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MONOTONE METHOD AND PERIODIC SOLUTION OF NON LINEAR PARABOLIC BOUNDARY VALUE PROBLEM FOR SYSTEMS

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A system of parabolic equations is considered:

$$Lu_i = u_{it} - u_{ix} = f_i\left(x,\ t,\ u,\ u_{ix}\right) \quad \text{on} \quad Q\ ,$$

$$B_i u_i(j,\ t) = \omega_{ij}(t)\ ,\quad t \in (-\infty,\ \infty)\ ,\quad j = 0, 1\ ,\quad i = 1, 2, \ldots, n\ ,$$

where B_i is one of the boundary operators $B_i u_i = u_i$ or $B_i u_i = \partial u_i/\partial\nu + \beta_i(x,\ t)u_i$, $x = 0, 1$, $\Omega = (0, 1)$, $Q = \Omega \times R$, $u\left(=\left(u_1,\ \ldots,\ u_n\right)\right) : Q \to R^n$, $\nu(x)$ is the outward normal to the boundary $\partial\Omega$, f, u, ω_0, ω_1 are n-valued functions and f, ω_0, ω_1 are periodic in t with period T and β_i is a positive function.

The paper is classified into two parts. The first part deals with the existence and uniqueness of periodic solutions of the above system of parabolic equations. The second part deals with a monotone iterative method which develops a monotone iterative scheme for the solution of the above system of equations. In this paper we establish the existence of coupled quasi-solutions of the above equation. Also we give a monotone iterative scheme for the construction of such a solution.

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1. Introduction

The method of upper and lower solutions is one of the known techniques in the theory of non-linear boundary value problems and initial boundary value problems. In particular, when we are dealing with systems of equations, we require that the non-linearities be quasimonotone. But in applications such a restriction does not always obtain for example, consider a simple model governing the combustion of a single species described by the following system of equations [7]:

$$\begin{aligned} \frac{\partial T}{\partial t} &= k_1 \Delta T + qye^{-E/RT} , \\ \frac{\partial y}{\partial t} &= k_2 \Delta y - \varepsilon ye^{-E/RT} , \end{aligned} \tag{1.1}$$

where T denotes temperature, y the concentration of reactunt and $k_1, k_2, q, \varepsilon, E, R$ are all positive constants. However, a mixed quasi-monotonicity condition is satisfied. In this paper we establish the existence of coupled quasi-solutions of the periodic parabolic boundary value problem. Also we give a monotone iterative scheme for the construction of such a solution.

2. Existence theorem

Consider a system of parabolic equations:

$$Lu_i = u_{it} - u_{ixx} = f_i(x, t, u, u_{ix}) \quad \text{on} \quad Q , \tag{2.1}$$

$$B_i u_i(j, t) = \omega_{ij}(t) \ , \quad t \in (-\infty, \infty) \ , \quad j = 0, 1 \ , \quad i = 1, 2, \ldots, n , \tag{2.2}$$

where B_i is one of the boundary operators $B_i u_i = u_i$ or $B_i u_i = \partial u_i/\partial\gamma + \beta_i(x, t)u_i$, $x = 0, 1$, $\Omega = (0, 1)$, $Q = \Omega \times R$, $u(=(u_1 \ldots u_n)) : Q \to R^n$, $\gamma(x)$ is the outward normal to the boundary $\partial\Omega$, f, u, ω_0, ω_1 are n valued functions and f, ω_0, ω_1 are periodic in t with period T and β_i is a positive function.

Let $f(x, t) : \overline{Q} \to R^n$ be a continuous function which is periodic in t with period T and Holder continuous in $x \in [0, 1]$ uniformly with respect to t . It is well known that the linear problem

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$$(2.3)\qquad \begin{aligned} &L_{ui} = f_i(x, t) , \\ &B_i u_i(j, t) = 0 , \quad j = 0, 1 , \quad i = 1, 2, \ldots, n , \end{aligned}$$

has a unique periodic solution of period T [5] which may be written as

$$u_i(x, t) = \int_{-\infty}^{t} \int_0^1 G(x, t; \sigma, \theta) f_i(\sigma, \theta) d\sigma d\theta$$

where $G(x, t; \sigma, \theta)$ is the Green's function for $Lu_i = 0$ on a rectangle.

