor $\Delta h^i = \sum (h^i)_1 \otimes (h^i)_2$ simply as h^i_1 and h^i_2.

Definition 2.1 An r-multilinear function $\alpha\colon H \times \cdots \times H \to R$ is called an r-character if it is convolution invertible and, for any $i = 1, \dots, r$, the following two conditions are satisfied:

$$\alpha(h^1, \dots, h^{i-1}, 1, h^{i+1}, \dots, h^r) = \varepsilon(h^1 \cdots h^{i-1} h^{i+1} \cdots h^r), \tag{1}$$

$$\begin{aligned} &\alpha(h^1, \dots, h^{i-1}, kl, h^{i+1}, \dots, h^r) \\ &\quad = \sum \alpha(h^1_1, \dots, h^{i-1}_1, k, h^{i+1}_1, \dots, h^r_1)\, \alpha(h^1_2, \dots, h^{i-1}_2, l, h^{i+1}_2, \dots, h^r_2), \end{aligned} \tag{2}$$

for all $h^1, \dots, h^{i-1}, k, l, h^{i+1}, \dots, h^r \in H$.

It is easy to verify that all r-characters form a group under the convolution product (since H is cocommutative). We will denote this group by $\mathrm{Ch}^r(H, R)$.

Let S_r be the symmetric group on r elements. Then for any $\pi \in S_r$ we can consider an r-character

$$(\pi \circ \alpha)(h^1, \dots, h^r) = \alpha(h^{\pi(1)}, \dots, h^{\pi(r)}),$$

for any $h^1, \dots, h^r \in H$.

Definition 2.2 An r-character α is called *symmetric* if, for any $\pi \in S_r$, we have $\pi \circ \alpha = \alpha$. All symmetric r-characters form a group, which will be denoted by $\mathrm{Sym}^r(H, R)$.

Definition 2.3 An r-character α is called *skew-symmetric* if for any $\pi \in S_r$, we have $\pi \circ \alpha = \alpha^{\operatorname{sgn} \pi}$. All skew-symmetric r-characters form a group, which we denote $\mathrm{Alt}^r(H, R)$.

In the case of $r = 1$ we simply obtain the algebra maps from H to R, which we will simply call *characters*. In the case of $r = 2$ our definitions agree with those of bicharacters and skew-symmetric bicharacters given in [1].

The situation with the complete description of the group of all r-characters for the most popular cocommutative Hopf algebras is as follows.

If $H = \mathcal{K}G$ is a group algebra, then the r-characters of H with values in R are in one-to-one correspondence with the group r-characters of G with values in the group $U(R)$ of invertible elements of R.

The structure of the r-characters of (restricted) universal enveloping algebras will be described explicitly in the next section. For $r = 2$ the complete descriptions are given in [1].

Now we consider a Hopf algebra H with the coalgebra structure isomorphic to a divided power coalgebra D, that is, H possesses a $\mathcal{K}$-basis of the form $x^{(\mathbf{n})}$, where $\mathbf{n} = (n_1, \dots, n_d)$, n_j are nonnegative integers, for $j = 1, \dots, d$, such that

$$\Delta x^{(\mathbf{n})} = \sum_{\mathbf{k}+\mathbf{l}=\mathbf{n}} x^{(\mathbf{k})} \otimes x^{(\mathbf{l})}. \tag{3}$$

Then $H^* \cong \mathcal{K}[[t_1, \dots, t_d]] = \mathcal{K}[[\mathbf{t}]]$ as an algebra, and the multiplication map $H \otimes H \to H$ induces the homomorphism $\Delta\colon H^* \cong \mathcal{K}[[\mathbf{t}]] \to (H \otimes H)^* \cong \mathcal{K}[[\mathbf{u}, \mathbf{v}]]$. In its turn, Δ determines a formal group law $\mathbf{u} \circ \mathbf{v} = \Delta(\mathbf{t}) = (\Delta t_1, \dots, \Delta t_d)$ so that we obtain $(\Delta\phi)(\mathbf{u}, \mathbf{v}) = \phi(\mathbf{u} \circ \mathbf{v})$, for any $\phi \in \mathcal{K}[[t]]$. In fact, given a formal group law $\circ$, one can recover the corresponding divided power algebra H as the algebra of pointed distributions on the formal group (see [3]).

We have a natural isomorphism of $\mathcal{K}$-spaces $\operatorname{Hom}(H^{\otimes r}, R) \cong R[[\mathbf{t}^1, \dots, \mathbf{t}^r]]$, which sends $f \in \operatorname{Hom}(H^{\otimes r}, R)$ to

$$\hat{f}(\mathbf{t}^1, \dots, \mathbf{t}^r) = \sum_{(\mathbf{n}^1, \dots, \mathbf{n}^r)} f(x^{(\mathbf{n}^1)}, \dots, x^{(\mathbf{n}^r)})(\mathbf{t}^1)^{\mathbf{n}^1} \cdots (\mathbf{t}^r)^{\mathbf{n}^r}. \tag{4}$$

Actually, this is an algebra isomorphism by (3) and the definition of the convolution product in $\operatorname{Hom}(H^{\otimes r}, R)$.

