ON THE NUMBER OF FACES OF A CONVEX POLYTOPE

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1. Introduction. The following problem is as yet unsolved: Given a convex polytope with N vertices in n-space, what is the maximum number of $(n-1)$-faces which it can have? Aside from its geometric interest this question arises in connection with solving systems of linear inequalities and linear equations in non-negative variables. The problem is equivalent to asking for the best bound on the number of basic solutions for such problems and hence a bound (though a weak one) for the number of iterations needed in the simplex method for solving linear programmes.

If we denote by $\mu(n, N)$ the maximum number of faces for a convex polytope with N vertices in n-space, it has been conjectured that these numbers are given by the formula

$$\mu(n,N) = \begin{cases} \binom{N-\frac{1}{2}n}{N-n} + \binom{N-\frac{1}{2}n-1}{N-n} & \text{for } n \text{ even,} \\ 2\binom{N-\frac{1}{2}(n+1)}{N-n} & \text{for } n \text{ odd.} \end{cases} \tag{1}$$

The number of faces given by (1) are actually achieved by the so-called *cyclic polytopes* **(1)** defined as the convex hull of N points on the curve $\chi(t) = (t, t^2, \ldots, t^n)$ in n-space. Also one easily verifies that (1) is correct for the cases $n = 1, 2, 3$ and also the case $N = n + 1$. Our purpose here is to show that it is also true for $N = n + 2$ and $N = n + 3$. An earlier proof (unpublished) of this result was geometric and proceeded via duality methods for linear inequalities. The much simpler proof presented here depends on reducing the problem to one which is purely combinatorial. For the case $N = n + 3$ this is a problem in graph theory. The solution of this problem, Theorem 4, is due to Allan H. Clark, and it is his result which makes the combinatorial method succeed in this case.

2. Preliminaries. A *convex polytope* is the convex hull of a finite set S in n-space. For our purposes it will be sufficient to study the set S itself.

A subset $F = \{a_1, \ldots, a_k\}$ of S is called a *face* of S if F generates a hyperplane H (i.e., the affine space through F has dimension $n - 1$) and all points of $S - F$ lie on the same side of H (two points lie on the same side of H if the segment connecting them does not meet H).

Received July 9, 1962. This work was supported in part by the Office of Naval Research under Task NR 047-018.

https://doi.org/10.4153/CJM-1964-002-x Published online by Cambridge University Press

We are interested in choosing the points a_i so as to maximize the number of faces of S. For this purpose it is fairly clear that we may assume the points to be in general position, that is, no $n+1$ points of S lie in a hyperplane, for if such degeneracy did occur, one could perturb the position of the points of S slightly without reducing the number of its faces. General position will henceforth be assumed.

Notations. If $a = (\alpha_1, \ldots, \alpha_n)$ is a point of n-space, we denote by $\tilde{a}$ the point in $(n+1)$-space given by $\tilde{a} = (\alpha_1, \ldots, \alpha_n, 1)$. If $a_1, \ldots, a_n$ are points in n-space, we denote their determinant by $D(a_1, \ldots, a_n)$. For any number α we define $\operatorname{sg}(\alpha)$ to be the usual signum function equal to 1, -1, or 0 according as α is positive, negative, or zero.

Our starting point is the following probably well-known fact.

LEMMA 1. *If $a_1, \ldots, a_n, b, c$ are in general position in n-space, then b and c are separated by the hyperplane through $a_1, \ldots, a_n$ if and only if*

$$D(\tilde{b}, \tilde{a}_1, \ldots, \tilde{a}_n) \quad \textit{and} \quad D(\tilde{c}, \tilde{a}_1, \ldots, \tilde{a}_n)$$

have opposite signs.

