Can. J. Math., Vol. XXXVI, No. 3, 1984, pp. 470-494

A CLASS OF HOMOMORPHISMS OF PRE-HJELMSLEV GROUPS

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Introduction. E. Salow [**8**] introduced the concept of pre-Hjelmslev groups, a generalization of F. Bachmann's Hjelmslev groups [**1**] which leads to a more natural theory of homomorphisms and permits a simpler construction of algebraic models. Basically, both types of groups are the groups of motions of a metric plane, the so-called group plane. In such a plane there is a unique perpendicular through any point to any line and the product of three collinear points (three copunctal lines) is a point (a line). Our first section contains the precise definitions and some basic facts.

The homomorphic image of a pre-Hjelmslev group can be more complicated than the pre-image. For instance, there may always be a unique line through two distinct points of the pre-image but not of the image. We study regular homomorphisms of pre-Hjelmslev groups, i.e., homomorphisms with the following property: If two lines intersect at exactly one point, their images will also have precisely one point in common.

Let Q denote a proper subset of the point set of a pre-Hjelmslev group satisfying an enrichment axiom called (W). We call Q *complete* if the following holds: Suppose two lines have a unique intersection C and both of them are incident with points of Q. Then $C \in Q$. Our main result is the following:

THEOREM. *There is a regular homomorphism of the pre-Hjelmslev group such that Q consists of the pre-image points of a point if and only if Q is complete.*

The special cases that Q consists of the fixed points of a rotation or that Q is the set of the neighbors of some point have been dealt with in [**9**] and [**4**].

In a forthcoming paper we study pre-Hjelmslev groups over commutative rings and establish a one-to-one correspondence between the non-trivial ideals of the ring and the kernels of regular homomorphisms of the pre-Hjelmslev group.

Received September 24, 1982 and in revised form June 13, 1983.

1. Pre-Hjelmslev groups.

The basic assumption. The triplet (G, S, P) consists of a group $G = \{\alpha, \beta, \dots\}$ and two sets $S = \{a, b, \dots\}$ and $P = \{A, B, \dots\}$ of involutions in G such that (i) S and P are invariant under inner automorphisms of G and $S \cap P = \emptyset$, (ii) S generates G, and (iii) $\emptyset \neq P \subseteq S^2 = \{ab\}$.

We assign to such a triplet a geometric structure, the *group plane*. Its *points* (*lines*) are the elements of P (of S). The point A and the line b are *incident*, $A|b$ or $b|A$, if Ab is an involution. The lines a and b are *orthogonal* if $ab \in P$; notation: $a|b$.

Every $\alpha \in G$ induces a *motion*, i.e., an automorphism of the group plane, given by $X \mapsto X^\alpha$, $x \mapsto x^\alpha$ for $X \in P$ and $x \in S$. If $\alpha \in P \cup S$, this motion is a *reflection* in α. We do not always distinguish between the element α and the motion induced by α. Thus the set

$$\mathrm{F}(\alpha) := \{X \in P : X^\alpha = X\}$$

of "the fixed points of α" is that of those of the induced motion.

A *pre-Hjelmslev group* is a triplet (G, S, P) satisfying the basic assumption and the following axioms:

(A1) Given A, b, there is a c such that $A, b|c$.
(A2) $A, b|c, d$ implies $c = d$.
(A3) $A, B, C|d$ implies $ABC \in P$.
(A4) $a, b, c|d$ implies $abc \in S$.

By (A1) and (A2), there is a unique perpendicular (A, b) through any point A to any line b. (A3) and (A4) are the "Three-reflections axioms".
We shall frequently use the following enrichment axiom:

(W) There are lines a, b, c, d with $a|b$ and $c|d$ such that any two of them intersect in exactly one point.

We next collect some elementary results on pre-Hjelmslev groups. If no reference is given, the proof in **[2]** for Hjelmslev groups remains valid for (G, S, P).

1.1. (i) *$A|b$ if and only if $A^b = A$.*
(ii) *If $A|b$, c and $b|c$ then $A = bc$. If $A|b$ then $Ab \in S$ and $Ab = (A, b)$.*
(iii) *If $A, B, C|d$ then $ABC \in P$ and $ABC|d$.*
(iv) *If $a, b, c|D$ then $abc \in S$ and $abc|D$.*
(v) *If $a, b, c|d$ then $abc \in S$ and $abc|d$.*

1.2. *Let $Aa = Bb = cC$. Then $(A, a) = (B, b) = (C, c)$.*

Occasionally we need the following consequence of 1.2.

1.2′. *$AbC \in S$ if and only if $(A, b)\,|C$.*

https://doi.org/10.4153/CJM-1984-029-4 Published online by Cambridge University Press

Namely, the assumption $c: = AbC \in S$ implies $Ab = cC$, hence

$$(A, b) = (C, c)\,|C.$$

Conversely, let $(A, b)\,|C$. Then

$$B: = b(A, b) \in P \quad \text{and} \quad A, B, C|(A, b).$$

1.1 (iii) implies $D: = ABC \in P$ and $D|\,(A, b)$. Therefore by 1.1 (ii) $D(A, b) = d$, where $d: = (D, (A, b)\,)$. Hence $AbC = d \in S$.

An element $\alpha = Aa$ is a *glide reflection* with the *axis* $[\alpha]: = (A, a)$. If $\alpha \notin S$, then $\mathrm{F}(\alpha) = \emptyset$.

1.3. *The group G is the disjoint union of the subgroup* $S^{\text{even}}: = S^2 \cup S^4 \ldots$ *and its coset* $S^{\text{odd}}: = S \cup S^3 \ldots$.. *Let* $\alpha \in S^{\text{even}}$ *and* $\mathrm{F}(\alpha) \neq \emptyset$. *Then* α *is a rotation. If* $A \in \mathrm{F}(\alpha)$ *and* $u|A$ *then* $\alpha = uv$ *for some* v *with* $v|A$.

REPRESENTATION THEOREM. *Let* $A \in P$. *Every* $\alpha \in S^{\text{even}}$ *has a unique decomposition* $\alpha = \beta C$ *where* β *is a rotation with* $A \in \mathrm{F}(\beta)$ *and* $C \in P$. *Every* $\alpha \in S^{\text{odd}}$ *has a unique decomposition* $\alpha = bC$ *where* $b|A$ *and* $C \in P$.

1.4. The point C is a *mid-point* of A and B if $A^C = B$. Two points have not more than one mid-point. Let $\alpha \in G$. By 1.3, A and A^α have a mid-point.

1.5. For any group H, let $\mathrm{Z}(H)$ denote its center. Then

$$\mathrm{Z}(S^{\text{even}}) = \{\alpha \in S^{\text{even}}: \mathrm{F}(\alpha) = P\}.$$

1.6. For every a define $\mathrm{P}_a = \{A:A|a\}$.

Let $a, b|c$. *Let* $A|a, g$ *and* $B|b, g$. *Then the mapping* $C \mapsto CAB$ *is a bijection of* $\mathrm{P}_a \cap \mathrm{P}_g$ *onto* $\mathrm{P}_b \cap \mathrm{P}_g$. *In particular,* $\mathrm{P}_a \cap \mathrm{P}_g = \{A\}$ *if and only if* $\mathrm{P}_b \cap \mathrm{P}_g = \{B\}$.

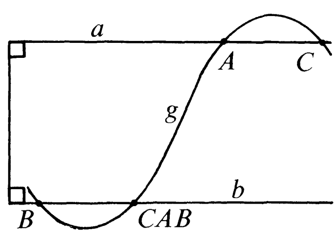

Figure 1

1.6′. COROLLARY. *Let* $a, b|c$. *Let* $b|d$. *Then* a *and* d *have at most one point in common.*

1.7. ([**8**], Lemma 1). *Let* $a, b|c$; $A|a, g$ *and* $B|b, g$. *Then* $\mathrm{F}(ag) = \{A\}$ *if and only if* $F(bg) = \{B\}$.

Applying 1.7 three times, we obtain

1.7′. COROLLARY. *Let* $A|a, b$; $B|c, d$; $a|c$ *and* $b|d$. *Then* $\mathrm{F}(ab) = \{A\}$ *if and only if* $\mathrm{F}(cd) = \{B\}$.

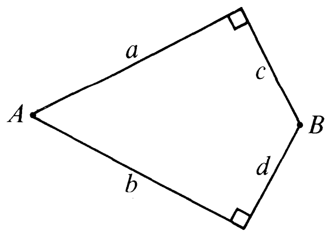

Figure 2

1.8. *Let* α *be a rotation*; $g \in S$. *Then* $\alpha g \in S$ *if and only if*

$$\mathrm{F}(\alpha) \cap \mathrm{P}_g \neq \emptyset.$$

In particular, let $\mathrm{F}(\alpha) = \{A\}$. *Then* $\alpha g \in S$ *if and only if* $A|g$.

