1 Fundamentals

1.1 Preliminaries

We have mentioned already at the beginning of the book that the fundamental role in elementary particle physics, that is, the SM and its extensions, is played by Lagrangians. They encode the information about the particle content of a given theory and of fundamental interactions between these particles that are characteristic for this theory. Therefore, it is essential to start our presentation by discussing the general structure of various Lagrangians that we will encounter in this book.

The theories we will discuss are relativistic quantum field theories, and it would appear at first sight that this first step of our expedition is extremely difficult. Yet, the seminal observation of Feynman that a given classical theory can be quantized by means of the path integral method simplifies things significantly. We can formulate the quantum field theory with the help of a Lagrangian of a classical field theory without introducing operators as done in canonical quantization. Having it, a simple set of steps allows us to derive the so-called Feynman rules and use them to calculate the implications of a given theory for various observables that can be compared with experiment.

A very important role in particle physics is played by symmetries. They increase significantly the predictive power of a given theory, in particular by reducing the number of free parameters. In this context a very good example is quantum chromodynamics (QCD), the theory of strong interactions. With eight gluons and three colors for quarks, there is a multitude of interactions that, without the SU(3) symmetry governing them, could be rather arbitrary. But the SU(3) symmetry implies certain conservation laws, and at the end there is only a single parameter in QCD: the value of the strong coupling evaluated at some energy scale that can be determined in experiment. Once this is done, all effects of strong interactions can be uniquely predicted, even if this requires often very difficult calculations.

The case of QCD is however special as it is based on an exact nonabelian symmetry. We will be more specific about this terminology later. In quantum electrodynamics (QED), which is based on an exact abelian symmetry U(1), in addition to the value of the electromagnetic coupling also the electric charges of quarks and leptons and generally fermions, scalars, and vector particles in a given theory are free. They have to be determined in experiment. In QCD all *color charges* are fixed by the SU(3) symmetry.

Yet QED, similar to QCD, is a very predictive theory because it is based on an exact symmetry. This is generally not the case in nature, and on many occasions the symmetries that we encounter in particle physics are only approximate, and the manner in which

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they are broken has an impact on physical implications. Moreover, models with broken symmetries often contain many new free parameters beyond the couplings, significantly lowering the predictive power of the theory. A very prominent example is supersymmetry.

Our goal for the next pages is to write down Lagrangians, in fact Lagrangian densities,[1] for simplest theories involving spin-0 particles (scalars), spin-$\frac{1}{2}$ particles (fermions), and spin-1 particles (vectors or vector bosons). In this context we will discuss various symmetries that we will encounter later at various places in our book.

1.2 Lagrangians for Scalar Fields

1.2.1 Real Scalar Field

Let us consider the real scalar field $\varphi \equiv \varphi(x)$ for which the Lagrangian, neglecting interactions, reads

$$\mathscr{L}(\varphi, \partial_\mu \varphi) = \frac{1}{2}\left(\partial_\mu \varphi\right)(\partial^\mu \varphi) - \frac{1}{2} m^2 \varphi^2. \tag{1.1}$$

Inserting this Lagrangian into the Euler–Lagrange equation,

$$\frac{\delta \mathscr{L}}{\delta \varphi} = \partial_\mu \frac{\delta \mathscr{L}}{\delta\left(\partial_\mu \varphi\right)}, \tag{1.2}$$

where

$$\delta \mathscr{L}(\varphi) = \mathscr{L}(\varphi + \delta\varphi) - \mathscr{L}(\varphi), \tag{1.3}$$

we find the Klein–Gordon (KG) equation

$$\left(\Box + m^2\right)\varphi = 0 \tag{1.4}$$

so that the only parameter in (1.1) m can be interpreted as the mass of the spin-0 particle corresponding to the field φ.

1.2.2 Complex Scalar Field

We next promote φ in (1.1) to a complex scalar field. The Lagrangian takes now the following form:

$$\mathscr{L}(\varphi, \partial_\mu \varphi, \varphi^*, \partial_\mu \varphi^*) = \frac{1}{2}\left(\partial_\mu \varphi^*\right)(\partial^\mu \varphi) - \frac{1}{2} m^2 \varphi^* \varphi. \tag{1.5}$$

[1] For simplicity, we will call these Lagrangian densities just Lagrangians.

If φ and φ^* are written as

$$\varphi = \frac{1}{\sqrt{2}}(\varphi_1 + i\varphi_2), \qquad \varphi^* = \frac{1}{\sqrt{2}}(\varphi_1 - i\varphi_2), \tag{1.6}$$

where $\varphi_{1,2}$ are real, then these real fields satisfy separately the KG equation in (1.4). Equivalently we have

$$\left(\Box + m^2\right)\varphi = 0, \qquad \left(\Box + m^2\right)\varphi^* = 0. \tag{1.7}$$

Before continuing, we have to discuss an important topic.

1.3 First Encounter with Symmetries

A symmetry is a transformation on the fields and x, which leaves the Lagrangian invariant. Thus if there is a transformation R:

$$\varphi(x) \xrightarrow{R} \varphi'(x'), \tag{1.8}$$

so that

$$\mathscr{L}(\varphi(x)) \xrightarrow{R} \mathscr{L}(\varphi'(x')) = \mathscr{L}(\varphi(x)), \tag{1.9}$$

then R is a symmetry of the Lagrangian.

