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THE DIRICHLET PROBLEM AT INFINITY AND COMPLEX ANALYSIS ON HADAMARD MANIFOLDS

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Introduction

In this paper we shall study hyperbolicity of Hadamard manifolds.

In Section 1 we shall define and solve the Dirichlet problem at infinity for Laplacian Δ, which gives a partial extension of the result of Anderson [1] and Sullivan [15] in Theorem 1 (cf. [4]). In Section 2 we apply the solution of the Dirichlet problem at infinity to a complex analysis on a Kähler Hadamard manifold whose metric restricted to every geodesic sphere is conformal to that of the standard sphere. It seems that the sphere at infinity of such a manifold admits a CR-structure. In fact we can define a CR-function at infinity on the sphere at infinity. We shall show in Theorem 2 that there exists a holomorphic extension from the sphere at infinity and it coincides with the solution of the Dirichlet problem at infinity, if the Dirichlet problem at infinity is solvable. So we see that such a manifold admits many bounded holomorphic functions. By the similar method we shall show in Theorem 3 that such a manifold is biholomorphic to a strictly pseudoconvex domain in C^n, if the holomorphic sectional curvature $K_h(x)$ is less than $-1/(1+r(x)^2)$, where $r(x)$ is a distance function from a pole. Theorem 3 is a partial answer to a conjecture raised by Green and Wu [8].

§1. Dirichlet problem at infinity

Let M be a Riemannian manifold of dimension n with metric g_{ij}. We denote by T_pM the tangent space at $p \in M$. For a C^2 function u, we define the Hessian D^2u of u at p by

$$D^2u(X, Y) = X(Yu) - (D_X Y)u$$

for $X, Y \in T_pM$, where D_x is the covariant derivative. The Laplacian Δu of u is the trace of D^2u, which is expressed by

$$\Delta u = \sum_{ij} g^{-1/2} \cdot \partial/\partial x_j \qquad (g^{1/2} g^{ij} \partial u/\partial x_i)$$

in a local coordinates $(x_1, \cdots, x_p)$, where $g = \det(g_{ij})$ and $(g^{ij}) = (g_{ij})^{-1}$. By the definition, for an orthonormal basis $X_1, \cdots, X_n$ of T_pM we see that $\Delta u_p = \sum_i (D^2u)(X_i, X_i)$.

A C^2 function u on M is said to be *harmonic* if $\Delta u = 0$. u is *subharmonic* if $\Delta u \geqq 0$, and u is *superharmonic* if $\Delta u \leqq 0$. A continuous function u is subharmonic if it is everywhere a subsolution of the Dirichlet problem [7]. The maximum principle and the Harnack's principle are valid for harmonic functions globally on M [2, 3].

Let M be a simply connected complete Riemannian manifold of non-positive sectional curvature, M is called a *Hadamard manifold.* By the well known theorem of Cartan-Hadamard, for any $p \in M$ $\exp: T_pM \to M$ is a diffeomorphism. We can construct the boundary of M following Everlein and O'Neil [5].

DEFINITION. Two normal geodesic rays $\gamma_1(t), \gamma_2(t)$ $(t > 0)$ in M are said to be *asymptotic* if there is a constant $c > 0$ such that $\operatorname{dist}(\gamma_1(t), \gamma_2(t)) < c$ for all $t > 0$.

We see that the asymptotic relation is an equivalence relation.

DEFINITION. *Sphere at infinity* $S(\infty)$ is the set of asymptotic classes of geodesic rays in M.

Let $\overline{M} = M \cup S(\infty)$ and fix a point $o \in M$. For $v \in T_oM$ we define the *cone* around v of angle δ by

$$C(v, \delta) = \{x \in M: \measuredangle_o (v, \dot{\gamma}_x(0)) < \delta\},$$

where $\gamma_x(t)$ is the normal geodesic rays through x starting from o, and $\measuredangle_o$ denotes angle in T_pM. Let $T(v, \delta, r) = C(v, \delta) \backslash B_o(r)$ be the *truncated cone* of rudious r, where $B_o(r)$ is the geodesic r-ball around o. The set of all $T(v, \delta, r)$, for all $v \in T_oM$, and $r > 0$, and $B_q(r)$, for all $q \in M$ and $r > 0$, defines a local basis of topology on $\overline{M}$ [5]. It is called the *cone topology.* The cone topology is independent of the choice of the origin $o \in M$. In this topology $\overline{M}$ is homeomorphic to a closed ball $\overline{B}$ in $\boldsymbol{R}^n$, and $S(\infty)$ is homeomorphic to the boundary ∂B.