1.9. *Suppose* (G, S, P) *satisfies* (W). (i) *Let* $a|b$. *Then there are lines* c, d *such that* $ab = cd|a, b, c, d$ *and not two of these lines intersect elsewhere.* (ii) *The lines* a *and* b *have a unique intersection if and only if* $|\mathrm{F}(ab)| = 1$. *In particular, let* $ab = cd$. *If* a, b *have a unique intersection then so will* c, d.

1.10. The pre-Hjelmslev group (H, T, Q) is a *pre-Hjelmslev subgroup* of the pre-Hjelmslev group (G, S, P) if H is a subgroup of G, $T \subseteq S$, $Q \subseteq P$. We then write $(H, T, Q) \leqq (G, S, P)$.

Let $(H, T, Q) \leqq (G, S, P)$. *Then* $T = S \cap H$ *and* $Q = P \cap H$. *Let* $a, b \in T$, $C \in Q$. *Then* $a|b$ (*Then* $a|C$) *in* (H, T, Q) *if and only if* $a|b$ $(a|C)$ *in* (G, S, P).

Proof. Since $T^{\text{even}} \subseteq S^{\text{even}}$ and $T^{\text{odd}} \subseteq S^{\text{odd}}$, we have

$$S \cap H \subseteq T^{\text{odd}} \quad \text{and} \quad P \cap H \subseteq T^{\text{even}}.$$

Let $a \in S \cap H$. Choose $A \in Q$. Then by 1.3, $a = bC$ for some $b \in T$, $C \in Q$ such that bA is an involution. Thus $a \in T$ by 1.1 (ii). Next, let $B \in P \cap H$. By 1.3 there are $g, h \in T$ and $C \in Q$ such that gA and hA are involutions and $B = ghC$. Here gh and C are uniquely determined. As $B = 1 \cdot B$, this yields $B = C \in Q$. The remaining assertions are obvious.

1.11. For any set $Q \subseteq P$ let $\mathrm{S}(Q)$ consist of those lines in S which meet points of Q.

Let $(H, T, Q) \leqq (G, S, P)$. *Thus* $T \subseteq \mathrm{S}(Q)$. *Suppose* (i) *If* $B \in P$ *and* A, $A^B \in Q$, *then* $B \in Q$, (ii) $\mathrm{S}(Q) \subseteq T$ (*thus* $\mathrm{S}(Q) = T$). *Then* (H, T, Q) *is called a spot of* (G, S, P). *In this case,*

$$H = \mathrm{N}_G(Q) := \{\alpha \in G : \alpha^{-1} Q \alpha \subseteq Q\}.$$

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Proof. As (H, T, Q) satisfies the basic assumption, we have $H \subseteq \mathrm{N}_G(Q)$. Conversely, let $\alpha \in \mathrm{N}_G(Q)$. Choose $A \in Q$. By 1.3 there are $\beta \in G$ and $C \in P$ such that $\alpha = \beta C$. Here β is a product of lines through A. Hence $\beta \in H$, by (ii). As $\alpha \in \mathrm{N}_G(Q)$, we have

$$A^{\alpha} = A^{\beta C} = A^C \in Q.$$

Thus $C \in Q \subseteq H$, by (i), and $\alpha = \beta C \in H$.

The final propositions of this section aim at Hjelmslev groups (without a "pre"). They will not be used in the sequel.

The pre-Hjelmslev group (G, S, P) is a *Hjelmslev group* if

$$P = \{ab{:}a, b \in S \text{ and } ab \text{ is an involution}\}.$$

1.12. (cf. [**9**], 2.8). *Let (G, S, P) be a pre-Hjelmslev group; $ab = ba$; $A|a$. Then $(A, b)b|a$. In particular, any two commuting lines in a pre-Hjelmslev group have a point in common.*

Proof. Let $(A, b) = c$. Thus

$$A|c, c^a \quad \text{and} \quad bc, b \cdot c^a = (bc)^a \in P.$$

Thus $c = c^a$, by (A2), and hence $bc|a, b$.

1.13. *Let $A \in P$. Then the pre-Hjelmslev group (G, S, P) is a Hjelmslev group if and only if* $\mathrm{D}(A){:} = \{bc{:}b, c|A\}$ *contains only one involution.*

Proof. By (1.1) (ii), A is the only involution in $\mathrm{D}(A)$ if (G, S, P) is a Hjelmslev group. Conversely, suppose A is the only involution in $\mathrm{D}(A)$. We first show that this remains true if A is replaced by any point B. We may assume that $A, B|g$ for some g. Let $\beta \in \mathrm{D}(B)$, $\beta^2 = 1$. By 1.1 (iv), βg is a line c through B. Since $\beta^2 = c^2 = 1$, c and g commute. By 1.12, $C{:} = (A, c)c|g$. Hence $(A, c) = Cc$ and g commute. Thus, by our assumption, either $(A, c) = g$ or $(A, c) = Ag$. Hence $(A2)$ yields that either $Cc = g$ and $\beta = C = B$, or $g = c$ and $\beta = 1$.

Next, let $a, b \in S$ and $ab = ba \neq 1$. By 1.12, there is a point $C|a, b$. As $\mathrm{D}(C)$ contains only one involution, viz. C, we obtain $ab = C \in P$.

We mention without a proof

1.14. (cf. [**9**], Lemma 2). *Suppose the pre-Hjelmslev group (G, S, P) satisfies* $\mathrm{Z}(S^{\mathrm{even}}) = 1$ *and has the following property*:

(Z) *If $a|b$ and $ab|c$ then c has a unique intersection with a or b.*

Then (G, S, P) is a Hjelmslev group.

2. Complete point sets.

2.1. Let $\Pi = (\mathscr{P}, \mathscr{L}, I)$ denote any incidence structure. The set $Q \subseteq \mathscr{P}$ is *complete* (in Π) if it satisfies the following condition: If two lines both meet Q and have a unique intersection, then that point belongs to Q. A substructure of Π is called *complete* (in Π) if its point set is complete in Π. The sets $\mathscr{P}$ and $\emptyset$ are complete; the intersection of complete sets is complete.

Examples. (a) Suppose through any two distinct points of Π there is always a unique line. If there are three non-collinear points, the complete sets in Π are $\mathscr{P}$, $\emptyset$, and the one-point sets.

(b) Suppose the homomorphism φ maps Π into an incidence structure $\Pi' = (\mathscr{P}', \mathscr{L}', I)$, and any pair of lines intersecting uniquely in Π is mapped by φ onto a pair of lines intersecting uniquely in Π'. Then $C\varphi^{-1}$ is complete in Π for every $C \in \mathscr{P}'$.

In the remainder of this section, (G, S, P) denotes a pre-Hjelmslev group satisfying (W).

2.2. Let $\alpha \in S^2$; $Q := \mathrm{F}(\alpha) \neq \emptyset$. Hence α is a rotation. It is well known that Q is complete and that $(\mathrm{N}_G(Q), S(Q), Q)$ is a spot; cf. [**2**, page 111 Section 9.4, Folgerung 7, and page 78, 6.3]. We wish to prove the following:

THEOREM. *Let $Q \subseteq P$ be complete in (G, S, P). Then* $(\mathrm{N}_G(Q), S(Q), Q)$ *is a spot of* (G, S, P).

Proof. The set Q being complete, we have

$$Q = \{ab{:}a, b \in \mathrm{S}(Q) \text{ and } ab \in P\}.$$

We wish to show

(A3*) If $B, C, D \in Q$; $A, B, C, D|g$ and $AB = CD$ then $A \in Q$.