Using this isomorphism we can rewrite in terms of the formal group law $\circ$ the conditions (1), (2) of $\alpha \in \operatorname{Hom}(H^{\otimes r}, R)$ being an r-character:

$$\hat{\alpha}(\mathbf{t}^1, \dots, \mathbf{t}^{i-1}, 0, \mathbf{t}^{i+1}, \dots, \mathbf{t}^r) = 1, \tag{5}$$

$$\begin{aligned} &\hat{\alpha}(\mathbf{t}^1, \dots, \mathbf{t}^{i-1}, \mathbf{u} \circ \mathbf{v}, \mathbf{t}^{i+1}, \dots, \mathbf{t}^r) \\ &\quad = \hat{\alpha}(\mathbf{t}^1, \dots, \mathbf{t}^{i-1}, \mathbf{u}, \mathbf{t}^{i+1}, \dots, \mathbf{t}^r)\hat{\alpha}(\mathbf{t}^1, \dots, \mathbf{t}^{i-1}, \mathbf{v}, \mathbf{t}^{i+1}, \dots, \mathbf{t}^r), \end{aligned} \tag{6}$$

for any $i = 1, \dots r$.

In other words, α is an r-character on H if and only if $\hat{\alpha}$ is an r-character on the corresponding formal group. Obviously, α is symmetric (skew-symmetric) if and only if $\hat{\alpha}$ is symmetric (skew-symmetric).

We will later need the following general results about the r-characters, which are straightforward generalizations of the bicharacter versions given in [1].

Proposition 2.1 *Let H be a cocommutative Hopf algebra, R a commutative algebra over $\mathcal{K}$. Let $\alpha \in \operatorname{Ch}^r(H, R)$. Then α vanishes on the ideal $I \lhd H$, generated by the commutators $kl - lk$, for all $k, l \in H$, i.e., $\alpha(H, \dots, I, \dots, H) = 0$, for any position of I among the arguments of α.* ∎

Proposition 2.2 *Let $I \lhd H$ be any Hopf ideal of H, $\bar{H} = H/I$. Then the r-characters $\bar{\alpha}$ of $\bar{H}$ are in one-to-one correspondence with the r-characters α of H such that $\alpha(H, \dots, I, \dots, H) = 0$, for any position of I. This correspondence is given by*

$$\alpha(h^1, \dots, h^r) = \bar{\alpha}(h^1 + I, \dots, h^r + I),$$

for any $h^1, \dots, h^r \in H$. ■

The proofs are left to the reader.

These two propositions allow us to reduce the study of r-characters of any cocommutative Hopf algebra to those of a Hopf algebra which is both commutative and cocommutative.

3 Polycharacters of Enveloping Algebras and Divided Power Algebras

The main aim of this section is an explicit description of the groups of r-characters for universal enveloping, restricted enveloping and divided power algebras. For the first two cases we use an appropriate extension of the method given in [1].

At first we consider the polynomial algebra $H = \mathcal{K}[X]$, where X is any set of variables (not necessarily finite), and $\Delta x = x \otimes 1 + 1 \otimes x$, for any $x \in X$.

Proposition 3.1 *Let* $w^1 = x_1^1 \cdots x_{m^1}^1, \dots, w^r = x_1^r \cdots x_{m^r}^r$ *be monomials,* α *an* r-*character on* $\mathcal{K}[X]$. *Then* $\alpha(w^1, \dots, w^r) = 0$ *unless* $m^1 = \cdots = m^r = m$, *and in the latter case we have*

(7)
$$\alpha(w^1, \dots, w^r) = \sum_{\pi^2, \dots, \pi^r \in S_m} \alpha(x_1^1, x_{\pi^2(1)}^2, \dots, x_{\pi^r(1)}^r) \cdot \cdots \cdot \alpha(x_m^1, x_{\pi^2(m)}^2, \dots, x_{\pi^r(m)}^r).$$

In particular, α *is completely determined by its values on* X.

Proof Induction on $m^1, \dots, m^r$ using the definition of the r-character. ■

Proposition 3.2 *Let* α *be an* r-*character of* $\mathcal{K}[X]$, $\operatorname{char} \mathcal{K} = p > 0$, $r \geq 2$. *Then* α *vanishes on the ideal generated by* x^p, *for all* $x \in X$.

Proof Let $w^1 = x^p$, $w^2, \dots, w^r$ any other monomials. Then by Proposition 3.1 $\alpha(w^1, \dots, w^r) = 0$, unless all $w^2, \dots, w^r$ have degree p. In the latter case by (7) we obtain

$$\begin{aligned}
&\alpha(w^1, \dots, w^r) \\
&\quad = \sum_{\pi^2, \dots, \pi^r \in S_p} \alpha(x, x_{\pi^2(1)}^2, \dots, x_{\pi^r(1)}^r) \cdot \cdots \cdot \alpha(x, x_{\pi^2(p)}^2, \dots, x_{\pi^r(p)}^r) \\
&\quad = p! \sum_{\pi^3, \dots, \pi^r \in S_p} \alpha(x, x_1^2, x_{\pi^3(1)}^3 \dots, x_{\pi^r(1)}^r) \cdot \cdots \cdot \alpha(x, x_p^2, x_{\pi^3(p)}^3, \dots, x_{\pi^r(p)}^r) \\
&\quad = 0.
\end{aligned}$$

The proposition follows. ■

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Now let L be a Lie algebra over $\mathcal{K}$, $H = U(L)$ the universal enveloping algebra of L. By Propositions 2.1 and 2.2, the r-characters of H are in one-to-one correspondence with the r-characters of $\bar{H} = U(L/[L,L]) \cong \mathcal{K}[X]$, where X is any basis of $L/[L,L]$. Let α be any r-character of H, $\bar{\alpha}$ the corresponding r-character of $\bar{H}$ and $\mathcal{A}$ the restriction of $\bar{\alpha}$ on the subspace $L/[L,L]$. By Proposition 3.1, $\bar{\alpha}$ is completely determined by $\mathcal{A}$. On the other hand, it can be easily verified that (7) defines an r-character on $\mathcal{K}[X]$ for any given values on the elements of X. It follows, that the correspondence $\alpha \mapsto \mathcal{A}$ is one-to-one, and it is actually an isomorphism of the Abelian groups $\mathrm{Ch}^r\big(U(L), R\big)$ and $\mathrm{Hom}\big((L/[L,L])^{\otimes r}, R\big)$, since $\overline{\alpha * \beta}(x^1, \ldots, x^r) = (\bar{\alpha} * \bar{\beta})(x^1, \ldots, x^r) = \bar{\alpha}(x^1, \ldots, x^r) + \bar{\beta}(x^1, \ldots, x^r)$, for any $x^1, \ldots, x^r \in X$. Obviously, α is (skew-)symmetric if and only if $\mathcal{A}$ is (skew-)symmetric. So we have proved the following.