Proof. The hyperplane H through the a_i is given by

$$H = \{x | x = \sum \alpha_i a_i, \sum \alpha_i = 1\}$$

and the segment $[b, c]$ is given by

$$[b, c] = \{x | x = \beta b + \gamma c; \beta, \gamma \geqslant 0, \beta + \gamma = 1\}.$$

Thus H separates b and c if and only if the equation

$$\sum_{i=1}^{n} \alpha_i \tilde{a}_i = \beta \tilde{b} + \gamma \tilde{c}$$

has a solution with $\beta, \gamma > 0$ (neither can be zero because of their general position), or equivalently (letting $\alpha_i' = \alpha_i/\gamma, \beta' = \beta/\gamma$),

$$\sum \alpha_i' \tilde{a}_i = \beta' \tilde{b} + \tilde{c}$$

has a solution with $\beta' > 0$. But if we solve the last equation for β' by Cramer's rule, we have

$$\beta' = - \frac{D(\tilde{c}, \tilde{a}_1, \ldots, \tilde{a}_n)}{D(\tilde{b}, a_1, \ldots, \tilde{a}_n)}$$

so that $\beta' > 0$ if and only if the condition of the lemma is satisfied.

As a consequence of the lemma we obtain a criterion for a subset of S to be a face.

THEOREM 1. *The subset $F = \{a_1, \ldots, a_n\}$ is a face of S if and only if the determinants $D(\tilde{a}, \tilde{a}_1, \ldots, \tilde{a}_n)$ have the same sign for all a in $S - F$.*

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3. The case $N = n + 2$. We shall denote the points of S by $a_1, a_2, \ldots, a_{n+2}$ and we shall denote by $D(\hat{a}_i)$ the determinant $D(\tilde{a}_1, \ldots, \tilde{a}_{i-1}, \tilde{a}_{i+1}, \ldots, \tilde{a}_{n+2})$ in which $\tilde{a}_i$ has been omitted from the sequence. We denote by $D(\tilde{a}_j, \hat{a}_i)$ the determinant $D(\tilde{a}_j, \tilde{a}_1, \ldots, \tilde{a}_{i-1}, \tilde{a}_{i+1}, \ldots, \tilde{a}_{n+2})$ in which the jth term is moved to the beginning of the sequence and the ith term is omitted.

Now define a function ϕ on S which assigns to each a_i either 1 or -1 as follows:

$$\phi(a_i) = (-1)^i \operatorname{sg} D(\hat{a}_i).$$

Note that $(-1)^i \phi(a_i) = \operatorname{sg} D(\tilde{a}_i)$.

THEOREM 2. *For $i \neq j$ the set $F = S - \{a_i, a_j\}$ is a face if and only if $\phi(a_i) = -\phi(a_j)$.*

Proof. Say $i < j$. By Theorem 1, F is a face if and only if $\operatorname{sg} D(\tilde{a}_i, \hat{a}_j) = \operatorname{sg} D(\tilde{a}_j, \hat{a}_i)$. By the elementary properties of interchanging rows of a determinant we have

$$\operatorname{sg} D(\tilde{a}_i, \hat{a}_j) = (-1)^{i-1} \operatorname{sg} D(\hat{a}_j) = (-1)^{i-1}(-1)^j\phi(a_j) = (-1)^{i+j-1}\phi(a_j)$$

and

$$\operatorname{sg} D(\tilde{a}_j, \hat{a}_i) = (-1)^{j-2} \operatorname{sg} D(\hat{a}_i) = (-1)^{j-2}(-1)^i\phi(a_i) = (-1)^{i+j-2}\phi(a_i),$$

hence, $\phi(a_i) = -\phi(a_j)$, as asserted.

From Theorem 2 it follows at once that the maximum number of faces cannot exceed the number ν of pairs $\{a_i, a_j\}$ such that $\phi(a_i)$ and $\phi(a_j)$ have opposite signs. But ν is clearly maximized by having $\phi(a_i) = 1$ for "half" of the a_i or more precisely for $[\frac{1}{2}n + 1]$ of the a_i. This gives the numbers

$$\nu = \begin{cases} [\frac{1}{2}(n+2)]^2 & \text{for } n \text{ even,} \\ [\frac{1}{2}(n+1)][\frac{1}{2}(n+3)] & \text{for } n \text{ odd,} \end{cases}$$

which agrees with formula (1) when $N = n + 2$.