At first we prove (A3*) under the additional assumption

$$(+) \quad |\mathrm{F}(gh)| = |\mathrm{F}(ghD)| = 1 \quad \text{for some } h|C.$$

Let $d = Dg$. Thus $|\mathrm{F}(dh)| = 1$, say $\mathrm{F}(dh) = \{E\}$. By 1.8, $d = (dh)h \in S$ implies $E|h$. Thus $E|d, h$. By (+), $|\mathrm{F}(gh)| = 1$ and $g, h|C$. Hence by 1.9 (ii), the intersections of h with d and g are unique. Q being complete, this yields, in particular, $E \in Q$. Let $b := Bg$ and $m := (E, b)$. As $B, E \in Q$ and $b|m$, the completeness of Q also implies $bm \in Q$. Finally, let $k = mdh$ and $j = Ch$. Then

$$A = BDC = bdC = bdhj = bmkj.$$

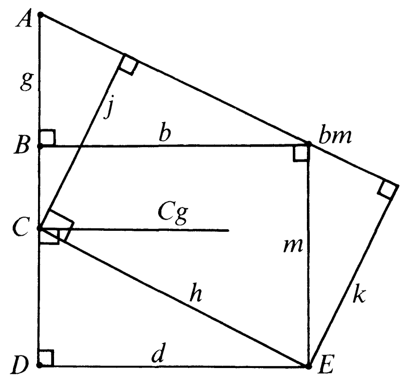

Figure 3

Hence, by 1.2, $(A, j) = (bm, k)$. The lines h and (A, j) have the common perpendicular j. Also h and g have a unique intersection. By 1.6, the intersection A of g and (A, j) is also unique. Since Q is complete, this yields $A \in Q$.

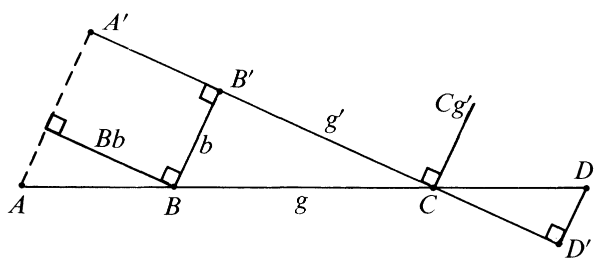

Figure 4

Now we prove (A3*) without assuming additional assumptions. Let B, C, $D \in Q$; A, B, C, $D|g$ and $AB = CD$. By 1.9, (i) and (ii), there is a line g' through C such that

$$F(gg') = F(Cgg') = \{C\}.$$

Let $b = (B, g')$, $d = (D, g')$, $B' = bg'$, $D' = dg'$. Thus Cg', b, $d|g'$; $B|b$, g; $D|d$, g. As $F(Cg'g) = \{C\}$, 1.7 implies $F(bg) = \{B\}$ and $F(dg) = \{D\}$. The completeness of Q yields B', C, $D' \in Q$. Let $A' = CD'B'$. Thus $A'|g'$; cf. 1.1 (iv). The special case of (A3*), which has already been proved, now yields $A' \in Q$. We have

$$A(Bb) = CDb = CDd \cdot db = CDdC \cdot CD'B' = (Dd)^C A'.$$

Therefore by 1.2, $(A, Bb)|A'$. Applying 1.7 once more, we obtain

$$F((A, Bb)g) = \{A\}.$$

As Q is complete, $(A, Bb) | A'$ and $A', B \in Q$ finally yield $A \in Q$.
Let $A \in Q$ and $g \in S(Q)$. Then $B: = (A, g)g \in Q$ and, by (A3*),

$$A^g = BAB \in Q.$$

Hence Q is invariant under inner automorphisms of the group $\langle S(Q) \rangle$. The same will apply to $S(Q)$. Thus $\mathscr{H} = (\langle S(Q) \rangle, S(Q), Q)$ satisfies the basic assumption; cf. Section 1. Obviously, the axioms (A1), (A2) and (A4) are satisfied, while (A3) follows from (A3*). Hence $\mathscr{H}$ is a pre-Hjelmslev subgroup of (G, S, P).

Obviously, $\mathscr{H}$ satisfies the second assumption of 1.11. We verify the first one:

(*) If $B \in P$ and $A, A^B \in Q$, then $B \in Q$.

For the present, let us assume $g|A, B$ for some line g. By 1.9 (i), there are lines h, j through B such that $B = hj$ and that no two of the lines g, h, j intersect elsewhere. By 1.9 (ii),

$$F(gh) = F(gj) = \{B\}.$$

Let $C = A^B$. Then

$$A^h = A^{Bj} = C^j | (A, h), (C, j).$$

Hence by 1.7, $F((A, h)(C, h)) = \{A^h\}$. As $A, C \in Q$ and Q is complete, this yields $A^h \in Q$. Thus $g, g^h \in S(Q)$. By 1.7,

$$F(ghB) = \{B\} = F(gh).$$

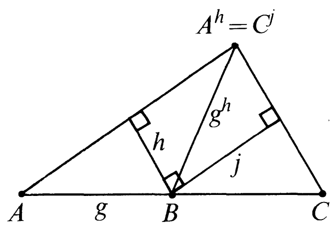

Figure 5

Therefore,

$$F(g \cdot g^h) = F((gh)^2) = \{B\};$$

cf. [2], Section 9, Lemma 1. Thus g and g^h intersect only at B. As Q is complete, we obtain $B \in Q$.

Now we are ready to prove (*). Let $B \in P$ and $A, A^B \in Q$.

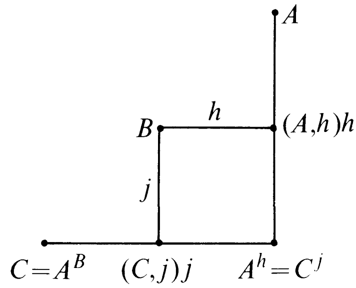

Figure 6

Choose h, j such that $B = hj$. Let $C = A^B$. As before, we obtain

$$F((A, h)(C, j)) = \{A^h\} = \{C^j\}$$

and thus $A^h \in Q$. As $A^h = A^{(A,h)h} = C^{(C,j)j}$, we can apply the special case of (*) which is proved above, and derive $(A, h)h, (C, j)j \in Q$. Thus $B \in Q$, by the completeness of Q.

Finally by 1.11, $S(Q) = N_G(Q)$.

2.3. *Let Q and R be complete sets in (G, S, P). Suppose $\emptyset \neq P_g \cap Q \subseteq R$ for some g. Then $Q \subseteq R$.*

Proof. Let $A \in P_g \cap Q$. By 1.9 (i), there is a line $h \neq g$ through A such that $F(gh) = F(Agh) = \{A\}$. Let $X \in P_h$. The sets Q and R being complete, we have

$$X \in Q \Leftrightarrow (X, g)g \in Q \quad \text{and} \quad X \in R \Leftrightarrow (X, g)g \in R.$$

Hence $P_h \cap Q \subseteq R$.

Next, let $Y \in Q$. Then

$$(Y, g)g \in P_g \cap Q \subseteq R \quad \text{and} \quad (Y, h)h \in P_h \cap Q \subseteq R.$$

By 1.7, Y is the unique intersection of (Y, g) and (Y, h). As R is complete, this yields $Y \in R$.

2.3′. Corollary. *Let Q and R denote complete sets in (G, S, P). Suppose $\emptyset \neq P_g \cap Q = P_g \cap R$ for some g. Then $Q = R$.*

3. Homomorphisms and coverings. Most of the results of this Section, collected for the readers' convenience, are known; cf. [**2**; **4**; **5**; **9**].

In this Section, (G, S, P) denotes an arbitrary pre-Hjelmslev group.

3.1. *Let φ be a homomorphism of G such that $1 \notin S\varphi \cup P\varphi$. Thus $(G\varphi, S\varphi, P\varphi)$ is well defined. Assume, in addition, that there is not more than one orthogonal line in $(G\varphi, S\varphi, P\varphi)$ through any point to any line. Then φ is a homomorphism of (G, S, P), i.e., $(G\varphi, S\varphi, P\varphi)$ is a pre-Hjelmslev group.*

Proof. Obviously, $(G\varphi, S\varphi, P\varphi)$ satisfies (A1) and (A2). We first verify

(∗) Let $A \in P$, $b \in S$ and $A\varphi | b\varphi$. Then

$$A\varphi = ((A, b)b)\varphi \quad \text{and} \quad b\varphi = (A(A, b))\varphi.$$

Let $g = (A, b)$. Then

$$A\varphi, g\varphi | (Ag)\varphi, b\varphi.$$

The normal of g through A being unique, we obtain $(Ag)\varphi = b\varphi$ and (∗). Choosing $a = Ag$ and $B = bg$, we obtain from (∗) that $a|A$; $B|b$; $A\varphi = B\varphi$; $a\varphi = b\varphi$. Thus the properties (A3) and (A4) of $(G\varphi, S\varphi, P\varphi)$ follow from the corresponding ones of (G, S, P).

We write $N \lhd G$ if N is a normal subgroup of G.

3.2. *Let* $N \lhd G$ *satisfy*
(N0*) $N \cap S = \emptyset$, *and*
(N1*) *If* $B, C|g$ *and* $B^C B \in N$, *then* $BC \in N$.
Then $N \subseteq S^{\text{even}}$.