Theorem 3.3 *For any Lie algebra L, commutative algebra R and $r \geq 1$*

$$\mathrm{Ch}^r\big(U(L), R\big) \cong \mathrm{Hom}\big(T^r(L/[L,L]), R\big),$$
$$\mathrm{Sym}^r\big(U(L), R\big) \cong \mathrm{Hom}\big(S^r(L/[L,L]), R\big),$$
$$\mathrm{Alt}^r\big(U(L), R\big) \cong \mathrm{Hom}\big(\Lambda^r(L/[L,L]), R\big),$$

as Abelian groups. ■

Here $T^r(V)$, $S^r(V)$ or $\Lambda^r(V)$ stand for the tensor, symmetric or skew-symmetric power of a vector space V, respectively.

Let L be a restricted Lie algebra over $\mathcal{K}$, $\operatorname{char} \mathcal{K} = p > 0$, and $H = u(L)$ the restricted enveloping algebra of L. We want to describe the r-characters of H, $r \geq 2$. By Propositions 2.1 and 2.2 we may assume L Abelian. Fix a basis X in L. Then $u(L) \cong \mathcal{K}[X]/\operatorname{ideal}(x^p - x^{[p]} \mid \forall x \in X)$. By Proposition 2.2, the r-characters of $u(L)$ are in one-to-one correspondence with the r-characters of $\mathcal{K}[X]$ that vanish on $\operatorname{ideal}(x^p - x^{[p]} \mid \forall x \in X)$. But by Proposition 3.2, any r-character of $\mathcal{K}[X]$ vanishes on $\operatorname{ideal}(x^p \mid \forall x \in X)$, so the r-characters of $u(L)$ are in one-to-one correspondence with the r-characters of $\mathcal{K}[X]$ that vanish on $\operatorname{ideal}(x^{[p]} \mid \forall x \in X)$, and the latter are in one-to-one correspondence with the r-characters of $\mathcal{K}[X]/\operatorname{ideal}(x^{[p]} \mid \forall x \in X) \cong u(L/L^{[p]})$. So we have proved the following.

Theorem 3.4 *For any restricted Lie algebra L, commutative algebra R and $r \geq 2$*

$$\mathrm{Ch}^r\big(u(L), R\big) \cong \mathrm{Hom}\Big(T^r\big(L/([L,L] + L^{[p]})\big), R\Big)$$
$$\mathrm{Sym}^r\big(u(L), R\big) \cong \mathrm{Hom}\Big(S^r\big(L/([L,L] + L^{[p]})\big), R\Big)$$
$$\mathrm{Alt}^r\big(u(L), R\big) \cong \mathrm{Hom}\Big(\Lambda^r\big(L/([L,L] + L^{[p]})\big), R\Big)$$

as Abelian groups. ■

Theorems 3.3 and 3.4 together cover the case of cocommutative Hopf algebras generated (as algebras) by their primitive elements, since any such Hopf algebra H is isomorphic to either $U(L)$ in characteristic zero or $u(L)$ in positive characteristic, where $L = P(H)$. If $\operatorname{char} \mathcal{K} = 0$, then such Hopf algebras exhaust the class of connected cocommutative Hopf algebras, but this is not true if $\operatorname{char} \mathcal{K} > 0$.

In the case $r = 1$ one can find non-trivial characters even in the case of $u(L)$ where L is a restricted Lie algebra with $L^{[p]} = L$. It suffices to take L of dimension 1 with a basis x and $x^{[p]} = x$. Then $u(L) \cong \mathcal{K}[x]/(x^p - x)$. Any of the maps $x \mapsto 1$, $x \mapsto 2, \dots, x \mapsto p-1$ extends to a homomorphism of $u(L)$ as above, providing us with $p-1$ pairwise different characters of this Hopf algebra.

There is no known complete description of connected cocommutative Hopf algebras in positive characteristic, although some partial results are known. For example, we have the following fact, which is a corollary of Theorems 2 and 3 in [5] (see also [6]).

First we need a definition.

Definition 3.1 Let H be a Hopf algebra. A sequence $x^{(0)}, x^{(1)}, \dots, x^{(T)}$ of elements of H, either finite or infinite ($T = \infty$), is called a sequence of divided powers if $\Delta x^{(n)} = \sum_{k+l=n} x^{(k)} \otimes x^{(l)}$, for any $n = 0, \dots, T$. In particular, $x^{(0)}$ is group-like and $x^{(1)}$ is $(x^{(0)}, x^{(0)})$-primitive.