The symmetry transformations that we will encounter in this book can be grouped in three classes:

- Continuous transformations in space-time

$$x \to x' + \delta x, \qquad \varphi(x) \to \varphi'(x'). \tag{1.10}$$

  These are the Lorentz transformations. We will not discuss them in this book as they are the topic of introductory lectures on field theory. However, we will make sure that our Lagrangians and their implications are consistent with Lorentz invariance. In the Lagrangians in (1.1) and (1.5) this is achieved by contracting the indices μ.
- Continuous internal symmetries

$$\varphi(x) \to \varphi(x) + \delta\varphi(x) = \varphi'(x). \tag{1.11}$$

  These symmetries will play a crucial role in our book. One distinguishes between *global* and *local* internal symmetries, and each of them can be either *abelian* or *nonabelian*. What this really means will be explained in detail as we proceed.
- Discrete symmetries

  A typical example of a discrete transformation is the flip of the sign of the field φ:

$$\varphi \to -\varphi. \tag{1.12}$$

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However, the best-known discrete transformations are

$$\text{Parity (P):} \qquad t \to t' = t, \quad \vec{x} \to \vec{x}' = -\vec{x}, \tag{1.13}$$

$$\text{Charge conjugation (C):} \qquad Q \to -Q, \tag{1.14}$$

$$\text{Time reversal (T):} \qquad t \to t' = -t, \quad \vec{x} \to \vec{x}' = \vec{x}. \tag{1.15}$$

As we will see, QED and QCD interactions are invariant under these three transformations, while this is not the case of weak interactions.

1.4 Checking the Symmetries of Scalar Lagrangians

Armed with this elementary knowledge of symmetries, we can now investigate which internal symmetries are present in the Lagrangians in (1.1) and (1.5). Specifically, let us ask whether these Lagrangians are invariant under the following transformations:

$$i) \qquad \varphi(x) \to \varphi'(x) = -\varphi(x) \qquad \text{(discrete)}, \tag{1.16}$$

$$ii) \qquad \varphi(x) \to \varphi'(x) = e^{i\theta r}\varphi(x) \qquad \text{(continuous global)}, \tag{1.17}$$

$$iii) \qquad \varphi(x) \to \varphi'(x) = e^{i\theta(x) r}\varphi(x) \qquad \text{(continuous local)}, \tag{1.18}$$

where θ is a phase that is either independent of x (global transformation) or dependent on x (local transformation). The parameter r is introduced to characterize a property of the field under this transformation, not the transformation itself, and can be interpreted as the conserved "charge," as we will see later on.

The Lagrangian in (1.1) is clearly invariant under the discrete transformation in (1.16) but fails completely with respect to the transformations in (1.17) and (1.18) as

$$\varphi^2 \to e^{2i\theta r}\varphi^2. \tag{1.19}$$

This is not surprising. The transformations in (1.17) and (1.18) promoted the real field to a complex field and thus changed the nature of the field. Because the Lagrangian for a complex field in (1.5) looks different than the Lagrangian in (1.1), it is not surprising that (1.1) is not invariant under (1.17) and (1.18).

Yet there is a solution to this problem. We just set $r = 0$ so that φ does not transform at all. One says it is a *singlet* under transformation (1.17) and (1.18), and its charge r vanishes.

We next investigate the Lagrangian (1.5) to find that indeed it is invariant under the global transformation (1.17) as

$$\varphi \to e^{i\theta r}\varphi, \qquad \varphi^* \to \varphi^* e^{-i\theta r} \tag{1.20}$$

leaves (1.5) unchanged provided θ is independent of x. In a more group theoretical language the phase transformation in (1.20) is the simplest unitary transformation related to the group U(1). The r can then be interpreted as a conserved charge. Indeed, the sign in front of r in case of φ^* is opposite to the one in the transformation of φ: the charges of φ

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and φ^* differ by sign and as known from elementary relativistic field theory φ^* represents the antiparticle to φ.

What about the last transformation (1.18) on our list? The last term in (1.5) is clearly invariant under (1.18), but the first one is not! Indeed

$$\partial_\mu \varphi' = \partial_\mu \left(e^{i\theta(x)r} \varphi(x) \right) = e^{i\theta(x)r} \partial_\mu \varphi(x) + ir\partial_\mu \theta(x) \left(e^{i\theta(x)r} \varphi(x) \right), \tag{1.21}$$

$$\partial_\mu \varphi'^* = \partial_\mu \left(\varphi(x)^* e^{-i\theta(x)r} \right) = \partial_\mu \varphi(x)^* e^{-i\theta(x)r} - ir\partial_\mu \theta(x) \left(\varphi(x)^* e^{-i\theta(x)r} \right). \tag{1.22}$$

Inserting these expressions into (1.5) we readily verify that the terms proportional to $\partial_\mu \theta(x)$ break the symmetry. Thus (1.5) is not invariant under the transformation (1.18), the local U(1) transformation.

There is no way out. We have to modify (1.5) to make it invariant under (1.18), or in other words we have to promote the global U(1) symmetry present already in (1.5) to a local U(1) symmetry.

There is a well-known procedure for how to perform this task with a very remarkable result. The requirement of a local U(1) symmetry implies automatically the existence of a new particle with spin-1 and a specific structure of the interaction of this new particle with the original complex field φ that we introduced from the start. Let us present this procedure that consists of four steps. We refrain from profound geometrical interpretations of transformations encountered here, as they can be found in many textbooks on field theory.