Proof. Suppose $\alpha \in N \cap S^{\text{odd}}$. By 1.3, $\alpha = cB$ where $c \in S$ and $B \in P$. Let $g: = (B, c)$. Then $B^{cg}B = \alpha^g\alpha \in N$. Hence by (N1*), $Bcg \in N$ and therefore $g = \alpha \cdot Bcg \in N$, contradicting (N0*).

3.3. *Let* $N \lhd G$ *and let* $\varphi: G \to G/N$ *denote the canonical homomorphism. Then* φ *is a homomorphism of* (G, S, P) *if and only if*
(N0) $N \cap S = \emptyset = N \cap P$, *and*
(N1) *If* $B, C|g$ *and* $A^{BC}A \in N$, *then* $BC \in N$.

(Note that (N0) and (N1) are stronger than (N0*) and (N1*).)

Proof. Obviously, (N0) is satisfied if $(G\varphi, S\varphi, P\varphi)$ is a pre-Hjelmslev group. Then also (N1) is true, because $A^{BC}A \in N$ implies $A\varphi^{B\varphi} = A\varphi^{C\varphi}$, hence $B\varphi = C\varphi$ by 1.4. Conversely, assume (N0) and (N1). Let $A\varphi, g\varphi | b\varphi$. On account of 3.1 it is sufficient to prove that $b\varphi = (A, g)\varphi$. As $A\varphi | b\varphi$, N contains

$$(bA)^2 = A^{A(A,b)b} \cdot A.$$

Hence by (N1),

$$A \cdot (A, b) \cdot b \in N.$$

This proves 3.1(∗). As $g\varphi | b\varphi$, there is a point $B \in P$ such that

$$g\varphi \cdot b\varphi = B\varphi.$$

On account of 3.1(∗), we may assume $B|g$. Since

$$A^{(A,g)gB}\varphi = A^{gB}\varphi = A^b\varphi = A\varphi,$$

(N1) yields $(A, g)gB \in N$ Thus

$$(A, g)\varphi = B\varphi g\varphi = b\varphi.$$

3.4. A set $\mathscr{F}$ of spots of (G, S, P) (cf. 1.11) is a *covering* of (G, S, P) if every point of P belongs to exactly one of the spots in $\mathscr{F}$. Such a covering $\mathscr{F}$ induces an equivalence relation $\sim$ in P. The *group of* $\mathscr{F}$ is equal to

$$\mathscr{N}(\mathscr{F}) := \{\alpha \in S^{\text{even}} : A^{\alpha} \sim A \text{ for every } A \in P\}.$$

A *homogeneous covering* $\mathscr{F}$ satisfies

(h) *Let $A \sim B$. Suppose $a|A, g$ and $b|B, g$. Then $ab \in \mathscr{N}(\mathscr{F})$.*

Let $\mathscr{F}$ denote a homogeneous covering of (G, S, P). Then $AB \in \mathscr{N}(\mathscr{F})$ if and only if $A \sim B$.

Proof. Let $AB \in \mathscr{N}(\mathscr{F})$. Then $A \sim A^{AB} = A^{B}$. Thus the spot containing A also contains B; cf. 1.11. Conversely, let $A \sim B$. Choose $a|A$ and $b|B$, a. A, B and ab are points belonging to the same spot. Hence $Aa \cdot b, a \cdot bB \in \mathscr{N}(\mathscr{F})$, by (h). Thus

$$AB = A(ab) \cdot (ab)B \in \mathscr{N}(\mathscr{F}).$$

3.5. *The covering $\mathscr{F}$ is homogeneous if and only if it satisfies*

(h1) *Suppose $A, B, C, D|g$; $AB = CD$; $A \sim B$. Then $C \sim D$; and*

(h2) *Let $a|A, g$ and $b|B, g$. Then*

$$A \sim B \Rightarrow ag \sim bg \Rightarrow A \sim (A, b)b.$$

Proof. Let $\mathscr{F}$ be homogeneous. Then (h1) follows immediately from 3.4. Next assume $a|A, g$ and $b|B, g$. We apply (h). If $A \sim B$, then

$$ag \cdot bg = ab \in \mathscr{N}(\mathscr{F}) \quad \text{and} \quad ag \sim bg.$$

If $ag \sim bg$, then $ab \in \mathscr{N}(\mathscr{F})$; hence

$$A \sim A^{ab} = A^{b} = A^{(A,b)b}$$

and therefore $A \sim (A, b)b$; cf. 1.11.

Conversely, assume (h1) and (h2). Let $A \sim B$; $a|A, g$; $b|B, g$. We have to show that $ab \in \mathscr{N}(\mathscr{F})$, i.e., $X^{ab} \sim X$ for every $X \in P$. By 1.1 (v),

$$c := (X, g)ab \in S \quad \text{and} \quad c|g.$$

Furthermore

$$(X, g)g \cdot cg = ag \cdot bg.$$

As $ag \sim bg$ by (h2), (h1) implies $(X, g)g \sim cg$. By (h2), $X \sim (X, c)c$. We have

$$X^{ab} = X^{(X,g)ab} = X^{c} = X^{(X,c)c};$$

hence

$$(X, c)c \cdot X^{ab} = X \cdot (X, c)c.$$

Also

$(X, c)c, X, X^{ab} | (X, c).$

Thus (h1) yields $X^{ab} \sim X$.

3.6. *Let* φ *be a homomorphism of* (G, S, P). *For any* $A \in P$ *let*

$Q_A: = \{B \in P: B\varphi = A\varphi\}.$

Then

$\mathcal{F}_\varphi: = \{ (N_G(Q_A), S(Q_A), Q_A): A \in P\}$

is a homogeneous covering of (G, S, P) *such that*

$\text{kernel}\varphi \subseteq \mathcal{N}(\mathcal{F}_\varphi).$

Equality holds if and only if $Z(S\varphi^{\text{even}}) = 1$.

Proof. Obviously, $\mathcal{F}_\varphi$ is a covering of (G, S, P); cf. 1.11. (h1) is satisfied because φ is a homomorphism. (h2) follows from the uniqueness of perpendiculars in $(G\varphi, S\varphi, P\varphi)$ and because orthogonal lines have a unique intersection. Thus $\mathcal{F}_\varphi$ is homogeneous. Finally

$$\mathcal{N}(\mathcal{F}_\varphi) = \{\alpha \in S^{\text{even}}: A^\alpha\varphi = A\varphi \text{ for all } A \in P\}$$
$$= \{\alpha \in G: \alpha\varphi \in Z(S\varphi^{\text{even}}) \};$$

cf. 1.5.

3.7. *Let* $\mathcal{F}$ *be a homogeneous covering of* (G, S, P); $|\mathcal{F}| \neq 1$. *Then* $\mathcal{N}(\mathcal{F}) \lhd G$ *and* $\varphi: G \to G/\mathcal{N}(\mathcal{F})$ *induces a homomorphism of* (G, S, P) *such that* $\mathcal{F} = \mathcal{F}_\varphi$; *cf.* 3.6. *Moreover,* $Z(S\varphi^{\text{even}}) = 1$.

Proof. Let $\sim$ denote the equivalence relation in P induced by $\mathcal{F}$. We first show

(*) Let $A \sim B$. Then $A^c \sim B^c$ for every c.

Define $a: = (A, c)$, $b: = (B, c)$, $C: = (B, a)a$. By 1.11, $A \sim B \sim C$.

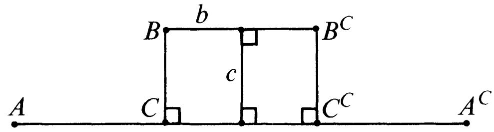

Figure 7

Since $AC = C^cA^c$, (h1) therefore yields $A^c \sim C^c$. Applying (h2) twice, we deduce from $B \sim C$ first $bc \sim ac$ and then $B^c \sim (B^c, a)a = C^c$. This yields (*).

Let $\alpha \in N: = \mathcal{N}(\mathcal{F})$. Let $c \in S$; $\beta: = \alpha^c$; $X \in P$. Then by the definition of N, $X^{c\alpha} \sim X^c$ and by (*),

$$X^{\beta} = (X^{c\alpha})^c \sim (X^c)^c = X.$$

Thus $N \lhd G$.

Suppose $A \in N \cap P$. Then $X^A \sim X$ and thus $A \sim X$ for all $X \in P$; thus $|\mathscr{F}| = 1$. As $N \subseteq S^{\text{even}}$, this yields the condition (N0) of 3.3.