Dirichlet problem at infinity. Given a continuous function f on $S(\infty)$, find $u \in C^0(\overline{M})$ satisfying $\Delta u = 0$ on M and $u = f$ on $S(\infty)$.

The maximum principle implies that if the Dirichlet problem at infinity is solvable, then there are many bounded harmonic functions on such a

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manifold. Anderson [1] and Sullivan [15] showed that the Dirichlet problem at infinity is solvable if the sectional curvature $K(x)$ satisfies $-a^2 \leqq K(x) \leqq -b^2$, where a and b are positive constants. Theorem 1 is a partial extension of the result of [1, 15], and the proof is based on [2]. The second inequality of (1) in Theorem 1 is a little similar to the inequality: curvature $(x) < r(x)^{-2}$, in fact the condition: curvature $(x) < r^{-2}$ implies several properties relating to hyperbolicity (cf. [9]).

THEOREM 1. *Let M be a Hadamard manifold and $K(x)$ be the sectional curvature at $x \in M$. Suppose relative to some $o \in M$,*

$$(1) \qquad -a^2 \leqq K(x) \leqq -1/(1 + r(x)^{2-\varepsilon}) \qquad \textit{for } x \in M$$

for two constants $a > 0$ and $2 > \varepsilon > 0$, then the Dirichlet problem at infinity is uniquely solvable, where $r(x) = \operatorname{dist}(o, x)$.

In the following of this section M always denotes a Hadamard manifold with metric $g = (g_{ij})$, and $o \in M$ is fixed.

LEMMA 1. *If the sectional curvature $K(x)$ satisfies*

$$(2) \qquad K(x) \leqq -(1 + r(x)^{2-\varepsilon})^{-1} \qquad \textit{for } x \in M$$

for a constant $2 > \varepsilon > 0$, then for any two normal geodesic rays $\gamma_1(t), \gamma_2(t)$ starting from $o \in M$ with angle $\theta = \measuredangle_o(\dot{\gamma}_1(0), \dot{\gamma}_2(0)) < \pi/4$, we have

$$(3) \qquad \operatorname{dist}(\gamma_1(t), \gamma_2(t)) > 2t + 2(2 + t)^{1-\varepsilon/2}(\log \theta - 1)$$

Proof. For every integer m, we see that $K(x) < -1/(1 + m)^{2-\varepsilon}$ on $B_o(m)$. Comparing with the space of constant curvature $-1/(1 + m)^{2-\varepsilon}$, by the Rauch's comparison theorem we obtain

$$\operatorname{dist}(\gamma_1(t), \gamma_2((t)) > 2t + 2(1 + m)^{1-\varepsilon/2}(\log \theta - 1) \qquad \text{for } 0 < t < m.$$

Define the function $f(t)$ on $t \in [0, \infty)$ by

$$f(t) = 2t + 2(1 + m)^{1-\varepsilon/2}(\log \theta - 1), \qquad \text{if } t \in [m - 1, m).$$

Clearly $\operatorname{dist}(\gamma_1(t), \gamma_2(t)) > f(t)$ on $t \in [0, \infty)$. On the other hand $f(t) \geqq 2t + 2(2 + t)^{1-\varepsilon/2}(\log \theta - 1)$ on $t \in [0, \infty)$ since $\theta < \pi/4$. Then we have (3) for all $t > 0$.

LEMMA 2. *If $K(x)$ satisfies (2) on M, then for any positive constant δ with $1 > \delta > 1 - \varepsilon/2$ there exist positive constants r_1 and C_1 such that*

$$(4) \qquad \Delta \exp(-r(x)^{1-\delta}) < -C_1 r(x)^{-2} \exp(-r^{1-\delta})$$

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on $M\backslash B_o/r_1)$.