Theorem 3.5 *Let H be a connected cocommutative Hopf algebra over a perfect field $\mathcal{K}$ of characteristic $p > 0$. Assume also that $d = \dim P(H) < \infty$. Then there exists a basis $x_1, \dots, x_d$ of $P(H)$, nonnegative integers N_j or $N_j = \infty$ and sequences of divided powers $x_j^{(0)} = 1$, $x_j^{(1)} = x_j, \dots, x^{(p^{N_j+1}-1)}$, for $j = 1, \dots, d$, such that the monomials $x_1^{(n_1)} \cdots x_d^{(n_d)}$, $0 \le n_j < p^{N_j+1}$, $j = 1, \dots, d$, form a basis of H.*

Setting $x^{(\mathbf{n})} = x_1^{(n_1)} \cdots x_d^{(n_d)}$ for $\mathbf{n} = (n_1, \dots, n_d)$, we see that H has the coalgebra structure isomorphic to the subcoalgebra $D(\mathbf{N})$, $\mathbf{N} = (N_1, \dots, N_d)$ of the divided power coalgebra D, spanned by $x^{(\mathbf{n})}$, $\mathbf{n} \le p^{\mathbf{N}}$, that is, $n_j < p^{N_j+1}$, for all $j = 1, \dots, d$.

As we already mentioned, in the case $N_1 = \cdots = N_d = \infty$ an r-character on H is the same as an r-character on the corresponding formal group.

Now we are going to describe the r-characters for the ordinary multiplication of divided powers, which is given by

$$x^{(\mathbf{k})} x^{(\mathbf{l})} = \binom{\mathbf{k}+\mathbf{l}}{\mathbf{k}} x^{(\mathbf{k}+\mathbf{l})} \tag{8}$$

and corresponds to the formal group law $\mathbf{u} \circ \mathbf{v} = \mathbf{u} + \mathbf{v}$. Then (6) takes the form:

$$\hat{\alpha}(\mathbf{t}^1, \dots, \mathbf{u}+\mathbf{v}, \dots, \mathbf{t}^r) = \hat{\alpha}(\mathbf{t}^1, \dots, \mathbf{u}, \dots, \mathbf{t}^r)\hat{\alpha}(\mathbf{t}^1, \dots, \mathbf{v}, \dots, \mathbf{t}^r). \tag{9}$$

Assume for a moment that $\operatorname{char} \mathcal{K} = 0$, and so H is isomorphic to $\mathcal{K}[x_1, \dots, x_d]$. Then the general solution of (9) is

$$\hat{\alpha}(\mathbf{t}^1, \dots, \mathbf{t}^r) = \exp\big(\mathcal{A}(\mathbf{t}^1, \dots, \mathbf{t}^r)\big),$$

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where $\mathcal{A}(\mathbf{t}^1, \dots, \mathbf{t}^r)$ is an arbitrary r-linear form. Obviously, α is (skew-) symmetric if and only if $\mathcal{A}$ is. Moreover, if $\hat{\beta} = \exp(\mathcal{B})$, then $\widehat{\alpha * \beta} = \hat{\alpha}\hat{\beta} = \exp(\mathcal{A} + \mathcal{B})$. So we have obtained another proof of Theorem 3.3 in the case of finite-dimensional abelian Lie algebras.

Now we return to $\operatorname{char} \mathcal{K} = p > 0$. Moreover, we will not restrict ourselves to the case $N_1 = \cdots = N_d = \infty$. From now up to the end of this section $H = D(\mathbf{N})$, the Hopf subalgebra of D spanned by $x^{(\mathbf{n})}$, $0 \le n_j < p^{N_j+1}$, $j = 1, \dots, d$, with comultiplication (3) and multiplication (8).

Note that since we passed from the whole algebra D to its Hopf subalgebra $D(\mathbf{N})$, we must replace $\operatorname{Hom}(D^{\otimes r}, R) \cong R[[\mathbf{t}^1, \dots, \mathbf{t}^r]]$ by

$$\operatorname{Hom}\big(D(\mathbf{N})^{\otimes r}, R\big) \cong R[[\mathbf{t}^1, \dots, \mathbf{t}^r]]/I,$$

where

$$I = \operatorname{ideal}\big((t_j^i)^{p^{N_j+1}} \mid i = 1, \dots, r,\ j = 1, \dots, d \text{ such that } N_j < \infty\big).$$

We keep the notation (4), but the indices should be appropriately restricted. Then the r-character condition (9) is still valid if understood modulo I.

Proposition 3.6 *Any r-character α on H is completely determined by its values on the elements of the form $x^{(p^s \mathbf{e}_j)}$, where $\mathbf{e}_j$ is the multi-index with 1 on the j-th place and 0 on the others.*

Proof Fix some $1 \le i \le r$. Let us collect the terms of $\hat{\alpha}$ containing $\mathbf{t}^i$:

$$\hat{\alpha}(\mathbf{t}^1, \dots, \mathbf{t}^r) = \sum_{\mathbf{n}} a^i_{\mathbf{n}}(\mathbf{t}^1, \dots, \mathbf{t}^{i-1}, \mathbf{t}^{i+1}, \dots, \mathbf{t}^r)(\mathbf{t}^i)^{\mathbf{n}}.$$

Then (9) is equivalent to

$$a^i_{\mathbf{k}} \cdot a^i_{\mathbf{l}} = \binom{\mathbf{k}+\mathbf{l}}{\mathbf{k}} a^i_{\mathbf{k}+\mathbf{l}}, \tag{10}$$

and also we must have $a^i_0 = 1$.

We need the following well-known lemma about the binomial coefficients modulo p, sometimes called The Lucas Theorem. We also keep the usual convention for the binomial coefficients $\binom{a}{b} = 0$ if $a \le b$.