We shall let S denote the Banach space of real-valued functions $v(x, t)$ which are periodic in t of period T, continuous on $[0, 1] \times [0, T]$, have a partial derivative with respect to x which is continuous on $[0, 1] \times [0, T]$. The norm of S will be given by

$$\|v\| = \max_{[0,1]\times[0,T]} |v(x, t)| + \max_{[0,1]\times[0,T]} |v_x(x, t)| .$$

Now let $X = S^n$. Let $u \in X$; then $u = (u_1 \ldots u_n)$ and $\|u\|_X = \|u_1\| + \ldots + \|u_n\|$.

We require the following hypotheses.

(i) $f_i(x, t, u, p_i)$ is a periodic in t of period T.

(ii) $f_i(x, t, u, p_i)$ is continuous for $0 \le x \le 1$ and $(t, u, p) \in R^{n+2}$ satisfies a Hölder condition (jointly in (x, t)) with exponent σ and satisfies local Hölder conditions (in u and p uniformly with respect to t) with exponent σ.

(iii) For $|u| \le M$ and all x, t, p,

$$|f_i(x, t, u, p_i)| \le \mu_{iM}(|p_i|) , \quad i = 1, 2, \ldots, n ,$$

where $\mu_{iM}(s)$ is defined for $s \ge 0$ and is positive non-decreasing in s with $\mu_{iM}(s) = o(s^2)$ as $s \to +\infty$.

(iv) f_i satisfies mixed quasi-monotonicity in u; that is, for each $j \ne i$ (fixed), f_i is monotonic (either non-decreasing or non-increasing) in u_j. Arranging the components of u (other than ith) in

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which f_i is non-decreasing in $[u]_1$, and those in which f_i is non-increasing in $[u]_2$, we write

$$f_i(x,\ t,\ u,\ u_{ix}) = f_i(x,\ t,\ u_i,\ [u]_1,\ [u]_2,\ u_{ix}) \ .$$

(v) ϕ, ψ are coupled lower and upper solutions of the problems (2.1), (2.2); namely, periodic functions $\phi, \psi \in C^{2,1}(\overline{Q},\ R^n)$ such that $\phi \le \psi$ and

$$(2.4)\quad \begin{aligned} &\phi_{it} - \phi_{ixx} \le f_i(x,\ t,\ \phi_i,\ [\phi]_1,\ [\phi]_2,\ \phi_{ix}) \quad \text{on} \quad Q \ , \\ &B_i\phi_i(j,\ t) \le \omega_{ij}(t) \ , \quad j = 0, 1 \ , \quad i = 1, 2, \ldots, n \ ; \end{aligned}$$

$$(2.5)\quad \begin{aligned} &\psi_{it} - \psi_{ixx} \ge f_i(x,\ t,\ \psi_i,\ [\psi]_1,\ [\psi]_2,\ \psi_{ix}) \quad \text{on} \quad Q \ , \\ &B_i\psi_i(x,\ t) \ge \omega_{ij}(t) \ , \quad j = 0, 1 \ , \quad i = 1, 2, \ldots, n \ . \end{aligned}$$

(vi) There is a function $q(x,\ t)$ in the Hölder class $C_T^{2+\sigma}(\overline{Q})$ such that $q_i(j,\ t) = \omega_{ij}(t)$, $t \in R$, $j = 0, 1$, $i = 1, \ldots, n$.

THEOREM 2.1. *Assume that* (i)-(vi) *above hold. Then problems* (2.1), (2.2) *has at least one solution* $u(x,\ t)$ *which satisfies* $\phi_i(x,\ t) \le u_i(x,\ t) \le \psi_i(x,\ t)$ *on* $\overline{\Omega} \times R$ *and moreover*

$$|u_{ix}(x,\ t)| \le N_i$$

for some constant $N = (N_1 \ \ldots \ N_n) > 0$.