Next, let B, $C|g$ and $A^{BC}A \in N$. Put $b: = (A, g)$, $c: = bBC$, and $D: = (A, c)c$. Then

$$A^{BC}A = A^{bc}A = A^cA = A^DA \in N.$$

This implies by 3.4 and 1.11 (i) that $A \sim A^D \sim D$. As $b|A, g$ and $c|D, g$, the homogeneity of $\mathscr{F}$ yields $BC = bc \in N$. Thus N also satisfies (N1), and φ is a homomorphism of (G, S, P); cf. 3.3.

By 3.4, $A \sim B$ if and only if $AB \in N$, i.e., $(AB)\varphi = 1$ or $A\varphi = B\varphi$. By 3.6, this is equivalent to A and B belonging to the same spot of $\mathscr{F}_{\varphi}$. Thus $\mathscr{F} = \mathscr{F}_{\varphi}$. The last statement now follows immediately from 3.6.

3.8. The sections 3.6, 3.7 and 3.4 yield the following result:

The mapping

$$\mathscr{F} \mapsto \mathscr{N}(\mathscr{F})$$

is a bijection of the set of the homogeneous coverings $\mathscr{F}$ *of* (G, S, P) *with* $|\mathscr{F}| \neq 1$ *onto the set of the kernels of those homomorphisms* φ *of* (G, S, P) *which satisfy* $Z(S\varphi^{\text{even}}) = 1$. *If* $\mathscr{F}$ *is such a covering, then* $AB \in \mathscr{N}(\mathscr{F})$ *if and only if* A *and* B *belong to the same spot of* $\mathscr{F}$.

3.9. A homomorphism of (G, S, P) is *regular* if the images of lines with unique intersections also have unique intersections.

A covering is *complete* if each of its spots is complete.

The homomorphism φ *of* (G, S, P) *is regular if and only if the induced covering* $\mathscr{F}_{\varphi}$ *is complete*; *cf.* 3.6.

Proof. Suppose φ is regular. Let A and B belong to the same spot of $\mathscr{F}_{\varphi}$. Let $a|A$; $b|B$ and suppose C is the unique intersection of a, b. Then

$$A\varphi = B\varphi \quad \text{and} \quad C\varphi|a\varphi, b\varphi.$$

Hence, φ being regular, $C\varphi = A\varphi = B\varphi$.

Conversely, let $\mathscr{F}_{\varphi}$ be complete. Let a, $b|C$. Suppose $a\varphi$ and $b\varphi$ have more than one point in common, say $a\varphi, b\varphi|C\varphi, D\varphi$ where $C\varphi \neq D\varphi$. The points $A: = (D, a)a$ and $B: = (D, b)b$ satisfy $A\varphi = D\varphi = B\varphi$. Thus A, D, B belong to the same spot F of $\mathscr{F}_{\varphi}$ while C does not belong to F. Since F is complete, the intersection of a and b is not unique.

3.10. *The mapping* $\mathcal{F} \mapsto \mathcal{N}(\mathcal{F})$ *is a bijection of the set of the complete homogeneous coverings* $\mathcal{F}$ *of* (G, S, P) *with* $|\mathcal{F}| \neq 1$ *onto that of the kernels of the regular homomorphisms* φ *of* (G, S, P) *which satisfy* $Z(S\varphi^{\text{even}}) = 1$.

The proof follows immediately from 3.8 and 3.9.

3.11. *If* $\mathcal{F}$ *is a homogeneous covering of* (G, S, P) *and* $A, B \in P$, *the following statements are equivalent*:

(i) $A \sim B$;
(ii) $AB \sim \mathcal{N}(\mathcal{F})$;
(iii) $C \sim C^{AB}$ *for some point* C.

On account of 3.4, it is sufficient to deduce (i) from (iii). We may assume $|\mathcal{F}| \neq 1$. By 3.7, $\mathcal{N}(\mathcal{F})$ is the kernel of a homomorphism φ of (G, S, P) such that $\mathcal{F} = \mathcal{F}_\varphi$. Hence (iii) and 3.4 imply

$$(C\varphi)^{A\varphi} = (C\varphi)^{B\varphi}.$$

Applying 1.4 to $(G\varphi, S\varphi, P\varphi)$, we obtain $A\varphi = B\varphi$, i.e., (i).

3.12. *Let* $|P| \neq 1$. *The normal subgroups* N *of* G *with* $N \subseteq Z(S^{\text{even}})$ *are precisely the kernels* N *of the homomorphisms of* (G, S, P) *which satisfy* $N \cap P^2 = 1$.

Proof. If $N \subseteq Z(S^{\text{even}})$, then $AB \in N$ implies

$$A^A = A = A^{AB} = A^B.$$

Hence $AB = 1$, by 1.4. Obviously, N satisfies the conditions (N0) and (N1) of 3.3. Conversely, suppose N is the kernel of a homomorphism of (G, S, P) and satisfies $N \cap P^2 = 1$. Let $\alpha \in N$. Then

$$A^\alpha A = \alpha^{-1}\alpha^A \in N \cap P^2 = 1 \quad \text{for any } A \in P.$$

Thus $F(\alpha) = P$ and, by 1.5, $\alpha \in Z(S^{\text{even}})$.

4. Semi-translations and transports. Let (G, S, P) again denote any pre-Hjelmslev group.

4.1. In [**9**, Section 7] E. Salow introduced semi-translations Γ_{AB} though only for pairs A, B joined by lines. Through our Lemma 4.2 we will be able to drop this restriction and give a definition of $\Gamma_{A,B}$ similar to that of semi-rotations in [**6**]. This will enable us to generalize Salow's beautiful results; cf. 4.3.

Let $\omega \in P^2$. The *semi-translation* Γ_ω is a pair of mappings of P and S into themselves: If $X \in P$ and $y \in S$, let $X\Gamma_\omega$ be the mid-point of X and X^ω and let $y\Gamma_\omega = [y\omega]$; cf. 1.4 and 1.2.

Note that $X^{(X\Gamma_\omega)} = X^\omega$ and $A\Gamma_{AB} = B$.

If $A, B|g$ for some g, the point $X\Gamma_{AB}$ can readily be constructed: Let $c: = (X, g)$. By 1.1 (iii), $cAB = g(gc \cdot AB)$ is the line through the point $gc \cdot AB$ on g perpendicular to g. Let $d = (X, cAB)$. Then

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$$X^{AB} = (X^{dc})^{AB} = X^{d \cdot cAB}.$$

Thus $X\Gamma_{AB} = d \cdot cAB$ is the intersection of d with cAB.

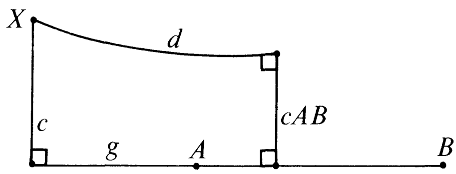

Figure 8

We require the following lemma.

4.2. *Let* $\alpha \in P^3$. *Then* $|\mathrm{F}(\alpha)| \leqq 1$.

Proof. Let $\alpha = ABC$ and $E \in \mathrm{F}(\alpha)$. Let $a|A$. Put $b: = (B, a)$, $e: = (E, a)$. Thus

$$A': = bAe = ba \cdot A \cdot ae|a.$$

Finally, let $b': = Bb$, $a': = (A', b')$ and $e': = (E, a')$. Thus

$$C': = b'A'e'|a' \quad \text{and} \quad \alpha C = AB = Abb' = eA'b' = ee'C'.$$

Applying 1.3 to αC, we obtain $\alpha = ee'$. By 1.7,

$$1 = |\mathrm{F}(a'b')| = |\mathrm{F}(aa')|.$$

Hence by 1.7′, $|\mathrm{F}(ee')| = 1$.

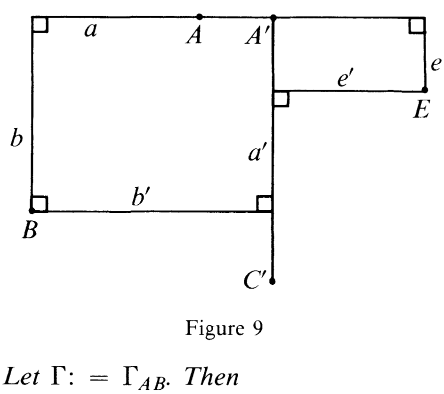

Figure 9

4.3. THEOREM. *Let* $\Gamma: = \Gamma_{AB}$. *Then*

(i) Γ *is injective both on* P *and on* S.

(ii) $C|g \Leftrightarrow C\Gamma|g\Gamma$, *for all* C *and* g.