Lemma 3.7 *Let a, b, c, d be nonnegative integers. Then*

$$\binom{a+bp}{c+dp} = \binom{a}{c}\binom{b}{d} \pmod{p}. \qquad \blacksquare$$

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From this lemma and the definition of the binomial coefficients of multi-indices it follows that, for any multi-index $\mathbf{n} \neq \mathbf{0}$ not of the form $p^s\mathbf{e}_j$, there are multi-indices $\mathbf{k}, \mathbf{l} \neq \mathbf{0}$ such that $\mathbf{k} + \mathbf{l} = \mathbf{n}$ and $\binom{\mathbf{n}}{\mathbf{k}} \neq 0 \pmod{p}$.

Hence, (10) defines all $a^i_{\mathbf{n}}$ by induction as soon as $a^i_{p^s\mathbf{e}_j}$ are known, for all $j = 1, \ldots, d$, $s = 0, \ldots, N_j$. Applying this fact for $i = 1, \ldots, r$ recursively we can calculate all coefficients of $\hat{\alpha}$ if we know the coefficients of the monomials of the form $(\mathbf{t}^1)^{p^{s_1}\mathbf{e}_{j_1}} \cdots (\mathbf{t}^r)^{p^{s_r}\mathbf{e}_{j_r}}$. ■

Let $\operatorname{rad} R$ denote the nil radical of the ring R.

Corollary 3.8 *Let $H = D\big((\infty, \ldots, \infty)\big)$ and $\operatorname{rad} R = 0$. Then the groups of r-characters $\mathrm{Ch}^r(H, R)$ are trivial, for all $r \geq 1$.*

Proof Keeping the notation of the previous proof we notice that (10) implies that

$$(a^i_{\mathbf{n}})^p = 0, \tag{11}$$

for any $\mathbf{n} \neq \bar{0}$. But since $N_1 = \cdots = N_d = \infty$ and $\operatorname{rad} R = 0$, $\operatorname{Hom}(D^r, R) \cong R[[\mathbf{t}^1, \ldots, \mathbf{t}^r]]$ does not have nonzero nilpotent elements, hence all $a^i_{\mathbf{n}} = 0$, except for $a^i_{\bar{0}} = 1$. ■

If $N_j < \infty$ for some j, then there are nontrivial r-characters for $r \geq 2$ even in the case $\operatorname{rad} R = 0$ (although 1-characters are obviously trivial). We will give the explicit description of the groups of bicharacters (*i.e.*, $r = 2$) with an additional assumption $\operatorname{rad} R = 0$.

Theorem 3.9 *Let $H = D(\mathbf{N})$ and $\operatorname{rad} R = 0$. Then*

$$\mathrm{Ch}^2(H, R) \cong \operatorname{Hom}\big(T^2(V), R\big)$$
$$\mathrm{Sym}^2(H, R) \cong \operatorname{Hom}\big(S^2(V), R\big)$$
$$\mathrm{Alt}^2(H, R) \cong \operatorname{Hom}\big(\Lambda^2(V), R\big)$$

as Abelian groups, where

$$V = \langle x^{(p^{N_j}\mathbf{e}_j)} \mid j = 1, \ldots, d \text{ and } N_j < \infty \rangle.$$

Proof We keep the notation of the proof of Proposition 3.6, but we will write $\hat{\alpha}(\mathbf{u}, \mathbf{v})$ instead of $\hat{\alpha}(\mathbf{t}^1, \mathbf{t}^2)$ for brevity.

Applying (11) for $i = 2$ we obtain that monomials of the form $u_1^{m_1} \cdots u_d^{m_d}$, $m_j < p^{N_j}$ for all $j = 1, \ldots, d$, cannot occur in $a^2_{\mathbf{n}}(\mathbf{u})$ (because taking a minimal of such monomials occuring in $a^2_{\mathbf{n}}$ we immediately get a contradiction with $\operatorname{rad} R = 0$). For the bicharacter α this implies that $\alpha(x^{(p^{s'}\mathbf{e}_{j'})}, x^{(p^{s''}\mathbf{e}_{j''})}) = 0$, for all $s' < N_{j'}$, $s'' < N_{j''}$. By Proposition 3.6 we conclude that α is completely determined by its restriction to the space V.

It suffices to prove now that for any bilinear form $\mathcal{A}$ on V there exists a bicharacter α such that $\alpha|_V = \mathcal{A}$, and that the convolution product of bicharacters corresponds to the summation of bilinear forms. We will achieve both aims by giving an explicit formula for α, as follows:

$$\hat{\alpha}(\mathbf{u}, \mathbf{v}) = \exp\big(\hat{\mathcal{A}}(\mathbf{u}, \mathbf{v})\big)\,, \tag{12}$$

where

$$\hat{\mathcal{A}}(\mathbf{u}, \mathbf{v}) = \sum_{j', j'': N_{j'}, N_{j''} < \infty} \mathcal{A}(x^{(p^{N_{j'}})}\mathbf{e}_{j'}, x^{(p^{N_{j''}})}\mathbf{e}_{j''})\mathbf{u}^{p^{N_{j'}}\mathbf{e}_{j'}}\mathbf{v}^{p^{N_{j''}}\mathbf{e}_{j''}}.$$

Note that the "exponential series" (which is actually finite in this case) still makes sense in spite of characteristic $p > 0$ because of the following lemma.

Lemma 3.10 *Suppose $c_1, \ldots, c_T$ are some elements of a commutative algebra C over a field $\mathcal{K}$ of characteristic $p > 0$. Let $(c_1)^p = \cdots = (c_T)^p = 0$. Then the "exponential series"*

$$\exp(a_1 + \cdots + a_T) = \sum_{n_1, \ldots, n_T} \frac{1}{n_1! \cdots n_T!} a_1^{n_1} \cdots a_T^{n_T}$$

is well-defined.