1.5 Promotion of a Global U(1) Symmetry to a Local U(1) Symmetry

- **Step 1**

  We introduce a vector particle A_μ to be called *gauge boson* in what follows. It is not surprising that we have to introduce a vector field $A_\mu(x)$. After all, we have to cancel terms involving $\partial_\mu \theta(x)$.
- **Step 2**

  We replace the derivative ∂_μ by a *covariant derivative* D_μ, which transforms under U(1) as the fields φ and φ^*:

  $$D_\mu \varphi \to e^{i\theta(x)r} D_\mu \varphi, \qquad \left(D_\mu \varphi \right)^* \to \left(D_\mu \varphi \right)^* e^{-i\theta(x)r}. \tag{1.23}$$

  It is given by

  $$D_\mu \varphi = \left(\partial_\mu - irgA_\mu \right) \varphi, \qquad \left(D_\mu \varphi \right)^* = \left(\partial_\mu + irgA_\mu \right) \varphi^*, \tag{1.24}$$

  where g is a real parameter, the gauge coupling characterizing the strength of the interaction, and r can be again interpreted as the charge of φ. It is not necessarily an electric charge but a charge related to a given local U(1) symmetry.
- **Step 3**

  In order to satisfy (1.23) also A_μ has to transform under U(1) in a special manner:

  $$A_\mu \to A_\mu + \frac{1}{g} \partial_\mu \theta(x). \tag{1.25}$$

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- **Step 4**

  For A_μ to be interpreted as a physical particle, a kinetic term describing the motion of this new particle has to be added to the original Lagrangian. This term must be invariant under the transformation (1.25). This additional term is familiar from QED:

$$\Delta\mathscr{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{1.26}$$

  The resulting Lagrangian takes the form

$$\mathscr{L}_{\text{gauged}} = \left(D_\mu\varphi\right)^* (D^\mu\varphi) - m^2\varphi^*\varphi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}. \tag{1.27}$$

  One can easily verify that it is invariant under the following set of transformations:

$$\varphi(x) \to e^{i\theta(x)r}\varphi(x), \qquad \varphi^*(x) \to \varphi^*(x)e^{-i\theta(x)r} \tag{1.28}$$

  for scalar fields and (1.25) for A_μ.

The requirement that the Lagrangian is invariant under such a local symmetry is called gauge principle. It is very restrictive for the Lagrangian, and some consequences are discussed in the next section. Nowadays gauge invariance is not just a property of the Lagrangian but rather the fundamental principle that determines the structure of the Lagrangian.

1.6 A Closer Look at the U(1) Gauge Theory

The Lagrangian in (1.27) has certain properties that we would like to emphasize here.

While the scalar particle has a mass term consistent with the U(1) gauge symmetry, the corresponding mass term for the gauge boson,

$$(\Delta\mathscr{L})^{\text{gauge}}_{\text{mass}} = \frac{M^2}{2}A_\mu A^\mu, \tag{1.29}$$

is clearly not invariant under (1.25) and is absent in (1.27). Consequently, A_μ is a massless gauge boson. The most prominent example of such a gauge boson is the photon.

The imposition of (local) gauge symmetry implies particular structure of the interactions between φ and A_μ. This structure can be found by decomposing (1.27) into a free and interacting Lagrangian:

$$\mathscr{L}_{\text{gauged}} = \mathscr{L}_{\text{free}} + \mathscr{L}_{\text{int}}, \tag{1.30}$$

where

$$\mathscr{L}_{\text{free}} = \partial_\mu\varphi^*\partial^\mu\varphi - m^2\varphi^*\varphi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}. \tag{1.31}$$

$$\mathscr{L}_{\text{int}} = -irgA_\mu\varphi\partial^\mu\varphi^* + irgA_\mu\varphi^*\partial^\mu\varphi + g^2r^2A_\mu A^\mu\varphi^*\varphi. \tag{1.32}$$

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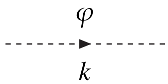
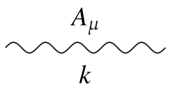
$\mathscr{L}_{\text{free}}$ describes the propagation of the fields φ and A_μ. These propagations can be represented by simple lines (propagators) for which mathematical expressions can be found. In the case at hand these propagators are given in momentum space as follows



$$= \frac{i}{k^2 - m^2 + i\varepsilon}$$

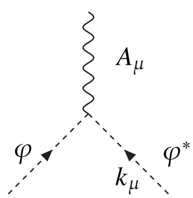


$$= -\frac{i g_{\mu\nu}}{k^2 + i\varepsilon}$$

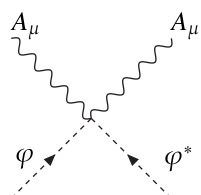
and what we have written down are the simplest Feynman rules with k_μ the four-momentum of the propagating particle. We do not derive these rules, as explicit derivations can be found in textbooks on quantum field theory, see, for instance [11].

In $\mathscr{L}_{\text{int}}$ all terms involve A_μ, φ, and φ^* at one point x, and these terms describe simply the local interactions between the particles in question. Then g is the gauge coupling describing the strength of the interaction and r the charge of φ.

With the help of path integral methods one can derive Feynman rules for the interactions in (1.32). In fact, these rules can be read of from (1.32) by simply multiplying $\mathscr{L}_{\text{int}}$ by i and replacing $i\partial_\mu$ by k_μ. We find then for the vertices representing the first and the last term in (1.32):



$$= -irgk_\mu$$



$$= ig^2r^2$$

and $irgk_\mu$ for the second term. From the propagator and vertices, Feynman diagrams can be constructed. Two examples are given in Figure 1.1. The first one represents the scattering of φ and φ^* with exchange of a photon, the second one the annihilation of these two fields into the photon followed by their regeneration.

Finally, we note that there are no vertices involving A_μ only. This is typical for a U(1) symmetry, which is an abelian symmetry: The gauge boson related to this symmetry carries no charge, and consequently there are no interactions between A_μ. In fact, in the absence of the field φ a theory based on a local U(1) symmetry is a free theory.