(iii) *If* $\mathrm{F}(gh) = \{C\}$, *then* $\mathrm{F}(g\Gamma \cdot h\Gamma) = \{C\Gamma\}$.

Proof. To (i). $\Gamma: P \to P$ is injective: Let $E = C\Gamma = D\Gamma$. Then $C^E = C^{AB}$ and $D^E = D^{AB}$. Thus $C, D \in \mathrm{F}(ABE)$ and hence $C = D$, by 4.2.

The injectivity of Γ on S will be dealt with later. Before proving (ii) we show

(*) For any C let $\gamma = AB \cdot C\Gamma$. Then $F(\gamma) = \{C\}$.

On account of 4.2 we need only show that $C \in F(\gamma)$. By 1.3, there are $\delta \in S^2$ and $D \in P$ such that $AB = \delta D$ and $C \in F(\delta)$. As $C^{C\Gamma} = C^{AB} = C^{\delta D}$, this implies $D = C\Gamma$ and thus $\delta = \gamma$.

To (ii). Let $D = C\Gamma$. As $\gamma = ABD$, (*) and 1.8 yield

$$C|g \Leftrightarrow \gamma g \in S \Leftrightarrow DgAB = (\gamma g)^{AB} \in S.$$

By 1.3, there are $e|D$ and E such that $gAB = eE$. Thus

$$g\Gamma = [eE] = (E, e)$$

and the assertion (ii) is equivalent to:

$$DeE \in S \Leftrightarrow (E, e)\,|D.$$

But this follows readily from 1.2′.

To (iii). Let $D: = C\Gamma$; $\gamma: = ABD$. Thus by (*), $F(\gamma) = \{C\}$. As $F(gh) = \{C\}$, 1.8 implies $C|g, h$. Hence, again by 1.8,

$$g': = g\gamma \in S \quad \text{and} \quad h': = h\gamma \in S.$$

Thus

$$\gamma = gg' = hh' \quad \text{and} \quad g'h' = g'g \cdot gh \cdot hh' = (gh)^{\gamma}.$$

This yields

$$F(g'h') = F(gh)^{\gamma} = \{C^{\gamma}\} = \{C\} \quad \text{and} \quad C|g', h'.$$

As $g\Gamma = [gAB] = [g'D] = (D, g')$ and $h\Gamma = (D, h')$, 1.7′ now yields (iii).

Finally, $\Gamma: S \rightarrow S$ is injective: Let $g\Gamma = h\Gamma$, i.e.,

$$[gAB] = [hAB] = [BAh].$$

The product of two glide reflections with the same axis being the product of two points, we obtain

$$gh = gAB \cdot BAh = XY$$

or $Xg = Yh$ for some X, Y; and, by 1.2, there is a line $j|g, h$. Let $C: = gj$ and $D: = hj$. Then by (iii),

$$\{C\Gamma\} = F(g\Gamma \cdot j\Gamma) = F(h\Gamma \cdot j\Gamma) = \{D\Gamma\}.$$

As Γ is injective on P, this implies $C = D$ and thus $g = h$.

4.4. Given $\omega \in P^2$, define the *transport* τ_ω through

$$Q\tau_\omega = \{C : C\Gamma_{\omega^{-1}} \in Q\} \quad \text{for every } Q \subseteq P.$$

As Γ_ω^{-1} need not be surjective, $Q\tau_\omega$ can be void for some ω, Q. If $A \in Q$, then $B \in Q\tau_{AB}$ because $B\Gamma_{BA} = A$.

Let $\mathscr{F}$ denote a homogeneous covering of (G, S, P). *Then* $Q_A\tau_{AB} = Q_B$ *for any two points* A, B; *here* $Q_X = \{Y \in P: Y \sim X\}$; *cf.* 3.4.

Proof. By our definition,

$$Q_A\tau_{AB} = \{C: C\Gamma_{BA} \sim A\} = \{C: C^M = C^{BA} \text{ for some } M \sim A\}.$$

Let $C \in Q_A\tau_{AB}$. By 3.4, $MA \in \mathscr{N}(\mathscr{F})$. Thus $C \sim C^{MA} = C^B \sim B$; cf. 1.11. In particular, $C \in Q_B$. Conversely, let $C \sim B$. Then by 3.4, $CB \in \mathscr{N}(\mathscr{F})$ and thus $C^B = C^{CB} \sim C$. If M denotes the mid-point of C and C^{BA}, then $C^{MA} = C^B \sim C$. Hence $M \sim A$ by 3.11.

4.5. *Let* τ_{AB} *be a transport in a pre-Hjelmslev group which satisfies* (W). *If the set* Q *is complete, so is* $Q\tau_{AB}$.

Proof. Let $C, D \in Q\tau_{AB}$; $c|C$; $d|D$; suppose c and d have the unique intersection E. We have to show $E \in Q\tau_{AB}$. Write $\Gamma = \Gamma_{BA}$. By 1.9 (ii), $\mathrm{F}(cd) = \{E\}$. Hence by 4.3 (iii),

$$\mathrm{F}(c\Gamma \cdot d\Gamma) = \{E\Gamma\}.$$

Thus $c\Gamma$ and $d\Gamma$ intersect precisely at $E\Gamma$. As $C\Gamma, D\Gamma \in Q$, and $C\Gamma|c\Gamma$ and $D\Gamma|d\Gamma$ by 4.3 (ii), and since Q is complete, this yields $E\Gamma \in Q$ and thus $E \in Q\tau_{AB}$.

4.5′. THEOREM. *Let* $\mathscr{F}$ *be a homogeneous covering of a pre-Hjelmslev group satisfying* (W). *If one spot of* $\mathscr{F}$ *is complete, then* $\mathscr{F}$ *is complete*; *cf.* 3.9.

The proof follows immediately from 4.4 and 4.5.

4.6. *Let* Q *be a complete point set in a pre-Hjelmslev group satisfying* (W). *If* $\alpha \in Q^3$, *then* $\mathrm{F}(\alpha) \subseteq Q$.

Proof. Let $\alpha = ABC$ where $A, B, C \in Q$. Let $E \in \mathrm{F}(\alpha)$. We repeat the construction in the proof of 4.2. As $\alpha = ee'$ and $\alpha C = ee'C'$, we obtain $C = C'$. Since A, B, C' lie in the complete set Q, the points ab, $a'b'$ and A' belong to Q. By 2.2, Q is the point set of a pre-Hjelmslev subgroup. As $ae = A \cdot ab \cdot A'$ and $a'e' = A' \cdot a'b' \cdot C'$, these products of three points on a and a', respectively, must also belong to Q. Since $|\mathrm{F}(ee')| = |\mathrm{F}(\alpha)| = 1$, this yields $E \in Q$.

4.7. *Let* Q *be a complete point set in a pre-Hjelmslev group satisfying* (W); $A, B \in Q$. *Then* $Q\tau_{AB} = Q$.

Proof. Let $C^{BA} = C^D$; i.e., $D = C\Gamma_{BA}$. Suppose $C \in Q\tau_{AB}$; thus $D \in Q$ and $BAD \in Q^3$. Hence by 4.6, $C = C^{BAD} \in Q$. Conversely let $C \in Q$. As $A, B, C \in Q$ and Q is the point set of a spot, we obtain $C^D = C^{BA} \in Q$ and thus $D \in Q$; c.f. 2.2 and 1.11 (i).

5. Embedding a complete point set into a homogeneous covering. In this section, (G, S, P) denotes a pre-Hjelmslev group satisfying (W).

5.1. *Let $h|A, B$; $D|Ah$; $g = DAh$; $E = (B, g)g$. Let $D', E'|g$; $D'E' = DE$; $v|B, E'$. Then there is a line u through A and D'. If v and g have a unique intersection, so will u and g.*

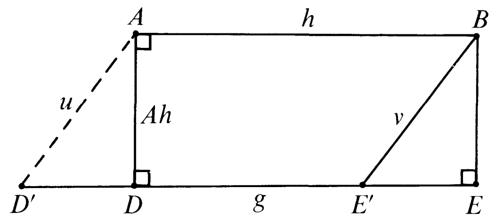

Figure 10

Proof. The elements $m: = E'v$ and $Egvh$ are lines since $E'|v$ and $Eg, v, h|B$. Thus

$$D'mA = DEvA = Dg \cdot Eg \cdot vA = Ah \cdot Egvh \cdot Ah$$
$$= (Egvh)^{Ah} \in S.$$

Hence by 1.2′,

$$u: = (D', m) = (A, D'mA)\,|A, D'.$$

If v and g have a unique intersection, 1.6 yields on account of $u, v|m$ that the intersection of u with g is unique too.