Proof. We shall let $d_{i0}(x,\ t)$ denote the unique periodic solution to the problem $Lu_i = 0$ with boundary condition (2.2). The existence and uniqueness of $d_{i0}(x,\ t)$ is well known [5]. Now let

$$(2.6)\quad F_i(x,\ t,\ u;\ p_i) = \begin{cases} f_i(x,\ t,\ \bar{u},\ p_i) - (u_i - \phi_i)/\left(1+u_i^2\right) & \text{for} \quad u_i < \phi_i \ , \\ f_i(x,\ t,\ \bar{u},\ p_i) & \text{for} \quad \phi_i \le u_i \le \psi_i \ , \\ f_i(x,\ t,\ \bar{u},\ p_i) - (u_i - \psi_i)/\left(1+u_i^2\right) & \text{for} \quad u_i > \psi_i \ , \end{cases}$$

where

$$\bar{u}_i = \begin{cases} \psi_i(x, t) & \text{if } u_i > \psi_i(x, t) , \\ u_i & \text{if } \phi_i(x, t) \le u_i \le \psi_i(x, t) , \\ \phi_i(x, t) & \text{if } u_i < \phi_i(x, t) ; \end{cases}$$

we can consider the equation

$$Lu_i = F_i\left(x, t, u, u_{ix}\right) \tag{2.7}$$

with the boundary conditions (2.2).

It is easy to show that ϕ and ψ are again coupled lower and upper solutions of (2.7), (2.2). We further define

$$g_i(x, t, u) = \begin{cases} F_i\left(x, t, \phi(x, t), \phi_{ix}(x, t)\right) - \left[u_i - \phi_i\right] & \text{if } u_i < \phi_i , \\ \left(u_i - \phi_i\right) / \left(\psi_i - \phi_i\right) \left[F_i\left(x, t, \psi(x, t), \psi_{ix}(x, t)\right) \right. & \\ \quad \left. - F_i\left(x, t, \phi(x, t), \phi_{ix}(x, t)\right)\right] & \\ \quad + F_i\left(x, t, \phi(x, t), \phi_{ix}(x, t)\right) & \text{if } \phi_i \le u_i \le \psi_i , \\ F_i\left(x, t, \psi(x, t), \psi_{ix}(x, t)\right) - \left[u_i - \psi_i\right] & \\ & \text{if } u_i > \psi_i(x, t) . \end{cases} \tag{2.8}$$

Observe that $g_i(x, t, u)$ is continuous on $Q \times R^n$ and satisfies local Hölder continuity in (x, t) with exponent σ. Moreover, being linear in u_i, g_i satisfies uniform Lipschitz conditions in the variable u. We form a family of parabolic equations

$$L_{ui} = \lambda F_i\left(x, t, u, u_{ix}\right) + (1-\lambda) g_i(x, t, u) \tag{2.9}$$

with boundary conditions (2.2) and $\lambda \in [0, 1]$.

We assert that any periodic solution $v(x, t)$ of (2.9), (2.2) satisfies

$$\phi_i(x, t) \le v_i(x, t) \le \psi_i(x, t) \quad \text{on } \bar{Q} . \tag{2.10}$$

To prove the right hand inequality, define

$$w_i(x, t) = v_i(x, t) - \psi_i(x, t) .$$

It is evident that w_i is continuous and periodic on $\bar{Q}$. Let $w_i(x, t)$

attain its maximum at (x^*, t^*). We can easily see that (x^*, t^*) is an interior point of $\overline{Q}$. Thus

$$
\begin{aligned}
0 &\le w_{it}(x^*, t^*) - w_{ixx}(x^*, t^*) \\
&= v_{it}(x^*, t^*) - v_{ixx}(x^*, t^*) - \psi_{it}(x^*, t^*) + \psi_{ixx}(x^*, t^*) \\
&\le \lambda F_i\left(x^*, t^*, v(x^*, t^*), v_{ix}(x^*, t^*)\right) + (1-\lambda)g_i\left(x^*, t^*, v(x^*, t^*)\right) \\
&\quad - \lambda F_i\left(x^*, t^*, \psi_i, [\psi]_1, [\phi]_2, \psi_{ix}(x^*, t^*)\right) - (1-\lambda)g_i\left(x^*, t^*, \psi(x^*, t^*)\right) \\
&= -(1-\lambda)\left[v_i-\psi_i\right] + \lambda f_i\left(x^*, t^*, \bar{v}, v_{ix}\right) \\
&\qquad - \lambda\left(\left(v_i-\psi_i\right)/\left(1+v_i^2\right)\right) - \lambda f_i\left(x^*, t^*, \psi_i, [\psi]_1, [\phi]_2, \psi_{ix}\right) \\
&< 0 \quad \left(\text{by the hypothesis (iv)}\right)
\end{aligned}
$$

which is a contradiction. Hence the above assertion. A similar argument can be used to complete the proof of (2.10).