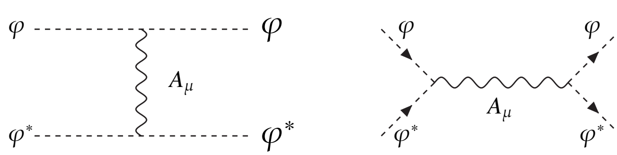


Figure 1.1 Feynman diagrams for a gauged U(1) theory for scalars where φ is a scalar particle and A_μ the gauge boson.

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On the other hand, pure interactions between the fields φ and φ^* consistent with U(1) symmetry can be introduced by adding the following term to $\mathscr{L}_{\text{int}}$:

$$\Delta\mathscr{L}_{\text{int}} = -\frac{1}{4}\lambda\left(\varphi^*\varphi\right)^2 \tag{1.33}$$

with λ describing the strength of this interaction. We will encounter this term elsewhere in this book.

1.7 Nonabelian Global Symmetries

1.7.1 General Considerations

So far we have considered a very simple theory that contained the scalar particle φ, its anti-particle φ^* and one gauge boson A_μ. In general, theories contain several fields $\varphi_1, \ldots \varphi_N$ and also several gauge bosons A_μ^a, where the index a distinguishes between different gauge bosons. If there are symmetries that involve these particles simultaneously, then the corresponding transformations of the fields are more complicated, and also the structure of the Lagrangian is modified relative to the U(1) theory considered until now.

To be able to efficiently discuss such theories, we have to recall certain elements of group theory. The most common groups encountered in particle physics are U(N), SU(N), and SO(N) groups:

- U(N): group of unitary $N \times N$ matrices,
- SU(N): as U(N) but with detU = 1,
- SO(N): group of orthogonal $N \times N$ matrices with determinant 1.

Here we will discuss only U(N) and SU(N) symmetries and postpone the discussion of orthogonal transformations until later.

While the U(1) transformation on a given field had the simple form of just multiplication by a phase factor

$$U(\theta) = e^{i\theta}, \tag{1.34}$$

in the case of U(N) and SU(N) transformations we have

$$U(\theta_1, \ldots \theta_m) = e^{i\sum_{a=1}^m \theta^a T^a}, \tag{1.35}$$

where θ^a are the parameters of the group and T^a the corresponding generators. We have $m = N^2$ and $m = N^2 - 1$ for U(N) and SU(N), respectively. The transformations $U(\theta_1, \ldots \theta_m)$ with θ_i being real satisfy the relations

$$U(\theta_1, \ldots \theta_m) \cdot U(\theta_1', \ldots \theta_m') = U(\theta_1'', \ldots \theta_m''), \tag{1.36}$$

where θ_i'' is analytic in θ_j and θ_k'. The T^a are the generators of SU(N) and satisfy commutation relations

$$\left[T^a, T^b\right] = i f^{abc} T^c, \tag{1.37}$$

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with f^{abc} being the group structure constants.[2] They are simply numbers that for a given group can be found in any book on group theory but I can also recommend the review in [12]. In practice summations over indices a, b, c like the one over c in (1.37) are performed, and we do not have to remember the values of f^{abc}, except that they vanish for two indices being equal. Useful formulas involving f^{abc} will be presented in subsequent chapters, when we will start doing explicit calculations. The fact that generally $f^{abc} \neq 0$ expresses the nonabelian nature of the transformation: T^a and T^b do not commute, and consequently the order of two unitary transformations matters. This fact has profound physical implications, in particular when the global nonabelian symmetry is promoted to the local one. Every set of $N^2 - 1$ $M \times M$ matrices T_r^a that fulfill (1.37) generates a representation r of the Lie algebra. The index r in T_r^a labels the representation. For example, for the fundamental representation of SU(N) one writes $r = \mathbf{N}$ and for the adjoint $r = G$. By an unitary transformation of T_r^a, every representation of SU(N) can be such that all T_r^a are block diagonal:

$$T_r^a = \begin{pmatrix} T_r^{a(1)} & 0 & \dots & 0 \\ 0 & T_r^{a(2)} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & T_r^{a(n)} \end{pmatrix}. \tag{1.38}$$

If there is only one block ($n = 1$), then the representation is called irreducible.

In contrast to U(1) transformations, which acted separately on each field φ, nonabelian transformations act simultaneously on a set of fields that from theory group point of view are the basis vectors of a given irreducible representation. These basis vectors denote a set of quantum mechanical states and are said to constitute a multiplet: doublet, triplet, octet, decouplet, etc. The transformations of the group transform a given field φ_i into linear combinations of the fields belonging to a given multiplet.

The important group theoretical property of a nonabelian symmetry is the existence of multiplets of a size characteristic for a given symmetry. For instance, while a doublet is the smallest multiplet that transforms nontrivially in the case of SU(2) symmetry, in the case of SU(3), it is a triplet. A singlet is of course always smaller. Moreover, while in the case of SU(3), triplets, sextets, octets, and decouplets and specific larger multiplets are present, a quartet or fiveplet is not possible in the case of SU(3).

This discussion shows that if already discovered particles do not fill out the full multiplet of a given nonabelian symmetry, this symmetry predicts the existence of new particles necessary to complete the multiplet in question. This was the case of Ω that was the missing member of the baryon decouplet of the global flavor SU(3) proposed by Gell–Mann or the case of the charm quark in the SU(2) doublet involving also the strange quark.

[2] One can also say that the T^a are the elements of a Lie algebra that generate the Lie algebra (1.37). The number of linear independent generators T^a is the dimension of the Lie algebra or Lie group. A Lie group is called nonabelian if at least one f^{abc} is nonvanishing.