Proof. If $K(x) \leqq -C^2$, then the Hessian comparison theorem of Greene and Wu [9] implies $D^2r(x) \geqq C\cdot\coth(Cr(x))\cdot G$, where $G = g - dr\otimes dr$. By the same reason of the proof of Lemma 1, we have

$$D^2r(x) \geqq (1+m)^{\varepsilon/2-1}\cdot G$$

if $m-1 \leqq r < m$. All of the above inequalities on each interval $[m, m+1)$ implies

$$D^2r(x) \geqq 1/(2+r)^{1-\varepsilon/2}\cdot G \tag{5}$$

on M. Direct computations give

$$\varDelta\exp(-r(x)^{1-\delta}) < (1-\delta)\exp(-r^{1-\delta})r^{-\delta}(-\varDelta r + r^{-\delta}), \qquad r(x) > 1\,.$$

By (5) we have

$$\varDelta\exp(-r^{1-\delta}) < (1-\delta)\exp(-r^{1-\delta})r^{-2\delta}[1 - Cr^{\delta}/(2+r)^{1-\varepsilon/2}]\,.$$

Since $1-\varepsilon/2 < \delta$ we obtain (4) for sufficiently large r_1.

Let h be a continuous function on the geodesic unit sphere $S_0(1)$ in M with center at $o\in M$. We extend h radially along rays from o to a function h_o on $M\backslash o$ with boundary values h on $S(\infty)$. Let λ: $[0,\infty)\to[0,1]$ be a C^2 function satisfying

$$\lambda(t) = \begin{cases}1, & t\in[0,1]\\ 0, & t\in[2,\infty)\,.\end{cases}$$

We define a C^2 function $H(x)$ on M by

$$H(x) = \int_M \lambda(r(x,y)^2)h_o(y)dy \Big/ \int_M \lambda/(r(x,y)^2)dy\,, \tag{6}$$

where $r(x,y) = \operatorname{dist}(x,y)$ and the integral is with respect to the volume form on M. We see that $H(x)$ is continuous on $\overline{M}$ and $H = h$ on $S(\infty)$. If we put $\lambda_1(t) = \lambda(t^2)$, we have $D^2\lambda(r^2) = \ddot{\lambda}_1 dr\otimes dr + \dot{\lambda}_1 D^2r$. If $K(x)$ satisfies $0 > K(x) \geqq -a^2$, then $0 < D^2r_y(x) \leqq a\cdot\coth(ar_y(x))\cdot G$ for any $x, y\in M$ by the Hessian comparison theorem [9], where $r_y(x) = r(x,y)$. Thus we obtain

$$-C_2 g \leqq D^2(r_y(x)^2) \leqq C_2 g\,, \qquad x, y\in M\,,$$

for a positive constant C_2. We see that

$$\varDelta H(x_0) = \varDelta[H - h_o(x_0)](x_0)$$

$$= \Delta\left[\int\int_M \lambda(r(x_0, y)^2)(h_o(y) - h_o(x_0))dy \Big/ \int_M \lambda(r(x_0, y)^2)dy\right].$$

The curvature bounds imply that the volumes of $B_x(1)$ and $B_x(2)$ are bounded from below and above for any $x \in M$. Then we have the following lemma.

LEMMA 3. *If* $0 > K(x) \geqq -a^2$ *on* M, *then we have*

$$(7) \qquad |\Delta H(x)| < C_3 \sup_{y \in B_x(2)} |h_o(y) - h_o(x)| \qquad \text{for } x \in M,$$

where C_3 *is a positive constant.*

Proof of the theorem. We identify $S(\infty)$ with the set of geodesic rays starting from o. We can approximate h of $C^0(S(\infty))$ by Lipshitz continuous function on $S_o(1) \approx S(\infty)$. By the maximum principle and the Harnack's principle, if a sequence of harmonic functions $u_k \in C^0(\overline{M})$ converges uniformly on $S(\infty)$, u_k converges uniformly on $\overline{M}$ to a harmonic function $u \in C^0(\overline{M})$. Thus we may assume that h is Lipshitz continuous on $S_o(1)$. We extend h radially on M. Define $H(x)$ by (6). From Lemma 3 we get $H(x) < C_4 \max_{y \in B_x(2)} \measuredangle_o(x, y)$ since H is Lipshitz continuous with respect to $\measuredangle_o(x, y)$. By Lemma 1 we obtain

$$\max_{y \in B_n(2)} \measuredangle_o(x, y) < \exp(5 - (2 + r(x))^{\varepsilon/2})$$

if $r(x) > 2$. Then

$$(8) \qquad |\Delta H(x)| < C_5 \exp(-(2 + r)^{\varepsilon/2}), \qquad r(x) > 2.$$