Next define integral operators

$$W_i : X \to S \text{ by } W_i(h) = h_{i\lambda}(x, t) ,$$

where

$$
\begin{aligned}
h_{i\lambda}(x, t) &= d_{i0}(x, t) + \int_{-\infty}^{t} \int_0^1 G(x, t; \sigma, \theta) \\
&\quad \left[\lambda F_i\left(\sigma, \theta, h(\sigma, \theta), h_{ix}(\sigma, \theta)\right)+(1-\lambda)g_i\left(\sigma, \theta, h(\sigma, \theta)\right)\right] d\sigma d\theta .
\end{aligned}
\tag{2.11}
$$

We can easily show that $v(x, t)$ and $v_x(x, t)$ are bounded functions where

$$
\begin{aligned}
v_i(x, t) = d_{i0}(x, t) &+ \int_{t-\delta}^{t} \int_0^1 G(x, t; \sigma, \theta) \\
&\left[F_i\left(\sigma, \theta, v, v_{ix}\right)+(1-\lambda)g_i\left(\sigma, \theta, v(\sigma, \theta)\right)\right] d\sigma d\theta \\
&+ \int_0^1 G(x, t; \sigma, t-\delta)v_i(\sigma, t-\delta)d\sigma
\end{aligned}
$$

and

$$v_{ix}(x, t) = d_{i0x}(x, t) + \int_{t-\delta}^{t} \int_0^1 G_x\left[\lambda F_i+(1-\lambda)g_i\right] d\sigma d\theta + \int_0^1 G_x v_i d\sigma .$$

Let M and N be such that $\left|v_i(x, t)\right| \le M_i$ and $\left|v_{ix}(x, t)\right| \le N_i$ on

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Q . See [1]. Now, when $\lambda = 0$,

$$Lu = g(x, t, u) . \tag{2.12}$$

Equation (2.12) with (2.2) has a unique periodic solution since g satisfies the Lipschitz condition (uniformly) in u (see [5], p. 202).

Therefore, by the Leray-Schauder theorem, the problems (2.1), (2.2) have periodic solution. This completes the proof.

For uniqueness of the solution, an additional hypothesis is imposed. This is the content of

COROLLARY 2.1. *In addition to the hypotheses of Theorem* 2.1, *suppose that, for all* t, x, u, v, p ,

$$\left(v_i-u_i\right)\left[f_i(x, t, u, p)-f_i(x, t, v, p)\right] \geq c\left(v_i-u_i\right)^2 , \quad i = 1, \ldots, n ,$$

for some positive constant c . *Then problem* (2.1), (2.2) *has a unique periodic solution* (*of period* T).

DEFINITION 2.1. The functions $u, v, C^{2,1}(Q, R^n)$ are said to be *coupled quasi-solutions* of (2.1), (2.2) if

$$\begin{aligned} &u_{it} - u_{ixx} = f_i\left(x, t, u_i, [u]_1, [v]_2, u_{ix}\right) , \\ &B_iu_i(j, t) = \omega_{ij}(t) , \quad j = 0, 1 , \quad i = 1, 2, \ldots, n , \end{aligned} \tag{2.13a}$$

and

$$\begin{aligned} &v_{it} - v_{ixx} = f_i\left(x, t, v_i, [v]_1, [u]_2, v_{ix}\right) , \\ &B_iv_i(j, t) = \omega_{ij}(t) , \quad j = 0, 1 , \quad i = 1, 2, \ldots, n . \end{aligned} \tag{2.13b}$$

In view of Theorem 2.1, we see that the problem (2.13) has a solution and hence (2.1), (2.2) have coupled quasi-solutions.

3. Monotone iterative method

To develop a monotone iterative scheme for the solution of the boundary value problem (2.1), (2.2), we need the following one-sided condition similar to Lipschitzian.

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(vii)

$$f_i(x,\ t,\ v_i,\ [v]_1,\ [u]_2,\ v_{ix}) - f_i(x,\ t,\ u_i,\ [u]_1,\ [v]_2,\ u_{ix}) \leq M_{1i}(B)(u_i-v_i) + M_{2i}(u_{ix}-v_{ix})$$

for $\phi_i(x,\ t) \leq v_i \leq u_i \leq \psi_i(x,\ t)$, where $B \subset R^{n+1}$ is any bounded set and $M_{1i} = M_{1i}(B)$, M_{2i} are positive constants.