Proof of Lemma Whenever $n_s! \equiv 0 \pmod{p}$ for some $s = 1, \ldots, T$ we also have $a_s^{n_s} = 0$. ■

Now α defined by (12) is obviously a bicharacter as well as $\alpha|_V = \mathcal{A}$. Finally $\widehat{\alpha * \beta} = \hat{\alpha}\hat{\beta} = \exp(\hat{\mathcal{A}})\exp(\hat{\mathcal{B}}) = \exp(\hat{\mathcal{A}} + \hat{\mathcal{B}}) = \exp(\widehat{\mathcal{A} + \mathcal{B}})$, for any bicharacters α, β such that $\alpha|_V = \mathcal{A}$, $\beta|_V = \mathcal{B}$. ■

From Theorems 3.3, 3.4 and 3.9 the reader might conjecture that the groups of r-characters always have the additional structure of $\mathcal{K}$-spaces; in particular, $\alpha^p = \varepsilon$ under the convolution product, for any r-character α. But this is not true, as is shown by the following example for $r = 1$. We can look into the formal law $u \circ v = u + v + uv$. Then the associated Hopf algebra will be a divided power coalgebra in one variable x, with multiplication

$$x^{(k)}x^{(l)} = \sum_{m=\min(k,l)}^{k+l} \binom{m}{k+l-m,\, m-k,\, m-l} x^{(m)}.$$

We set $\hat{\alpha} = 1 + t$, that is, $\alpha(x^{(0)}) = \alpha(x^{(1)}) = 1$ and $\alpha(x^{(i)}) = 0$ for $i \geq 2$. Then $\hat{\alpha}^n = (1 + t)^n \neq 1$ for all natural n. The case $r \geq 2$ remains open.

4 Scheunert's Theorem for Hopf Algebras and Generalizations

Now we will explore the structure of groups of r-characters, with a special interest in bicharacters. At first we assume that H is connected. Our aim is to prove the following result.

Theorem 4.1 *Let H be a cocommutative connected Hopf algebra over $\mathcal{K}$, m a positive integer such that* $\operatorname{char}\mathcal{K} \nmid m$. *Let α be an r-character on H with values in a commutative algebra R over $\mathcal{K}$. Then there exists a unique r-character β such that $\alpha = \beta^m$ (under the convolution product). Moreover, if α is symmetric (or skew-symmetric), then so is β.*

We first consider a slightly more general situation. Suppose we have a multilinear map of r variables $f\colon H \times \cdots \times H \to R$, which is normalized with respect to the i-th variable in the sense that

$$f(h^1, \dots, h^{i-1}, 1, h^{i+1}, \dots, h^r) = \varepsilon(h^1 \cdots h^{i-1} h^{i+1} \cdots h^r),$$

for any $h^1, \dots, h^{i-1}, h^{i+1}, \dots, h^r \in H$.

We define a multilinear map by setting

$$(13) \qquad f_0(h^1, \dots, h^r) = f(h^1, \dots, h^r) - \varepsilon(h^1 \cdots h^r), \quad \forall\, h^1, \dots, h^r \in H.$$

Given a coalgebra C and an algebra R, one can define a natural action of the algebra $\operatorname{Hom}(C, R)$ (under convolution) on the vector space $C \otimes R$:

$$f \circ (c \otimes a) = \sum c_1 \otimes f(c_2) a,$$

for any $f \in \operatorname{Hom}(C, R)$, $c \in C$, $a \in R$. This action is always faithful.

Lemma 4.2 *Let C be a coalgebra, R an algebra. Let $f \in \operatorname{Hom}(C, R)$, then the following conditions are equivalent:*

1. *for any $c \in C$ there exists an integer N such that $f^m(c) = 0$, for any $m \geq N$,*
2. *the action of f on $C \otimes R$ is locally nilpotent.*

Proof Assume the first holds. Let $a \in R$, $c \in C$, $\Delta c = \sum_j k_j \otimes l_j$. Then there is an integer N such that for any $m \geq N$, $f^m(l_j) = 0$, for all j. Therefore, $f^m \circ (c \otimes a) = \sum_j k_j \otimes f^m(l_j) a = 0$. This proves that the action of f on $C \otimes R$ is locally nilpotent.

Now, let the action of f be locally nilpotent. Then for any $c \in C$ there is an integer N such that $f^m \circ (c \otimes 1) = \sum c_1 \otimes f^m(c_2) = 0$, for any $m \geq N$. Applying $\varepsilon \otimes 1$ to both sides, we get $f^m(c) = 0$, for all $m \geq N$. ∎

In our situation, take $C = H^{\otimes r}$. Then $\operatorname{Hom}(H^{\otimes r}, R)$ acts on $H^{\otimes r} \otimes R$ as follows:

$$(14) \qquad f \circ (h^1 \otimes \cdots \otimes h^r \otimes a) = \sum h^1_1 \otimes \cdots \otimes h^r_1 \otimes f(h^1_2, \dots, h^r_2) a,$$

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for any $f \in \mathrm{Hom}(H^{\otimes r}, R)$, $h^1, \ldots, h^r \in H$, $a \in R$.

Since the action is faithful, we obtain an imbedding $\eta\colon \mathrm{Hom}(H^{\otimes r}, R) \hookrightarrow \mathrm{End}(H^{\otimes r} \otimes R)$, where $H^{\otimes r} \otimes R$ is viewed as a vector space.

In the following proposition H is not necessarily cocommutative (but it is still assumed connected). It should be mentioned that our way of arguing is fairly close to Lemma 5.2.10 in [2] belonging to Y. Takeuchi.