Choose a constant δ with $1 > \delta > 1 - \varepsilon/2$, and define

$$F^+(x) = H(x) + C_6 \exp(-r(x)^{1-\delta}),$$
$$F^-(x) = H(x) - C_6 \exp(-r(x)^{1-\delta})$$

From (4) and (8) we have

$$\Delta F^+(x) < C_5 \exp(-(2 + r)^{\varepsilon/2}) - C_1 C_6 r(x)^{-2\delta} \exp(-r^{1-\delta}),$$
$$\Delta F^-(x) > -C_5 \exp(-(2 + r)^{\varepsilon/2}) + C_1 C_6 r(x)^{-2\delta} \exp(-r^{1-\delta})$$

on $x \in M \backslash B_o(r_1)$. If we fix a constant r_2 with $r_2 > r_1$, F^+ and F^- is superharmonic and subharmonic respectively on $M \backslash B_o(r_2)$ since $\varepsilon/2 > 1 - \delta$. Moreover we choose C_6 such that

$$(9) \qquad \max_{x \in \overline{M}} H(x) - \min_{x \in \overline{M}} H(x) < C_6 \exp(-r_2^{1-\delta}).$$

Now we define $G^+(x)$ and $G^-(x)$ by

$$G^+(x) = \min\{\inf_{x\in B_o(r_2)} H(x) + C_6 \exp(-r_2^{1-\delta}), \qquad F^+(x)\}$$
$$G^-(x) = \max\{\sup_{x\in B_o(r_2)} H(x) - C_6 \exp(-r_2^{1-\delta}), \qquad F^-(x)\}.$$

Clearly $G^+(x)$ and $G^-(x)$ are continuous on $\overline{M}$ and constant on $B_o(r_2)$. Then $G^+(x)$ is superharmonic and $G^-(x)$ is subharmonic on M. By (9) we can check $G^+(x) > G^-(x)$ on M, moreover we can find a constant $r_3 > r_2$ such that

$$(10) \qquad C_6 \exp(-r_2^{1-\delta}) - (\max_{x\in\overline{M}} H(x) - \min_{x\in\overline{M}} H(x)) > C_6 \exp(-r_3^{1-\delta}).$$

(10) implies

$$F^+(x) < \inf_{x\in B_o(r_2)} H(x) + C_6 \exp(-r_2^{1-\delta})$$
$$F^-(x) > \sup_{x\in B_o(r_2)} H(x) - C_6 \exp(-r_2^{1-\delta})$$

for $r \in \overline{M}\backslash B_o(r_3)$. The above inequalities mean $F^+(x) = G^+(x)$ and $F^-(x) = G^-(x)$ on $\overline{M}\backslash B_o(r_3)$. Hence $G^+(x) = G^-(x) = h(x)$ on $S(\infty)$. $G^+(x)$ and $G^-(x)$ are barrier functions to solve the Dirichlet problem at infinity by the Perron method. Consequently there is the Perron solution which is exactly the solution of the Dirichlet problem at infinity. The uniqueness follows from the maximum principle. This completes the proof.

Remark. Recently Proffesor H. Wu informed the author that H. Wu and R. Schoen proved that *if* $-a\cdot r(x)^2 \leqq K(x) \leqq -b\cdot r(x)^{-2}$ $(b \geqq 2)$, *then the Dirichlet problem at infinity is solvable.*

§ 2. Complex analysis on Kähler Hadamard manifold

Now we prove the existence of bounded holomorphic functions on Kähler Hadamard manifold M in a special class. For this purpose we will consider the Dirichlet problem at infinity for $\bar\partial$ like that for Δ. If the sphere at infinity $S(\infty)$ should admit a CR-structure and M should be hyperbolic in a sense, there would be a holomorphic extension to M. However, in general $S(\infty)$ admits no differentiable structure. We shall define a CR-function on $S(\infty)$ for a special class of Kähler Hadamard manifolds, and extend to a holomorphic function on M. The boundedness of the extended function follows from the absolute maximum principle.