If we restrict our attention first to the simplest multiplets of the SU(2) and SU(3) groups, we can represent the doublets and triplets by column vectors

$$\vec{\varphi} = \begin{pmatrix} \varphi_1 \\ \vdots \\ \varphi_N \end{pmatrix}, \qquad \vec{\varphi}^\dagger = \left(\varphi_1^\dagger, \ldots \varphi_N^\dagger\right), \tag{1.39}$$

with $N = 2$ and $N = 3$ for SU(2) and SU(3), respectively. The SU(N) transformations of $\vec{\varphi}$ and $\vec{\varphi}^\dagger$ read

$$\vec{\varphi}' = \exp\left(i\theta^a T^a\right) \vec{\varphi}, \tag{1.40}$$

$$\vec{\varphi}^{\dagger\prime} = \vec{\varphi}^\dagger \exp\left(-i\theta^a T^a\right), \tag{1.41}$$

where we have used $T^{a\dagger} = T^a$ and summation over $a = 1, \ldots N^2 - 1$ is understood.

The generators T^a can be represented by Hermitian $N \times N$ matrices, which we recall here for completeness for $N = 2$ and $N = 3$:

$$T^a = \begin{cases} \sigma^a/2 & a = 1,2,3 \quad \mathrm{SU(2)}, \\ \lambda^a/2 & a = 1, \ldots 8 \quad \mathrm{SU(3)} \end{cases}. \tag{1.42}$$

Here σ^a are the Pauli matrices and λ^a satisfying

$$\mathrm{Tr}(\lambda^a \lambda^b) = 2\delta^{ab} \tag{1.43}$$

are Gell–Mann matrices. Their explicit expressions are given in Appendix A.2. These matrices are Hermitian and traceless, which follows from the unitarity of the transformations U and the requirement that in the case of SU(N), detU = 1:

$$\det\left(e^{iT}\right) = e^{i\mathrm{Tr}(T)} = 1 \quad \Rightarrow \quad \mathrm{Tr}(T) = 0. \tag{1.44}$$

As

$$\mathrm{U(N)} = \mathrm{SU(N)} \otimes \mathrm{U}(1), \tag{1.45}$$

a U(N) symmetry has an additional generator that is represented by a unit matrix.

We have used here the words *representation* and *multiplet*. As a given irreducible representation of a given group implies automatically the size of a multiplet, both names are used often in particle physics to denote a multiplet. We will follow this terminology here as well.

The so-called adjoint representation is important for particle physics because the gauge bosons A_μ^a of a SU(N) gauge theory belong to it. In the case of $\mathrm{SU(3)}_C$, with the subscript C standing for color, we have eight gluons and in the case of $\mathrm{SU(2)}_L$ three weak gauge bosons. The generators of the adjoint representation are simply given by the structure constants of the group. One can check that with $(T_r^b)_{ac} := i f^{abc}$ the commutation relation in (1.37) is fulfilled. The complex conjugated representation $\bar{r}$ to a given representation r is generated by

$$T_{\bar{r}}^a := -T_r^{a*}. \tag{1.46}$$

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In the case of the fundamental representation of SU(N), which we denote by **N**, the complex conjugated representation is denoted by $\bar{\mathbf{N}}$. For example quarks transform as triplets **3** under $SU(3)_C$ while their antiparticles transform as $\bar{\mathbf{3}}$. If two representations r and $\bar{r}$ are equivalent, i.e., there exists a unitary matrix U such that $T^a_{\bar{r}} = UT^a_r U^\dagger$, then we call r real. For example, all representations of SU(2), especially the fundamental representation **2**, are real, whereas **3**, the fundamental representation of SU(3), is not. A consequence is that something like an "anti(iso)spin," related to SU(2), does not exist, whereas antiquarks have anticolors. Particles that are their own antiparticles transform under real representations of the symmetry group.

1.7.2 Lagrangian and First Implications

The Lagrangian invariant under the transformations (1.40) and (1.41) is given as follows

$$\mathscr{L} = \left(\partial_\mu \vec{\varphi}^\dagger\right)(\partial^\mu \vec{\varphi}) - \vec{\varphi}^\dagger \mathsf{M}^2 \vec{\varphi}, \tag{1.47}$$

where M^2 is the mass matrix (squared) of the field $\vec{\varphi}$. The fields in (1.47) belong to a single multiplet. If there are several multiplets, for each of them (1.47) applies except that M^2 could be different for different multiplets. There is an immediate consequence of a nonabelian symmetry. The masses of particles belonging to a given multiplet must be degenerate. This means that M^2 must be a unit matrix multiplied by m^2. In order to see this, we perform the transformations (1.40) and (1.41) on the last term in (1.47):

$$\vec{\varphi}^\dagger \mathsf{M}^2 \vec{\varphi} \to \vec{\varphi}^\dagger e^{-iT^a\theta^a} \mathsf{M}^2 e^{iT^a\theta^a} \vec{\varphi} \tag{1.48}$$

and find that this term is invariant under these transformations if and only if M^2 commutes with all generators T^a (Schur's lemma):

$$\left[T^a, \mathsf{M}^2\right] = 0. \tag{1.49}$$

This is only possible if M^2 is proportional to the unit matrix.