Proposition 4.3 *Let f_0 be defined as in (13). Then f_0 is locally nilpotent in the sense of either of the equivalent conditions of Lemma 4.2.*

Proof Without loss of generality we may assume $i = 1$.

Let $H_0 \subset H_1 \subset \cdots \subset H$ be the coradical filtration of H. It is known (see [2]) that $H = \bigcup_{n=0}^{\infty} H_n$ and $\Delta H_n \subset \sum_{i+j=n} H_i \otimes H_j$. Iterating the latter formula yields

$$\Delta_s H_n \subset \sum_{i_1+\cdots+i_s=n} H_{i_1} \otimes \cdots \otimes H_{i_s}, \tag{15}$$

where $\Delta_s\colon H \to H^{\otimes s}$ is the iterated comultiplication map.

Since H is connected, we have $H_0 = \mathcal{K} \cdot 1$. It follows that $f_0(H_0, H, \ldots, H) = 0$, because $f_0(1, h^2, \ldots, h^r) = 0$. Now look at

$$f_0^2(h^1, \ldots, h^r) = \sum f_0\big((h^1)_1, \ldots, (h^r)_1\big)\, f_0\big((h^1)_2, \ldots, (h^r)_2\big)\,.$$

If $h^1 \in H_1$ then $\Delta h^1 = \sum (h^1)_1 \otimes (h^1)_2 \in H_1 \otimes H_0 + H_0 \otimes H_1$, which forces

$$f_0^2(h^1, \ldots, h^r) = \sum f_0\big((h^1)_1, \ldots, (h^r)_1\big)\, f_0\big((h^1)_2, \ldots, (h^r)_2\big) = 0,$$

for any $h^2, \ldots, h^r \in H$.

More generally, if $h^1 \in H_n$, then applying (15) with $s > n$ we see that every summand in $\Delta_{n+1} h^1$ must contain at least one factor from H_0, which forces $f_0^s(h^1, h^2, \ldots, h^r) = 0$, for all $h^2, \ldots, h^r \in H$. So we have proved that $f_0^s(H_n, H, \ldots, H) = 0$. Since $H = \bigcup_{n=0}^{\infty} H_n$, it follows that f_0 is locally nilpotent (take $N = n + 1$ if $h^1 \in H_n$). ■

Let $\mathcal{N}$ denote the set of all $f \in \mathrm{Hom}(H^{\otimes r}, R)$ that are locally nilpotent in the sense of Lemma 4.2. Then, given a formal power series $A(t) = a_0 + a_1 t + a_2 t^2 + \cdots$ with coefficients in R, we can define a linear map $A(f_0) = a_0 + a_1 f_0 + a_2 (f_0)^2 + \cdots : H^{\otimes r} \to R$, for any $f_0 \in \mathcal{N}$, since for any particular element of $H^{\otimes r}$ only finitely many terms of the series take nonzero values. Moreover, we will have

$$\begin{aligned} A(f_0) \circ (h^1 \otimes \cdots \otimes h^r \otimes a) &= \sum h_1^1 \otimes \cdots \otimes h_1^r \otimes A(f_0)(h_2^1, \ldots, h_2^r) a \\ &= a_0 \circ (h^1 \otimes \cdots \otimes h^r \otimes a) + a_1 f_0 \circ (h^1 \otimes \cdots \otimes h^r \otimes a) \\ &\quad + a_2 (f_0)^2 \circ (h^1 \otimes \cdots \otimes h^r \otimes a) + \cdots, \end{aligned}$$

for any $h^1, \dots, h^r \in H$, $a \in R$. Therefore,

$$\eta\big(A(f_0)\big) = a_0 + a_1\eta(f_0) + a_2\big(\eta(f_0)\big)^2 + \cdots$$

in $\operatorname{End}(H^{\otimes r} \otimes R)$.

Note that the identity element of $\operatorname{Hom}(H^{\otimes r}, R)$ under the convolution product is $\varepsilon(h^1 \cdots h^r)$, so a_0 means $a_0\varepsilon(h^1 \cdots h^r)$ in the definition of $A(f_0)$.

Lemma 4.4 *The set $1 + \mathcal{N}$ is an Abelian group under the convolution product.*

Proof Let $f_0, g_0 \in \mathcal{N}$, then $(1 + f_0)(1 + g_0) = 1 + f_0 + g_0 + f_0g_0$ and we want to prove that $f_0 + g_0 + f_0g_0 \in \mathcal{N}$. By Lemma 4.2, this is equivalent to showing that $\eta(f_0 + g_0 + f_0g_0) = \eta(f_0) + \eta(g_0) + \eta(f_0)\eta(g_0)$ is a locally nilpotent operator. But it is straightforward that a sum or a product of commuting locally nilpotent operators is again a locally nilpotent operator.

Finally, if $f_0 \in \mathcal{N}$, then $1 + f_0 + (f_0)^2 + \cdots$ is the inverse of $1 - f_0$, since we have $\eta\Big(\big(1 + f_0 + (f_0)^2 + \cdots\big)(1 - f_0)\Big) = \Big(1 + \eta(f_0) + \big(\eta(f_0)\big)^2 + \cdots\Big)\big(1 - \eta(f_0)\big) = 1$, and $f_0 + (f_0)^2 + \cdots \in \mathcal{N}$, since $\eta(f_0) + \big(\eta(f_0)\big)^2 + \cdots$ is obviously a locally nilpotent operator. ■

Proposition 4.5 *For any multilinear map $f\colon H \times \cdots \times H \to R$, normalized with respect to the i-th variable (or, more generally, such that $f - \varepsilon$ is locally nilpotent), there exists a unique multilinear map $g\colon H \times \cdots \times H \to R$, normalized with respect to the i-th variable, such that $f = g^m$ under the convolution product, provided* $\operatorname{char} k \nmid m$.