Consider now the boundary value problem

$$Lu_i = Y_i(x,\ t,\ u,\ u_{ix}) \tag{3.1}$$

with the boundary conditions (2.2) where

$$\begin{aligned} &Y_i(x,\ t,\ u,\ u_{ix}) \\ &= H_i(x,\ t,\ \eta_{1i},\ [\eta_1]_1,\ [\eta_2]_2,\ \eta_{1ix}) - M_{1i}(u_i-\eta_{1i}) - M_{2i}(u_{ix}-\eta_{1ix}) \end{aligned} \tag{3.2}$$

for any periodic function $\eta_1,\ \eta_2 \in C[Q,\ R^n]$ such that $\phi(x,\ t) \leq \eta_1,\ \eta_2 \leq \psi(x,\ t)$, $|\eta_{1ix}| \leq N_i$ on Q and H_i is truncated in p_i; that is

$$H_i(x,\ t,\ u,\ p_i) = f_i(x,\ t,\ u,\ \bar{p}_i)$$

where

$$\bar{p}_i = \begin{cases} N_i & \text{if } p_i > N_i, \\ p_i & \text{if } -N_i \leq p_i \leq N_i, \\ -N_i & \text{if } p_i \leq -N_i, \end{cases}$$

where N_i is the bound of the derivative of the solution of (2.1), (2.2). We can easily show that Y_i satisfies all the hypotheses (i)-(vi) of Theorem 2.1 (see [3]).

LEMMA 3.1. *Let the assumptions* (i)-(vi) *hold. Then there exists a unique periodic solution* $u \in C^{2,1}(Q,\ R^n)$ *to the boundary value problem* (3.1), (2.2) *such that*

$$\phi_i(x,\ t) \leq u_i(x,\ t) \leq \psi_i(x,\ t) \quad \text{on } Q$$

and there exists a $N_0 > 0$ *such that*

$$|u_{ix}(x, t)| \le N_{0i} \quad \text{on} \quad Q .$$

Proof. The existence is assumed by Theorem 2.1. The uniqueness can be proved on lines similar to the part of the proof in Theorem 2.1 where any solution v is shown to satisfy (2.10). Hence the lemma.

Now for each pair (η_1, η_2) as used in defining Y_i in (3.2), let $A[\eta_1, \eta_2]$ denote the unique $v \in C^{2,1}(Q, R^n)$ which is the solution of the boundary value problem (3.1), (2.2).

LEMMA 3.2. *Under the assumptions of Lemma* 3.1 *and* (vii) *above, the mapping* $A[\cdot, \cdot]$ *satisfies*

$$A[\eta_1, \eta_2] \le A[\eta_2, \eta_1] \tag{3.3}$$

where η_1, η_2 *are any periodic functions with* $(\phi \le)\ \eta_1 \le \eta_2\ (\le \psi)$ *and* $|\eta_{1ix}|, |\eta_{2ix}| \le N_{0i}$.

Proof. Let $A[\eta_1, \eta_2] = w_1$, $A[\eta_2, \eta_1] = w_2$. Then writing $z_i = w_{1i} - w_{2i}$, if possible, let $(x_0, t_0) \subset Q$ be the interior point such that z_i attains a positive maximum there:

$$Lw_{1i} - Lw_{2i} = -z_{ixx} ,$$

left hand side

$$\begin{aligned}
&= H_i(x_0, t_0, \eta_1, [\eta_1]_1 [\eta_2]_2, \eta_{1ix}) - M_{1i}(w_{1i}-\eta_{1i}) \\
&\qquad - M_{2i}(w_{1ix}-\eta_{1ix}) - H_i(x_0, t_0, \eta_{2i}, [\eta_2]_1, [\eta_1]_2, \eta_{2ix}) \\
&\qquad\qquad + M_{2i}(w_{2i}-\eta_{2i}) + M_{2i}(w_{2ix}-\eta_{2ix}) \\
&= f_i(x_0, t_0, \eta_{1i}, [\eta_1]_1, [\eta_2]_2, \eta_{1ix}) + M_{1i}(\eta_{1i}-\eta_{2i}) \\
&\qquad - f_i(x_0, t_0, \eta_{2i}, [\eta_2]_1, [\eta_1]_2, \eta_{2ix}) + M_{2i}(\eta_{1ix}-\eta_{2ix}) \\
&\qquad\qquad + M_{1i}(w_{2i}-w_{1i}) + M_{2i}(w_{2ix}-w_{1ix}) \\
&< 0 \quad (\text{by (vii)})
\end{aligned}$$

which is a contradiction. From boundary conditions we have that $z_i(j, t) \le 0$ for all t in R , $j = 0, 1$, $i = 1, 2, \ldots, n$ (see [5],

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page 53). Therefore $z_i(x, t) \le 0$ on $\bar{Q}$. Hence the lemma.