1.7.3 Explicit Breakdown of a Global Nonabelian Symmetry

In reality the masses of physical particles belonging to a given multiplet are not exactly equal to each other even if the mass splittings could be very small. This means that whereas in this case the first term in (1.47) is still invariant under global nonabelian transformation, the mass term in this equation is not. This type of symmetry breaking is called explicit because it takes place at the level of the Lagrangian. In contrast, in Section 1.10.4 we will discuss spontaneous symmetry breaking where the Lagrangian is still symmetric but only the ground state breaks the symmetry. It is instructive to consider two examples of explicit symmetry breaking: one of SU(2) symmetry, the other of SU(3). In the SU(2) case let us assume that

$$\mathsf{M}^2 = \begin{pmatrix} m_1^2 & 0 \\ 0 & m_2^2 \end{pmatrix}, \qquad m_1 \neq m_2. \tag{1.50}$$

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Inserting this matrix into (1.48) with $T^a = \sigma^a/2$ we find

$$\left[\sigma^{1,2}, \mathsf{M}^2\right] \neq 0, \qquad \left[\sigma^3, \mathsf{M}^2\right] = 0. \tag{1.51}$$

Evidently only one generator commutes with M^2, and the symmetry has been reduced from SU(2) down to U(1):

$$\mathrm{SU}(2) \to \mathrm{U}(1), \tag{1.52}$$

with $T^3 = \sigma^3/2$ playing the role of the generator of the leftover U(1) symmetry.

One easily finds that the nonvanishing commutator in (1.51) is proportional to $m_1^2 - m_2^2$ and if this difference is small the breaking of symmetry can be regarded as a small perturbation. Now the disparity of masses in the mass matrix led to the breakdown of SU(2) to a U(1) symmetry and so at first sight the situation looks like in an abelian U(1) theory: the fields φ_1 and φ_2 are unrelated to each other. However not everything has been lost and the presence of the initial SU(2) symmetry left footprints which we would like to identify. Indeed the charges of the fields φ_1 and φ_2 with respect to the leftover U(1) group are not independent of each other, which would not be the case if φ_1 and φ_2 did not sit in an SU(2) doublet together. In order to see this, we note that the leftover U(1) symmetry is summarized by

$$\vec{\varphi} \to \exp\left(i\frac{\sigma_3}{2}\theta^{(3)}\right)\vec{\varphi}, \qquad \vec{\varphi} = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}, \tag{1.53}$$

or equivalently

$$\varphi_1 \to \exp\left(i\frac{1}{2}\theta^{(3)}\right)\varphi_1, \qquad \varphi_2 \to \exp\left(-i\frac{1}{2}\theta^{(3)}\right)\varphi_2. \tag{1.54}$$

Comparing with the U(1) transformation in (1.20) we note that the charges r are fixed and related to each other. Denoting this quantum number by T_3 we have

$$T_3(\varphi_1) = \frac{1}{2}, \qquad T_3(\varphi_2) = -\frac{1}{2}. \tag{1.55}$$

This result is not surprising. T_3 is the third component of the isospin.

This consideration can be extended to SU(3) by choosing for the mass matrix

$$\mathsf{M}^2 = \begin{pmatrix} m^2 & 0 & 0 \\ 0 & m^2 & 0 \\ 0 & 0 & m_3^2 \end{pmatrix}. \tag{1.56}$$

Inserting this matrix into (1.48) with $\vec{\varphi}$ being this time three-dimensional vector, we find that λ^1, λ^2, λ^3, and λ^8 commute still with M^2, while this is not the case of λ^4, λ^5, λ^6, and λ^7. Thus the symmetry has been reduced as follows

$$\mathrm{SU}(3) \to \mathrm{SU}(2) \otimes \mathrm{U}(1), \tag{1.57}$$

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with $(\lambda_1, \lambda_2, \lambda_3)$ being the generators of the leftover SU(2) and λ_8 of U(1). Writing then

$$\vec{\varphi} \to \exp\left(i\frac{\lambda_8}{2}\theta^{(8)}\right)\vec{\varphi}, \qquad \vec{\varphi} = \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix}, \tag{1.58}$$

with λ_8 given in (A.21), we find the charges of $\varphi_{1,2,3}$ with respect to the leftover U(1) symmetry:

$$T_8(\varphi_1) = T_8(\varphi_2) = \frac{1}{2\sqrt{3}}, \qquad T_8(\varphi_3) = -\frac{1}{\sqrt{3}}. \tag{1.59}$$

Note that the T_8 charges of φ_1 and φ_2 are equal to each other as these two fields form a doublet under the leftover SU(2) symmetry.

1.7.4 More Complicated Global Symmetries

In general there can be several multiplets. For instance, three doublets

$$\vec{\varphi}_A = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}, \qquad \vec{\varphi}_B = \begin{pmatrix} \varphi_3 \\ \varphi_4 \end{pmatrix}, \qquad \vec{\varphi}_C = \begin{pmatrix} \varphi_5 \\ \varphi_6 \end{pmatrix}. \tag{1.60}$$

The SU(2) symmetric Lagrangian is given then by

$$\mathscr{L} = \sum_{s=A,B,C} \left(\partial_\mu \vec{\varphi}_s^\dagger\right)\left(\partial^\mu \vec{\varphi}_s\right) - \sum_{s=A,B,C} m_s^2 \vec{\varphi}_s^\dagger \vec{\varphi}_s, \tag{1.61}$$

with the mass m_s different for different doublets.

These examples should be sufficient to illustrate the basic features of global nonabelian symmetries involving spinless particles. Some new features appear in the case of fermions, as the left-handed fermions can generally transform differently than the right-handed ones. We will discuss this issue soon.

1.8 Promotion of a Global Nonabelian Symmetry to a Local One

In Section 1.5 we have made this promotion for U(1) finding that this required the introduction of a gauge boson that was massless and neutral with respect to the conserved charge connected with the U(1) symmetry. We now want to see what happens when a nonabelian global symmetry becomes a local symmetry, or in short it is gauged.