By the same idea we shall show in Theorem 3 that a manifold in the special class is biholomorphic to a bounded domain in C^n under some curvature condition.

Let M be a complex manifold of dimension n, $n \geqq 2$. Let J be the complex structure of M. For a real C^∞ hypersurface N of M, we define the vector subspace $\overline{H}_p(N)$ of $T_pN \otimes C$ by

$$\overline{H}_p(N) = \{Z \in T_pN \otimes C: JZ = -\sqrt{-1}Z\}.$$

It is obvious that $\dim_c \overline{H}_p = n - 1$. Let h be a complex valued function on N. If $Zh = 0$ for every $Z \in \overline{H}_p(N)$, we call that h satisfies the *tangential Cauchy-Riemann equation* at p. If h satisfies the tangential Cauchy-Riemann equation at every point of N, we call h a *CR-function on N*.

In the following let M be a Kähler Hadamard manifold of complex dimension n, $n \geqq 2$. Suppose that the metric of M is of the form

$$(11) \qquad ds^2 = dr^2 + g(r, \theta)^2\{d\theta_2^2 + \sin^2\theta_2 d\theta_3^2 + \cdots + \sin^2\theta_2 \cdots \sin^2\theta_{2n-1} d\theta_{2n}^2\},$$

where in terms of the geodesic polar coordinates at o, $\theta = (\theta_2, \cdots, \theta_{2n})$ is a spherical angle of $S_o(1)$, and r denotes the distance from o, i.e. each geodesic sphere with center at o is conformal to the standard sphere in R^{2n}. The Dirichlet problem at infinity on such manifolds is studied by Choi [4]. For example, every rotationally symmetric manifold satisfies this condition (cf. Milnor [12], Shiga [14]).

Identifying $S_o(1)$ with $S(\infty)$, for any $h \in C^0(S(\infty))$ we define a continuous function h_o on $\overline{M}\backslash o$ by

$$h_o(r, \theta) = h(\theta) \quad \text{for} \quad \theta \in S(\infty) \approx S_o(1).$$

DEFINITION. We call $h \in C^0(S(\infty))$ a *CR-function at infinity with respect to* $o \in M$, if $h_o(r, \theta)$ is differentiable on $M\backslash o$ and $h_o(1, \theta)$ is a CR-function on $S_o(1)$.

The following lemma shows that our definition is natural for the above manifolds. We denote by $\mathrm{CR}_o(\infty)$ the set of all CR-functions at infinity with respect to $o \in M$. Note that there exists a bijection between $\mathrm{CR}_o(\infty)$ and the set of all CR-functions on $S_0(1)$. Regarding $B_o(2)$ a domain in C^n, we see that $\mathrm{CR}_o(\infty)$ is not empty.

LEMMA 4. *Let M be a Kähler Hadamard manifold of complex dimension n $(n \geqq 2)$. Assume that the Kähler metric in terms of the geodesic polar coordinates at o is of the form* (11). *If $h \in \mathrm{CR}_o(\infty)$, then $h_o|_{S_o(t)}$ is a CR-function on $S_0(t)$ for all $t > 0$.*

Proof. It is sufficient to show that for any rays $\gamma(t)$ starting from

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$o \in M$, $Zh = 0$ at $\gamma(t)$ for all $Z \in \overline{H}_{\gamma(t_0)}(S_o(t))$ and $t > 0$. Then we fix a ray $\gamma(t)$ and $t_0 > 0$. In the geodesic polar coordinates we denote $\gamma(1)$ by $(1, \theta')$, and we may assume that $\sin\theta'_2, \cdots, \sin\theta'_{2n}$ are not 0.