5.2. *Suppose g and h have a common perpendicular a. For any C let $B = (C, h)h$ and $E' = (C, g)g$. Then there is a line v through B and E' such that*

$$|\mathrm{F}(vg)| = |\mathrm{F}(vh)| = 1.$$

Proof. Let $c = (C, a)$. Then

$$d: = gch \in S \quad \text{and} \quad E'dB = (C, g)c(C, h) \in S$$

since $(C, g), c, (C, h)\,|C$. Thus by 1.2′,

$$v: = (E', d) = (B, E'db)\,|E', B, d.$$

As $|\mathrm{F}(dv)| = 1$, $a|d, g, h$ implies that

$$|\mathrm{F}(gv)| = |\mathrm{F}(hv)| = 1;$$

cf. 1.7.

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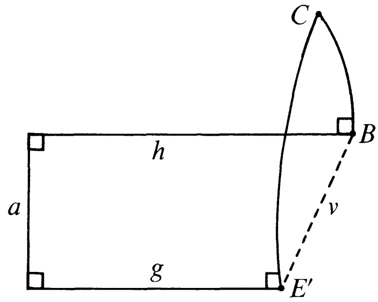

Figure 11

5.3. *Let* $A = ah$, $D = ag$, $B = (C, h)h$, $E' = (C, g)g$; $E|\,(C, h), g$; $D'E' = DE$. *Let* Q *be complete*; $A \in Q$. *Then* $D' \in Q \Leftrightarrow D \in Q$.

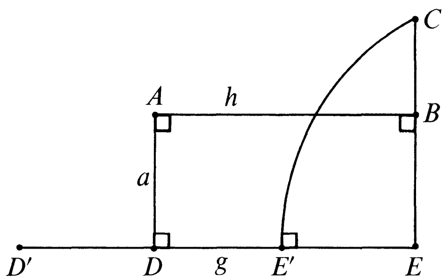

Figure 12

Proof. The intersection of a and g being unique, $A, D' \in Q$ implies $D \in Q$. Conversely, let $A, D \in Q$. By 5.2, there is a line $v|B, E'$ such that $F(vg) = \{E'\}$. Let $E'' := (B, g)g$ and $D'' := DE''E'$. By 5.1, there is a line $v'|A, D''$ such that $F(v'g) = \{D''\}$. Finally, let $D^* := DE''E$.

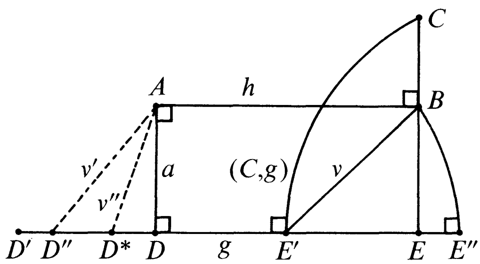

Figure 13

Apply 5.1, replacing E by E'' and E' by E. This yields a line $v^*\,|A, D^*$ such that $F(v^*g) = \{D^*\}$. As Q is complete, we obtain $D'', D^* \in Q$. Hence by 2.2,

$$D' = DEE' = D \cdot EE''D \cdot DE''E' = DD^*D'' \in Q.$$

5.4. *Let* $B = hu$; $A|h$ *and* $U|u$. *If the complete set* Q *contains* A, *then*

$$Q\tau_{AB}\tau_{BU} = Q\tau_{AU}.$$

Proof. By 4.5, $Q\tau_{AB}\tau_{BU}$ and $Q\tau_{AU}$ are complete. As they contain U, it is therefore sufficient to prove

$$\mathrm{P}_u \cap Q\tau_{AB}\tau_{BU} = \mathrm{P}_u \cap Q\tau_{AU};$$

cf. 2.3′.

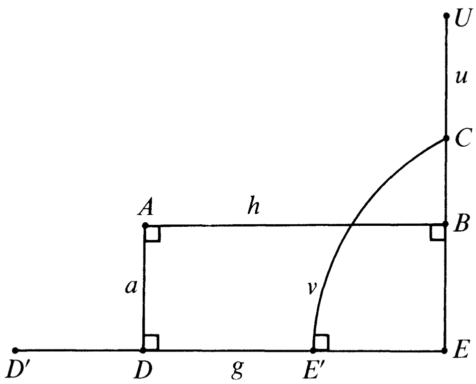

Figure 14

Let $C \in \mathrm{P}_u$, $E{:} = BUC$, $a{:} = Ah$, $g{:} = (E, a)$, $v{:} = (C, g)$, $E'{:} = gv$, $D{:} = ag$, $D'{:} = E'ED$. Then

$$UA = Uha = UBua = CEua = CuEa = CuEgD = CugED$$
$$= CuvE'ED = CuvD'.$$

Thus

$$C^{UA} = C^{CuvD'} = C^{D'}.$$

Hence $C\Gamma_{UA} = D'$ and:

$$C \in Q\tau_{AU} \Leftrightarrow D' \in Q.$$

As $C^{UB} = C^{CE} = C^E$ and $E^{BA} = E^{uhha} = E^a = E^{ga} = E^D$ we have

$$C\Gamma_{UB} = E \text{ and } E\Gamma_{BA} = D.$$

Therefore:

$$C \in Q\tau_{AB}\tau_{BU} \Leftrightarrow E \in Q\tau_{AB} \Leftrightarrow D \in Q.$$

By 5.3, the last relation is also equivalent to $D' \in Q$, i.e., to $C \in Q\tau_{AU}$.

5.5. *Let* $U, V|j$. *If the complete set* Q *contains* A, *then*

$$Q\tau_{AU}\tau_{UV} = Q\tau_{AV}.$$

Proof. Let $B: = (A, j)j$. By 4.5, $Q^*: = Q\tau_{AB}$ is a complete set containing B. Let C be any point of the line j. Then V, B, $C|j$ implies $CVB \in \mathrm{P}_j$ and $C^{CVB} = C^{VB}$, hence $C\Gamma_{VB} = CVB$. The following statements are therefore equivalent:

$$C \in Q^*\tau_{BV};$$

$$C\Gamma_{VB} = CVB \in Q^*;$$

$$(CVU)\Gamma_{BU} = CVU \cdot UB \in Q^*;$$

$$C\Gamma_{VU} = CVU \in Q^*\tau_{BU};$$

$$C \in Q^*\tau_{BU}\tau_{UV}.$$

Thus by 2.3′,

$$Q^*\tau_{BV} = Q^*\tau_{BU}\tau_{UV};$$

cf. 4.5. By 5.4,

$$Q^*\tau_{BV} = Q\tau_{AV} \quad \text{and} \quad Q^*\tau_{BU} = Q\tau_{AU}.$$

This proves our assertion.

5.6. We can now prove the

THEOREM. *Let Q be complete; $A \in Q$. Then $Q\tau_{AU}\,{}_{UV} = Q\tau_{AV}$ for any two points U, V.*

Proof. Choose a point W joined by lines to U and V. By 4.5, $Q^* = Q\tau_{AU}$ is complete and, by 5.5,

$$Q^*\tau_{UW}\tau_{WV} = Q^*\tau_{UV}.$$

Applying 5.5 twice more, we obtain

$$Q\tau_{AU}\tau_{UW} = Q\tau_{AW} \quad \text{and} \quad Q\tau_{AW}\tau_{WV} = Q\tau_{AV}.$$

Hence

$$\begin{aligned} Q\tau_{AU}\tau_{UV} &= Q^*\tau_{UV} = Q^*\tau_{UW}\tau_{WV} = Q\tau_{AU}\tau_{UW}\tau_{WV} \\ &= Q\tau_{AW}\tau_{WV} = Q\tau_{AV}. \end{aligned}$$

5.7. *Let Q denote any complete set; M, $N \in Q$. Then by* 5.6 *and* 4.7

$$(*) \quad Q\tau_{MB} = Q\tau_{MN}\tau_{NB} = Q\tau_{NB} \textit{ for every } B.$$

Suppose now that Q is any non-void complete set. On account of $(*)$ we may define

$$Q_B: = Q\tau_{MB},$$

where $M \in Q$ is arbitrary.

5.8. THEOREM. *Let* (G, S, P) *satisfy* (W). *Let* $Q \subseteq P$ *be non-void and complete. Then*

$$\mathscr{F}_Q := \{(\mathrm{N}_G(Q_B), \mathrm{S}(Q_B), Q_B) : B \in P\}$$

is a homogeneous covering. $\mathscr{F}_Q$ *is complete; cf.* 3.9.