In order to have a transparent discussion let us just consider SU(N) symmetry. The transformations in (1.40) and (1.41) become local transformations

$$\vec{\varphi}'(x) = \exp\left(i\theta^a(x)T^a\right)\vec{\varphi}(x), \tag{1.62}$$

$$\vec{\varphi}^{\dagger\prime}(x) = \vec{\varphi}^\dagger(x)\exp\left(-i\theta^a(x)T^a\right), \tag{1.63}$$

and we find that the Lagrangian (1.47) is not invariant under these transformations due to the appearance of the terms $\partial_\mu\theta^a(x)$. These terms have to be cancelled, and this requires the introduction of new particles, gauge bosons, one for every $\partial_\mu \theta^a(x)$ or equivalently one for every generator of SU(N). The four steps of Section 1.5 can also be made here. Only formulas are more complicated and, as we will see the resulting dynamics, differs profoundly from the one of an abelian gauge theory. The four steps in the nonabelian case are then as follows:

- **Step 1**
  We introduce a vector particle A_μ^a for every generator T^a. Thus in the case of the SU(N) group the local gauge invariance requires the existence of $N^2 - 1$ gauge bosons.
- **Step 2**
  We replace the derivatives $\partial_\mu \vec{\varphi}$ and $\partial_\mu \vec{\varphi}^\dagger$ by covariant derivatives
$$D_\mu \vec{\varphi} = \left(\partial_\mu - igA_\mu^a T^a\right) \vec{\varphi}, \qquad \left(D_\mu \vec{\varphi}\right)^\dagger = \vec{\varphi}^\dagger \left(\partial_\mu + igA_\mu^a T^a\right), \tag{1.64}$$
  with g denoting the gauge coupling corresponding to the SU(N) group.
- **Step 3**
  The transformation for the gauge fields is given by
$$A_\mu^a \to A_\mu^{a\prime} = A_\mu^a + \frac{1}{g}\partial_\mu \theta^a(x) - f^{abc}\theta^b(x)A_\mu^c \tag{1.65}$$
  with f^{abc} being the structure constants introduced in (1.37).
- **Step 4**
  The gauge invariant strength tensor corresponding to A_μ^a is given as follows
$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^{abc} A_\mu^b A_\nu^c. \tag{1.66}$$
  The resulting Lagrangian that is invariant under the transformations (1.62), (1.63), and (1.65) is finally given as follows:

$$\mathscr{L}_{\text{gauged}} = \left(D_\mu \vec{\varphi}\right)^\dagger \left(D^\mu \vec{\varphi}\right) - m^2 \vec{\varphi}^\dagger \vec{\varphi} - \frac{1}{4} F_{\mu\nu}^a F^{\mu\nu,a}. \tag{1.67}$$

At this point we want to mention that there exist at least two different conventions in the literature, which results in some sign flips. More details on conventions can be found in Appendix B. Here we only discuss two of them. If the transformation in (1.62) has opposite sign in the exponential, then also the sign in front of g in the covariant derivative has to be changed. We collect here the differences between these two conventions

- Convention 1 (used by us)

$$\varphi(x) \to \varphi(x)' = \exp\left(i\theta^a(x)T^a\right)\varphi(x) \tag{1.68}$$
$$D_\mu = \partial_\mu - igA_\mu^a T^a \tag{1.69}$$
$$A_\mu^a(x) \to A_\mu^{a\prime}(x) = A_\mu^a(x) + \frac{1}{g}\partial_\mu \theta^a(x) - f^{abc}\theta^b(x)A_\mu^c \tag{1.70}$$
$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^{abc} A_\mu^b A_\nu^c \tag{1.71}$$

- Convention 2

$$\varphi(x) \to \varphi(x)' = \exp\left(-i\theta^a(x)T^a\right)\varphi(x) \tag{1.72}$$

$$D_\mu = \partial_\mu + igA_\mu^a T^a \tag{1.73}$$

$$A_\mu^a(x) \to A_\mu^{a\prime}(x) = A_\mu^a(x) + \frac{1}{g}\partial_\mu\theta^a(x) + f^{abc}\theta^b(x)A_\mu^c \tag{1.74}$$

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a - gf^{abc}A_\mu^b A_\nu^c \tag{1.75}$$

Conventions 1 and 2 can be transformed into each other by flipping simultaneously the signs of θ^a and g.

Let us then list the properties of the Lagrangian in (1.67) paying in particular attention to those properties that distinguish it from the one in (1.27):

- The appearance of several gauge bosons, one for each generator of the symmetry group.
- All these gauge bosons are massless as the mass term

$$\Delta\mathscr{L}_{\text{mass}} = \frac{M^2}{2}\sum_a A_\mu^a A^{\mu,a} \tag{1.76}$$

is clearly not invariant under (1.65), even if all these gauge bosons were degenerate in mass. Thus the exact nonabelian gauge symmetry SU(3) of strong interactions implies that all eight gluons are massless. On the other hand, this is a problem for heavy $W^{\mu\pm}$ and Z^μ bosons. Indeed, the gauge bosons were introduced to be able to change the phase of the wave function of all particles independently at each space-time point without any observable consequence. Thus the gauge boson has to reconcile such phase changes over arbitrary large distances. A force with an infinite range is associated with a massless gauge boson. However, the weak force is short-ranged due to the masses of $W^{\mu\pm}$ and Z^μ bosons. We will address this problem in Sections 1.10 and 1.11.
- In order to see the structure of the interactions present in (1.67), we separate the free Lagrangian

$$\mathscr{L}_{\text{free}} = \partial_\mu\vec{\varphi}^\dagger\partial^\mu\vec{\varphi} - m^2\vec{\varphi}^\dagger\vec{\varphi} - \frac{1}{4}\left(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a\right)\left(\partial^\mu A^{\nu,a} - \partial^\nu A^{\mu,a}\right). \tag{1.77}$$