4. The case $N = n + 3$. We write $S = \{a_1, \ldots, a_{n+3}\}$ and denote by $D(\hat{a}_i, \hat{a}_j)$ the determinant obtained by omitting $\tilde{a}_i$ and $\tilde{a}_j$ from the sequence $\tilde{a}_1, \ldots, \tilde{a}_{n+3}$, and we denote by $D(\tilde{a}_k, \hat{a}_i, \hat{a}_j)$ the determinant obtained from the same sequence but in which $\tilde{a}_k$ has been moved to the beginning of the sequence.

Now define ϕ on pairs $\{a_i, a_j\}$ by the rule

$$\phi\{a_i, a_j\} = (-1)^{i+j} \operatorname{sg} D(\hat{a}_i, \hat{a}_j)$$

and note that $(-1)^{i+j}\phi\{a_i, a_j\} = \operatorname{sg} D(\hat{a}_i, \hat{a}_j)$.

THEOREM 3. *If $i < j < k$, then $F = S - \{a_i, a_j, a_k\}$ is a face of S if and only if*

$$\phi\{a_i, a_j\} = \phi\{a_j, a_k\} = -\phi\{a_i, a_k\}. \tag{2}$$

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Proof. By Theorem 1, F is a face if and only if

$$\operatorname{sg} D(\tilde{a}_i, \hat{a}_j, \hat{a}_k) = \operatorname{sg} D(\tilde{a}_j, \hat{a}_i, \hat{a}_k) = \operatorname{sg} D(\tilde{a}_k, \hat{a}_i, \hat{a}_j).$$

Now

$$\begin{aligned}
\operatorname{sg} D(\tilde{a}_i, \hat{a}_j, \hat{a}_k) &= (-1)^{i-1} \operatorname{sg} D(\hat{a}_j, \hat{a}_k) = (-1)^{i+j+k-1}\phi\{a_j, a_k\},\\
\operatorname{sg} D(\tilde{a}_j, \hat{a}_i, \hat{a}_k) &= (-1)^{j-2} \operatorname{sg} D(\hat{a}_i, \hat{a}_k) = (-1)^{i+j+k-2}\phi\{a_i, a_k\},\\
\operatorname{sg} D(\tilde{a}_k, \hat{a}_i, \hat{a}_j) &= (-1)^{k-3} \operatorname{sg} D(\hat{a}_i, \hat{a}_j) = (-1)^{i+j+k-3}\phi\{a_i, a_j\},
\end{aligned}$$

so (2) is verified.

Of course, Theorems 2 and 3 are special cases of a general theorem for arbitrary N, but since we shall only need the results in special cases, we have chosen to avoid the calculations and definitions needed in the general case.

Because of Theorem 3 our problem can be reduced to a problem in graph theory. We consider a graph whose vertices are the points a_i and whose edges are all pairs $\{a_i, a_j\}$. Such a graph is called a *complete graph* with N vertices. We now *orient* the edges of the graph as follows: for $i < j$ we orient $\{a_i, a_j\}$ in the direction from a_i to a_j (from a_j to a_i) if $\phi\{a_i, a_j\} = 1$ (if $\phi\{a_i, a_j\} = -1$). Condition (2) is then equivalent to the requirement that $\{a_i, a_j, a_k\}$ be the vertices of a *cyclic triangle*. If we can find an upper bound for the number of such triangles, this will, in view of Theorem 3, give an upper bound to the number of faces of our polytope.

THEOREM 4 (Clark). *The maximum number of cyclic triangles in an oriented complete graph with N vertices is*

$$2\binom{\frac{1}{2}(N+2)}{3} \qquad \text{for } N \text{ even},$$

$$\binom{\frac{1}{2}(N+3)}{3} + \binom{\frac{1}{2}(N+1)}{3} \qquad \text{for } N \text{ odd}.$$

Proof. We observe that every non-cyclic triangle contains exactly one vertex at which two edges terminate. Let T_i be the number of edges terminating at the vertex a_i. Then any pair of such edges will belong to a non-cyclic triangle. Hence, there are $T_i(T_i - 1)/2$ non-cyclic triangles associated with a_i and, therefore, there are in all

$$\frac{1}{2}\sum_{i=1}^{N} T_i(T_i - 1) = \frac{1}{2}\sum T_i^2 - \frac{1}{2}\sum T_i \tag{3}$$

non-cyclic triangles in the graph.