For any $Z_0 \in \overline{H}_{\gamma(t_0)}(S_o(t_0))$ we denote by $Z(t)$ the parallel vector field along $\gamma(t)$ with $Z(t_0) = Z_0$. Since J is parallel and $Z(t)$ is always orthogonal to $\dot\gamma(t)$, we see that $Z(1) \in \overline{H}_{(\gamma)(1)}(S_o(1))$.

We define the vector field $X_i(t)$ along $\gamma(t)$ by

$$X_i(t) = \{g(t, \theta')\sin\theta'_2\cdots\sin\theta'_{i-1}\}^{-1}\cdot\partial/\partial\theta_i\,, \tag{12}$$

$i = 2, \cdots, 2n$. Therefore

$$\begin{aligned}\nabla_{\dot\gamma(t)}X_i(t) = &-\partial g/\partial t\cdot\{g^2\cdot\sin\theta'_2\cdots\sin\theta'_{i-1}\}^{-1}\cdot\partial/\partial\theta_i\\ &+\textstyle\sum_{k=2}^{2n}\{g\cdot\sin\theta'_2\cdots\sin\theta'_{i-1}\}^{-1}\Gamma^k_{1i}\partial/\partial\theta_k\\ &+\{g\cdot\sin\theta'_2\cdots\sin\theta'_{i-1}\}^{-1}\Gamma^1_{1i}\partial/\partial r,\end{aligned}$$

where we put $\nabla_{\dot\gamma(t)}\partial/\partial\theta_i = \Gamma^1_{1i}\partial/\partial r + \sum_{k=2}^{2n}\Gamma^k_{1i}\partial/\partial\theta_k$. We see that $\Gamma^1_{1i} = 0$ and $\Gamma^i_{1i} = f^{-1}\cdot\partial f/\partial r$. Since the metric tensor is diagonal with respect to the polar coordinates, other Γ^k_{1i}'s are vanished. Then $\nabla_{\dot\gamma(t)}X_i(t) = 0$, that is, $X_i(t)$ is parallel for all $i \geqq 2$.

$\{X_i(t_0)\}$ is an orthonormal frame of $T_{\gamma(t_0)}(S_o(t))$. So we may set $Y(t) = \sum_{k=2}^{2n} a^k X_k(t)$, $J(Y(t)) = \sum_{k=2}^{2n} b^k X_k(t)$. Thus

$$Z(t) = \textstyle\sum_{k=2}^{2n}\{a^k X_k(t) + \sqrt{-1}\,b^k X_k(t)\} \tag{13}$$

$h \in \mathrm{CR}_o(\infty)$ implies $Z(1)h_o = 0$ at $\gamma(1)$. In the geodesic polar coordinates we have

$$\begin{aligned}g^{-1}\cdot\textstyle\sum_{k=2}^{2n}[&a^k(\sin\theta'_2\cdots\sin\theta'_{k-1})^{-1}\cdot\partial h_o(1,\theta')/\partial\theta_k\\ &+\sqrt{-1}\,b^k(\sin\theta'_2\cdots\sin\theta'_{k-1})^{-1}\cdot\partial h_o(1,\theta')/\partial\theta_k] = 0\end{aligned} \tag{14}$$

by (12) and (13). Similarly

$$\begin{aligned}&Z(t_0)h_o(t_0, \theta')\\ &\quad= g^{-1}\textstyle\sum_{k=2}^{2n}[a^k(\sin\theta'_2\cdots\sin\theta'_{k-1})^{-1}\cdot\partial h_o(t_0,\theta')/\partial\theta_k\\ &\qquad+\sqrt{-1}\,b^k(\sin\theta'_2\cdots\sin\theta'_{k-1})^{-1}\cdot\partial h_o(t_0,\theta')/\partial\theta_k]\end{aligned}$$

Recall that $h_o(t_0, \theta) = h_o(1, \theta)$, hence $Z(t_0)h_o = 0$ at $\gamma(t_0)$ by (14). This completes the proof.