Proof Since $\operatorname{char} k \nmid m$, we can consider the formal power series

$$A(t) = (1 + t)^{\frac{1}{m}} = 1 + \frac{1}{m}t + \cdots.$$

Set $g = A(f_0)$. Obviously, g is normalized with respect to the i-th variable, and $f = g^m$ by construction. It remains to prove the uniqueness of such g.

Suppose we have two normalized multilinear maps $g', g''\colon H \times \cdots \times H \to R$ such that $g'^m = g''^m$. Then we can write $g' = 1 + g_0'$, $g'' = 1 + g_0''$ and g_0', g_0'' are locally nilpotent. So we have

$$(1 + g_0')^m = (1 + g_0'')^m, \tag{16}$$

and we want to show that $g_0' = g_0''$.

By Lemma 4.4, we can divide the left side of (16) by the right side, so it suffices to prove that $(1 + g_0)^m = 1$ implies $g_0 = 0$ for g_0 locally nilpotent. But

$$1 = (1 + g_0)^m = 1 + mg_0 + \binom{m}{2}g_0^2 + \cdots$$

yields $g_0 = -\frac{1}{m}\binom{m}{2}g_0^2 - \cdots$ (recall $\operatorname{char} k \nmid m$), which contradicts the local nilpotency of g_0 if $g_0 \neq 0$. ■

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Proof of Theorem Since the r-characters are normalized with respect to any variable, we can apply Proposition 4.5 to $\alpha\colon H \times \cdots \times H \to R$, obtaining a normalized r-linear function $\beta\colon H \times \cdots \times H \to R$ such that $\alpha = \beta^m$. It remains to prove that β is an r-character on H. To do so we have to show (2) for β, for any $i = 1, \ldots, r$. Since $\alpha = \beta^m$ is an r-character, we have

$$\beta^m(h^1, \ldots, kl, \ldots, h^r) = \sum \beta^m(h_1^1, \ldots, k, \ldots, h_1^r)\beta^m(h_2^1, \ldots, l, \ldots, h_2^r). \tag{17}$$

Set

$$f(h^1, \ldots, k, l, \ldots, h^r) = \beta(h^1, \ldots, kl, \ldots, h^r) \quad \text{and}$$
$$g(h^1, \ldots, k, l, \ldots, h^r) = \sum \beta(h_1^1, \ldots, k, \ldots, h_1^r)\beta(h_2^1, \ldots, l, \ldots, h_2^r).$$

Then f and g are multilinear maps, normalized with respect to any variable except the $(i+1)$-st, and (17) can be rewritten as $f^m = g^m$ (we use the cocommutativity). Applying Proposition 4.5 again, we conclude that $f = g$, which is exactly (2) for β. Note that in the case $r = 1$ f and g are not normalized, but they are still nilpotent (the proof is analogous to that of Lemma 4.3), so we can apply Proposition 4.5 in this case as well.

The proof of the invertibility, symmetry or skew-symmetry of β is similar (and simpler). ■

Let us define two operators:

$$\mathrm{sym}\colon \quad \mathrm{Ch}^r(H, R) \to \mathrm{Sym}^r(H, R)\colon \alpha \mapsto \prod_{\pi \in S_r} (\pi \circ \alpha),$$

and

$$\mathrm{alt}\colon \quad \mathrm{Ch}^r(H, R) \to \mathrm{Alt}^r(H, R)\colon \alpha \mapsto \prod_{\pi \in S_r} (\pi \circ \alpha)^{\mathrm{sgn}\,\pi},$$

for any cocommutative Hopf algebra H.

Corollary 4.6 *If H is connected and $r < \mathrm{char}\,\mathcal{K}$ or $\mathrm{char}\,\mathcal{K} = 0$, then* sym *and* alt *are projections of* $\mathrm{Ch}^r(H, R)$ *on* $\mathrm{Sym}^r(H, R)$ *and* $\mathrm{Alt}^r(H, R)$*, respectively.* ■

In particular, we obtain the following corollary, which extends Proposition 3.16 of [1] on the skew-symmetric bicharacters on universal enveloping and restricted enveloping algebras.

Corollary 4.7 *Let H be a connected cocommutative Hopf algebra over a field $\mathcal{K}$ of characteristic not equal to 2. Let α be a skew-symmetric bicharacter on H with values in a commutative algebra R. Then there exists a (unique) skew-symmetric bicharacter β such that*

$$\alpha(h, k) = \beta^2(h, k) = \beta(h, k) * \beta^{-1}(k, h).$$ ■

Now we can pass to arbitrary pointed cocommutative Hopf algebras, in Theorem 3.20 of [1]:

Theorem 4.8 *Let H be a pointed cocommutative Hopf algebra over a field $\mathcal{K}$ of characteristic not equal to 2. Let α be a skew-symmetric bicharacter on H with values in a commutative algebra R. Then α can be written in the form*

$$\alpha(h,k) = \sigma(h,k) * \sigma^{-1}(k,h),$$

for some 2-cocycle σ if and only if $\alpha(g,g) = 1$ for any group-like element $g \in H$. If this condition is not satisfied then we can write α in the form

$$\alpha(h,k) = \alpha_0(h,k) * \sigma(h,k) * \sigma^{-1}(k,h)$$

where σ is a 2-cocycle and α_0 is the sign bicharacter of H. ∎

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