On the other hand, the sum of the T_i is the total number of edges, i.e.,

$$\sum_{i=1}^{N} T_i = N(N-1)/2, \tag{4}$$

and combining this with (3) we see that in order to maximize the number of cyclic triangles we must minimize

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$$\sum_{i=1}^{N} T_i^{\,2}$$

subject to (4).

The simple "calculus" solution to this constrained minimum problem is given by $T_i = \frac{1}{2}(N-1)$, and, therefore, this is also the diophantine solution when N is odd. In this case substituting in (3) gives for the number of non-cyclic triangles

$$N(N-1)(N-3)/8$$

and subtracting from the total number of triangles, $N(N-1)(N-2)/6$, gives $(N+1)N(N-1)/24$. A simple calculation shows that this agrees with the assertion of the theorem.

In case N is even, the minimum is achieved by setting all T_i equal to $\frac{1}{2}N$ or $\frac{1}{2}N - 1$, for if, say, $T_1 = \frac{1}{2}N + p$, $p > 0$, then because of (4) some T_i is at most $\frac{1}{2}N - 1$, say, $T_2 = \frac{1}{2}N_2 - q$, $q > 0$. But then $\sum T_i^2$ could be further decreased by letting $T_1 = \frac{1}{2}N + p - 1$, $T_2 = \frac{1}{2}N - q + 1$.

Again from (4) it is necessary that $T_i = \frac{1}{2}N$ for $\frac{1}{2}N$ values of i and $T_i = \frac{1}{2}N - 1$ for the other $\frac{1}{2}N$. Hence, a lower bound for the number of non-cyclic triangles is

$$\tfrac{1}{2}N \times \tfrac{1}{2}N \times \tfrac{1}{2}(\tfrac{1}{2}N - 1) + \tfrac{1}{2}N \times \tfrac{1}{2}(\tfrac{1}{2}N - 1)(\tfrac{1}{2}N - 2) = N(N-2)^2/8$$

and subtracting from the total number of triangles gives

$$(N+2)N(N-2)/24,$$

which is the desired number.

We finally note that the numbers given in Theorem 4 give an upper bound to the number of faces of a polytope with N vertices in $(N-3)$-space. On the other hand, substituting $n = N - 3$ in formula (1) gives precisely these numbers, and since we know that these numbers are realized by the cyclic polytopes, it follows that the numbers are also realized by suitable graphs.

5. The general case. Although the conjecture on the maximum number of faces has not been proved in general, the procedure for constructing the related combinatorial problem can be generalized. We conclude by stating this problem and the corresponding conjecture, a proof of which would give an affirmative answer to the original conjecture on convex polytopes.

Let S be the set of integers $\{1, 2, \ldots, N\}$ and let ϕ be a function which assigns to every $(m-1)$-element subset of S either 1 or -1.

An m-element subset $\{a_1, \ldots, a_m\}$ of S is said to be *oriented* by ϕ if the numbers

$$(-1)^k \phi\{a_1, \ldots, \hat{a}_k, \ldots, a_m\}$$

are all equal.

Problem. Find ϕ which maximizes the number of oriented m-element sets.

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Conjecture. The maximum is achieved by defining ϕ according to the rule

$$\phi\{a_1, \ldots, a_{m-1}\} = \begin{cases} +1 \\ -1 \end{cases} \text{if} \quad \sum_{i=1}^{m-1} a_i \text{ is } \begin{matrix}\text{even,}\\ \text{odd.}\end{matrix}$$

For this case one can easily show that the number of oriented m element sets is

$$\binom{\frac{1}{2}(m+N)}{m} + \binom{\frac{1}{2}(m+N-2)}{m} \qquad \text{if } m+N \text{ is even,}$$

$$2\binom{\frac{1}{2}(m+N-1)}{m} \qquad \text{if } m+N \text{ is odd.}$$

We have here proved the conjecture for $m = 2$ and $m = 3$.

Added in proof. Since this was written, Klee has shown **(2)** that formula (1) holds provided $N \geqslant (n/2)^2 - 1$. Some results announced by Motzkin **(3)** are also relevant.

Reference

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https://doi.org/10.4153/CJM-1964-002-x Published online by Cambridge University Press