THEOREM 2. *Let M be a Kähler Hadamard manifold of complex dimension n, $n \geqq 2$. Assume that the Dirichlet problem at infinity is solvable on M, and the Kähler metric in terms of the geodesic polar coordinates at*

$o \in M$ is of the form

$$ds^2 = dr^2 + g(r, \theta)^2\{d\theta_2^2 + \sin^2\theta_2 d\theta_3^2 + \cdots + \sin^2\theta_2 \cdots \sin^2\theta_{2n-1} d\theta_{2n}^2\}.$$

Then for any $h \in \mathrm{CR}_o(\infty)$, there exists a holomorphic function H on M with boundary values h, and H coincides with the solution of the Dirichlet problem at infinity.

Remark. Trivial examples of Kähler manifolds as above are C^n and the unit ball B in C^n with the invariant metric. For the ball B we may identify a CR-function at infinity with respect to the origin as a CR-function on ∂B, hence we can extend it to a holomorphic function on B by the well known method (Hörmander [10, Theorem 2.3.2′]). On one hand by Liouville's theorem we see that any CR-function at infinity on the sphere at infinity of C^n can not be extended to a holomorphic function on C^n. So in order to extend a function of $\mathrm{CR}_o(\infty)$ to a holomorphic function, we need some hypothesis on N relating to hyperbolicity.

The hypothesis that the Dirichlet problem at infinity be solvable is fulfilled if, for example, the sectional curvature $K(x)$ satisfies $-a^2 \leqq K(x) \leqq -1/(1 + r^{2-\varepsilon})$ by Theorem 1.

Proof. We denote h by $h = h^1 + \sqrt{-1}h^2$, where $h^1 = \mathrm{Re}\, h$, $h^2 = \mathrm{Im}\, h$. Since the Dirichlet problem at infinity is solvable on M, there exist harmonic functions H^1 and H^2 on M with $H^1 = h^1$ and $H^2 = h^2$ on $S(\infty)$. Thus we have only to show that $H = H^1 + \sqrt{-1}H^2$ is holomorphic on M.

It is shown in [9] that a Kähler Hadamard manifold is a Stein manifold. By Lemma 4 h_o is a CR-function on $S_o(r)$ for all $r > 0$. We see that the boundary $S_o(r)$ of $B_o(r)$ is connected and $B_o(r)$ is relatively compact in M. Then we can find a holomorphic function H_r on $B_o(r)$ with $H_r = h_o$ on $S_o(r)$ (Shiga [13, Theorem 2-5]). So we have a sequence of holomorphic functions $\{H_k\}$ with $H_k = h_o$ on $S_o(k)$ for $k \in N$. Put $H_k^1 = \mathrm{Re}\, H_k$ and $H_k^2 = \mathrm{Im}\, H_k$. Then H_k^1 and H_k^2 are harmonic on $B_o(r)$ since M is Kähler. In the polar coordinates we have $h_o^1(k, \theta) = H_k^1(k, \theta)$, and $h_o^2(k, \theta) = H_k^2(k, \theta)$ on $S_o(k)$. Since H^1 and H^2 are continuous on M, for any $\varepsilon > 0$ there is a large integer k_0 such that

$$|H_k^j(k, \theta) - H^j(k, \theta)| < \varepsilon \quad \text{for} \quad j = 1, 2$$

on $S_o(k)$ for all $k > k_0$. The maximum principle implies

$$|H_k^j - H^j| < \varepsilon \quad \text{for} \quad j = 1, 2$$

on $B_o(k)$ for all $k > k_0$. This means that $\{H_k\}$ converges to H uniformly on every compact subset of M. Then H is holomorphic since $\{H_k\}$ is a sequence of holomorphic functions.

If the Dirichlet problem at infinity is solvable on a Hadamard manifold M, then we see that there is the *harmonic measure* μ^x on $S(\infty)$ from the Riesz representation theorem. Then we have the following corollary (cf. [1, 2]).

COROLLARY. *Let M be as in Theorem 2. Let μ^x is the harmonic measure on $S(\infty)$. Then for every $h \in \mathrm{CR}_o(\infty)$,*

$$h(x) = \int_{S(\infty)} h d\mu^x$$

is a holomorphic function on M with boundary values h.