We define the sequences

$$v_n = A[v_{n-1}, w_{n-1}], \quad w_n = A[w_{n-1}, v_{n-1}], \tag{3.4}$$

where $v_0 = \phi$, $w_0 = \psi$. In view of Lemma 3.2, we get

$$v_0 \le v_1 \le \ldots \le v_n \le w_n \le \ldots \le w_0,$$

$$\begin{aligned} v_{n,i}(x, t) = d_{i0}(x, t) &+ \int_{t-\delta}^{t} \int_0^1 G(x, t; \sigma, \theta) \\ &\left[H_i\left(\phi, \theta, v_{n-1,i}, [v_{n-1}]_1, [w_{n-1}]_2, v_{n-1,ix}\right) + M_{1i}\left(v_{ni}-v_{n-1,i}\right)\right. \\ &\left.+ M_{2i}\left(v_{nix}-v_{n-1,ix}\right)\right]d\sigma d\theta + \int_0^1 G(x, t; \sigma, t-\delta)v_{n,i}(\sigma, t-\delta)d\sigma \end{aligned}$$

where G is Green's function and

$$\begin{aligned} v_{n,ix}(x, t) = d_{i0x} + \int_{t-\delta}^{t} \int_0^1 G_x\left[H_i+M_{1i}\left(v_{ni}-v_{n-1,i}\right) + M_{2i}\left(v_{nix}-v_{n-1,ix}\right)\right]d\sigma d\theta \\ + \int_0^1 G_x v_{ni} d\sigma . \end{aligned}$$

Since for each i, H_i is a bounded function, the sequence $\{v_n\}$ is uniformly bounded and equicontinuous. Then by the Arzela-Ascoli theorem $\{v_n\}$ contains a subsequence which is uniformly convergent. Since $\{v_n\}$ is monotone, the full sequence converges uniformly. $\{v_{nx}\}$ contains a subsequence which is uniformly convergent. We can find subsequences which we again denote by $\{v_n\}$, $\{w_n\}$ converging uniformly and monotonically.

Let $\alpha(x, t) = \lim_{n\to\infty} v_n(x, t)$, $\beta(x, t) = \lim_{n\to\infty} w_n(x, t)$. Then

$$L\alpha_i = H_i\left(x, t, \alpha_i, [\alpha]_1, [\beta]_2, \alpha_{ix}\right),$$

$$B_i\alpha(j, t) = \omega_{ij}(t), \quad j = 0, 1, \quad i = 1, 2, \ldots, n,$$

$$L\beta_i = H_i\left(x, t, \beta_i, [\beta_1]_1, [\alpha]_2, \beta_{ix}\right),$$

$$B_i\beta(j, t) = \omega_{ij}(t), \quad j = 0, 1, \quad i = 1, 2, \ldots, n.$$

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For each i, using the continuation arguments as outlined in Theorem 1.5.1 of Bernfeld and Lakshmikantham [2], we arrive at the conclusion that α, β are coupled quasi-solutions of (1.1), (1.2).

Using similar arguments, we can conclude that

$$v_n(x, t) \leq u, \; v \leq w_n(x, t) \quad \text{on} \quad Q$$

where u, v are any coupled quasi-solutions of (1.1), (1.2).

We claim that (α, β) are coupled minimal and maximal quasi-solutions. Let (u, v) be any coupled quasi-solutions of (1.1), (1.2) such that $\phi_i \leq u_i$, $v_i \leq \psi_i$ on Q. Since $v_0 = \phi$, $w_0 = \psi$, $v_0 \leq u$ and $w_0 \geq v$, then $v_1 = A\left[v_0, w_0\right] \leq A[u, v] = u$ and so on. This implies that $v_n \leq u$ and similarly $w_n \geq v$ for all n. Therefore $\lim\limits_{n\to\infty} v_n = \alpha(x, t) \leq u$ and $\lim\limits_{n\to\infty} w_n(x, t) = \beta(x, t) \geq v$. Hence (α, β) is coupled minimal and maximal quasi-solutions.

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