Remark. By 4.7, $(\mathrm{N}_G(Q), \mathrm{S}(Q), Q) \in \mathscr{F}_Q$.

Proof. Let $M \in Q$. We have $B \in Q_B$ for every B; also by 5.6 and 4.7,

$$Q_C = Q\tau_{MC} = Q\tau_{MB}\tau_{BC} = Q_B\tau_{BC} = Q_B \quad \text{for every } C \in Q_B.$$

Thus $\{Q_B : B \in P\}$ is a partition of P into complete sets; cf. 4.5. By 2.2, $\mathscr{F}_Q$ is a complete covering of (G, S, P). It remains to show that $\mathscr{F}_Q$ is homogeneous. Let $\sim$ denote the equivalence relation induced by $\mathscr{F}_Q$. We apply 3.5 to (h1). Let $A, B, C, D|g$; $AB = CD$; $A \sim B$. Thus

$$D\Gamma_{CA} = DCA = B \in Q_A \quad \text{and} \quad D \in Q_A\tau_{AC} = Q_C$$

and hence $D \sim C$. To (h2). Let $a|A, g$; $b|B, g$. Put $A' := (A, b)b$, $C := ag$, and $D := bg$. We have

$$A^{A'D} = A^{(A,b)g} = A^g = A^C.$$

Hence $A\Gamma_{A'D} = C$. From 5.6,

$$Q_{A'} = Q\tau_{MA'} = Q\tau_{MD}\tau_{DA'} = Q_D\tau_{DA'} = \{X : X\Gamma_{A'D} \in Q_D\}.$$

Hence $A \sim A'$ if and only if $C \sim D$. Since Q_A is complete, $A \sim B$ implies $A \sim A'$.

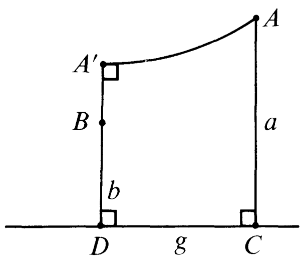

Figure 15

5.8′. We restate our result in terms of homomorphisms.

THEOREM. *Let* (G, S, P) *satisfy* (W). *Let* $Q \subset P$ *be complete;* $M \in Q$. *Then there exists a homomorphism* φ *of* (G, S, P) *such that*

$$Q = \{B \in P : B\varphi = M\varphi\} \quad \textit{and} \quad \mathrm{Z}(S\varphi^{\text{even}}) = 1.$$

The homomorphism φ *is unique (up to isomorphism), and* φ *is regular; cf.* 3.9.

Proof. According to 5.8, $\mathscr{F}_Q$ is a homogeneous covering having more than one element. Hence by 3.7 the canonical homomorphism

$$\varphi:G \to G/\mathscr{N}(\mathscr{F}_Q)$$

induces a homomorphism of (G, S, P) with the desired properties; cf. 3.4. Furthermore, by 3.10, this homomorphism is regular; cf. 3.9. Now, let us assume that φ is a homomorphism of (G, S, P) satisfying the properties of the theorem. Then by 3.6 the kernel of φ is $\mathscr{N}(\mathscr{F}_\varphi)$. Q is the point set of some spot of $\mathscr{F}_Q$ (cf. 5.8), and Q is also the point set of some spot of $\mathscr{F}_\varphi$; cf. 3.8. Therefore, $\mathscr{F}_Q$ and $\mathscr{F}_\varphi$ induce the same equivalence relation $\sim$ on P; cf. 4.4. Hence $\mathscr{F}_Q = \mathscr{F}_\varphi$ by 1.11 (ii). Thus we have

$$\text{kernel } \varphi = \mathscr{N}(\mathscr{F}_Q).$$

5.9. As an elementary application of 5.8 we state a result for which we could not find a direct proof.

Let $A|a, h$; $a|g$; $b|g, h$. If Q is complete and A, $bh \in Q$, then $(ag, h)h \in Q$.

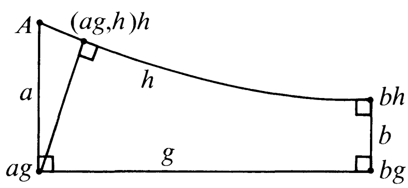

Figure 16

Our proof uses the existence of the homogeneous covering $\mathscr{F}_Q$. Applying (h2) twice, we deduce from $A \sim bh$ first $ag \sim bg$, then $(ag, h)h \sim bh$.

In the remaining two sections we consider two special cases of our Theorem 5.8′. We continue to assume that (G, S, P) is a pre-Hjelmslev group satisfying (W).

5.10. ([**9**]). *Let α be a rotation and $M \in \mathrm{F}(\alpha) \neq P$. There is a unique homomorphism φ of (G, S, P) such that*

$$\mathrm{F}(\alpha) = \{X:X\varphi = M\varphi\} \quad \textit{and} \quad \mathrm{Z}(S\varphi^{\text{even}}) = 1;$$

φ is regular.

Obviously, this theorem is a consequence of 5.8′, because the set $\mathrm{F}(\alpha)$ is complete; cf. 2.2. Let $Q_A: = \{X:X\varphi = A\varphi\}$, for any point A.

ADDENDUM. *For every A let α_A denote the rotation which satisfies $A \in \mathrm{F}(\alpha_A)$ and $\alpha = \alpha_A AC$ for some $C \in P$; cf. 1.3. Then $Q_A = \mathrm{F}(\alpha_A)$.*

Outline of the proof. The construction of φ in 5.8′ and the definition of $\mathscr{F}_Q$ in 5.7 and 5.8 yield $Q_A = Q\tau_{MA}$, where $Q: = \mathrm{F}(\alpha)$. Hence we have to show $Q\tau_{MA} = \mathrm{F}(\alpha_A)$ for every $A \in P$. This is easy.

Remark. Salow's proof for 5.10 is rather complicated. In [7] we give a short direct proof for 5.10 and the addendum, that does not use 5.8 and the tools developed in Sections 4 and 5.

5.11. Points A, B are called *neighbors* if $A, B \in \mathrm{F}(\alpha) \neq P$ for some rotation α. The neighbor relation is reflexive and symmetric, provided that $|P| \neq 1$. The following theorem is proved in [**4**].

If the neighbor relation is transitive and $|P| \neq 1$ *then there is a unique homomorphism* ψ *of* (G, S, P) *such that* $A\psi = B\psi$ *if and only if* A, B *are neighbors, and* $\mathrm{Z}(S\psi^{\text{even}}) = 1$; ψ *is regular.*

Proof. For $A \in P$ let

$$Q_A: = \{X : X \text{ is a neighbor of } A\}.$$

(i) Q_A is complete.

Let $B, C \in Q_A$ and $b|B$; $c|C$ such that b, c have a unique intersection D. Then B is a neighbor of C, i.e., $B, C \in \mathrm{F}(\alpha) \neq P$ for some rotation α. Hence $B, D \in \mathrm{F}(\alpha) \neq P$ by 2.2. Thus A, B and also B, D are neighbors. This implies $D \in Q_A$.

Now 2.2 and the transitivity of the neighbor relation yield

(ii) $F_A: = (\mathrm{N}_G(Q_A), \mathrm{S}(Q_A), Q_A)$ is a spot for every $A \in P$,

and

$$\mathscr{F}: = \{F_A : A \in P\} \text{ is a complete covering of } (G, S, P);$$

furthermore, $|\mathscr{F}| \neq 1$; cf. 3.4, 3.9.

(iii) $\mathscr{F}$ is homogeneous.

We prove property (h) of 3.4. Let A, B be neighbors, $a|A, g$ and $b|B, g$. Then $A, B \in \mathrm{F}(\alpha) \neq P$ for some rotation α. Let φ denote the homomorphism assigned to α by 5.10. Since $A\varphi = B\varphi$, $g\varphi | a\varphi, b\varphi$, the uniqueness of perpendiculars in $(G\varphi, S\varphi, P\varphi)$ implies $a\varphi = b\varphi$, hence $(X^{ab})\varphi = X\varphi$ for every point X. Thus, by the addendum in 5.10,

$$X, X^{ab} \in \mathrm{F}(\alpha_X),$$

where α_X is a rotation satisfying $\alpha = \alpha_X XY$ for some Y. Finally, $\mathrm{F}(\alpha_X) \neq P$, since otherwise $A = A^{\alpha_X} = A^{\alpha YX}$, hence $A^X = A^Y$; but 1.4 implies $X = Y$ and $\alpha = \alpha_X$. Thus we proved $X^{ab} \in Q_X$ for every X, i.e., $ab \in \mathscr{N}(\mathscr{F})$.

The assertion now follows from 3.10.

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