Let M_1 and M_2 be complex manifolds of complex dimension n, $n \geqq 2$. Let D_1 and D_2 be bounded domains with smooth boundaries $\partial D_1, \partial D_2$ respectively. We call a C^∞ mapping f of ∂D_1 to ∂D_2 a *CR-mapping* if $f_*(\overline{H}_p((\partial D_1)) \subset \overline{H}_{f(p)}(\partial D_2)$ for all $p \in \partial D_1$. Note that f is a CR-mapping if and only if for any CR-function h on ∂D_2, $f \circ h$ is a CR-function on ∂D_1 [13].

Let M be a complex manifold and d_M the Kobayashi pseudodistance [11]. If d_M is a distance and M is complete with respect to d_M, M is said to be *complete hyperbolic.*

Let D be a domain in C^n. D is called a *strictly pseudoconvex domain with C^k boundary* if there exist an open neighborhood U of D and a strictly plurisubharmonic function $r(z)$ on U of class C^k such that $D = \{z \in U: r(z) < 0\}$ and $\operatorname{grad} r(z) \neq 0$ for all $z \in \partial D$.

THEOREM 3. *Let M be a Kähler Hadamard manifold of complex dimension n, $n \geqq 2$. Assume that the Kähler metric in terms of the geodesic polar coordinates at o is of the form*

$$ds^2 = dr^2 + g(r, \theta)^2\{d\theta_2^2 + \sin^2\theta_2 d\theta_3^2 + \cdots + \sin^2\theta_2 \cdots \sin^2\theta_{2n-1} d\theta_{2n}^2\},$$

and the holomorphic sectional curvature $K_h(x)$ satisfies $K_h(x) < -1/(1 + r(x)^2)$. Then M is biholomorphic to a strictly pseudoconvex domain in C^n.

Remark. It is shown in Shiga [14] that if $g(r, \theta) = g(r)$, and the holomorphic radial curvature $K(x)$ satisfies $K(x) < -(1 + \varepsilon)/r^2 \log r$, then M is biholomorphic to the unit ball in C^n (cf. Milnor [12]).

The following Lemma is given in Fridman [6].

LEMMA 5. *Let $D \subset C^n$ be a bounded strictly pseudoconvex domain with C^3 boundary, and M is a completely hyperbolic manifold of complex dimension n. Suppose that M can be exhausted by biholomorphic images of D, that is, for any compact $K \subset M$ there is a biholomorphic imbedding $F_K: D \to M$ such that $F_K(D) \subset K$. Then M is biholomorphically equivalent either to D or to the unit ball in C^n.*

Proof of the theorem. Recall that M is a Stein manifold. Choose a holomorphic coordinate neighborhood U of M such that $B_o(\varepsilon) \subset\subset U$ for a positive ε. By the Hessian comparison theorem [9], $r(x)^2$ is strictly plurisubharmonic on M since M is Kähler. Clearly $\operatorname{grad} r(x)^2 \neq 0$ on $M \backslash o$. Then we may regard $B_o(\varepsilon)$ as a strictly pseudoconvex domain with C^∞ boundary in C^∞. We define a diffeomorphism f_k from $S_o(\varepsilon)$ to $S_o(k)$, $k \in N$, by

$$f_k(\varepsilon, \theta) = (k, \theta)$$

where (r, θ) is the polar coordinates at o. Lemma 4 implies that f_k is a CR-diffeomorphism. Obviously $S_o(\varepsilon)$ and $S_o(k)$ are connected. From the Bochner-Hartogs' theorem on Stein manifolds (Shiga [13]) we see that $B_o(\varepsilon)$ is biholomorphic to $B_o(k)$ for all integer k. For any compact set K in M, there exists an integer k so that $B_o(k) \supset K$ since $\exp_o: T_oM \to M$ is a diffeomorphism. So M is exhausted by biholomorphic images of the strictly pseudoconvex domain $B_o(\varepsilon)$. Since $K_h(x) < -1/(1 + r^2)$ and M is complete, M is complete hyperbolic from the theorem of Green and Wu [9, Theorem E].

It follows that M is biholomorphically equivalent either to the unit ball B in C^n or to $B_o(\varepsilon)$ from Lemma 5. Both $B_o(\varepsilon)$ and B are strictly pseudoconvex, then the theorem is proved.

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