The interactions of $\vec{\varphi}$ with gauge bosons are then described by

$$\mathscr{L}_{\text{int}}^{(1)} = -ig\partial_\mu\vec{\varphi}^\dagger T^a\vec{\varphi}A^{\mu,a} + ig\vec{\varphi}^\dagger T^a\partial_\mu\vec{\varphi}A^{\mu,a} + g^2\vec{\varphi}^\dagger T^aT^b\vec{\varphi}A_\mu^a A^{\mu,b}, \tag{1.78}$$

where summation over repeated indices is understood. It is convenient to rewrite this Lagrangian in the component form $(i,j = 1,\dots N)$

$$\mathscr{L}_{\text{int}}^{(1)} = -ig\partial_\mu\varphi_i^\dagger T_{ij}^a\varphi_j A^{\mu,a} + ig\varphi_i^\dagger T_{ij}^a\partial_\mu\varphi_j A^{\mu,a} + g^2\varphi_i^\dagger\left(T^aT^b\right)_{ij}\varphi_j A_\mu^a A^{\mu,b}. \tag{1.79}$$

Using the matrix representations for the generators T^a, like the ones in terms of σ^a and λ^a in (1.42), it is an easy matter to derive the Feynman rules for the relevant vertices. One just uses $k_\mu = i\partial_\mu$, multiplies $\mathscr{L}_{\text{int}}^{(1)}$ by i, and simply reads of the coefficients in front of the products of fields. While at first sight, in view of $i,j = 1,\dots N$, these interactions look rather complicated, they have a very simple structure due to the SU(N) symmetry. We will exhibit this in a moment. Moreover the indices i,j, the colors in the case of

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$SU(3)_C$ for the strong interactions, are seldom seen in practical calculations as one can use "color algebra," which we will develop in Section 2.7.

The remarkable facts about these interactions are

– there is only a single coupling g describing them,
– all "nonabelian" charges entering these interactions are fixed by the symmetry. They are simply given in terms of the elements of the matrices T^a.

• Finally, we discuss the last term in (1.67), which involves the gauge bosons only. We find two types of interactions

$$\mathscr{L}_{\text{int}}^{(2)} = -\frac{1}{4} g f^{abc} \left(\partial_\mu A_\nu^a\right) A^{\mu,b} A^{\nu,c} + \cdots \tag{1.80}$$

$$\mathscr{L}_{\text{int}}^{(3)} = -\frac{1}{4} g^2 f^{abd} f^{ade} A_\mu^b A_\nu^c A^{\mu,d} A^{\nu,e}. \tag{1.81}$$

Evidently $\mathscr{L}_{\text{int}}^{(2)}$ and $\mathscr{L}_{\text{int}}^{(3)}$ describe triple and quartic gauge boson vertices, which again are fully fixed by the symmetry up to the coupling g.

The presence of such couplings is a very important difference from the case of abelian gauge theories, which are free theories in the absence of matter fields φ_i. The nonabelian theories are interacting even when the matter fields are absent. This is because the gauge bosons A_μ^a carry "color charges." The presence of interactions between gauge bosons in nonabelian theories has very profound dynamical implications to which we will return at several places in this book.

1.9 Lagrangians for Fermions

1.9.1 Preliminaries

Until now we have considered only scalars as matter fields. Even if the Higgs particle has been discovered recently, most of the known elementary particles are fermions: quarks and leptons. In order to incorporate them into the framework presented until now, we have to construct Lagrangians involving spin-$\frac{1}{2}$ particles that possess global and local, abelian and nonabelian symmetries. To this end it will be useful to recall some of the important properties of fermions that distinguish them from scalars. In this context we follow the conventions of Bjorken and Drell [13]. See also [14]. They differ from Weinberg's well-known book [15].

A fermion is described by a four-component spinor and its adjoint:

$$\psi, \qquad \bar{\psi} \equiv \psi^\dagger \gamma^0, \tag{1.82}$$

where γ^0 is one of the Dirac matrices. There are different representations of these matrices, which can be found in many textbooks. In particular we have

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$$\gamma^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}, \qquad [\gamma^i] = \vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}. \tag{1.83}$$

Here $\vec{\sigma} = \left[\sigma^1, \sigma^2, \sigma^3\right]$ are 2×2 Pauli matrices given in Appendix A.2, and $\mathbb{1}$ stands for the 2×2 unit matrix. We will use

$$\gamma^\mu \quad (\mu = 0, 1, 2, 3), \qquad \gamma_5 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} = i\gamma^0\gamma^1\gamma^2\gamma^3 \tag{1.84}$$

with

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}, \qquad \{\gamma_5, \gamma^\mu\} = 0. \tag{1.85}$$

Note that

$$\gamma_5^\dagger = \gamma_5, \qquad \gamma_5^2 = \mathbb{1}. \tag{1.86}$$

Some other properties of Dirac matrices are collected in Appendix A, and the full set can be found in any textbook for quantum field theory, in particular in [13, 14].

We next introduce left-handed (LH) and right-handed (RH) fermion fields

$$\psi_L = \frac{1}{2}\left(1 - \gamma_5\right)\psi \equiv P_L\psi, \qquad \psi_R = \frac{1}{2}\left(1 + \gamma_5\right)\psi \equiv P_R\psi. \tag{1.87}$$

Their adjoints are given by

$$\bar{\psi}_L = \bar{\psi}\frac{1}{2}\left(1 + \gamma_5\right) = \bar{\psi}P_R, \qquad \bar{\psi}_R = \bar{\psi}\frac{1}{2}\left(1 - \gamma_5\right) = \bar{\psi}P_L. \tag{1.88}$$

Consequently, using

$$\psi = \psi_L + \psi_R, \qquad \bar{\psi} = \bar{\psi}_L + \bar{\psi}_R, \tag{1.89}$$

we find very important properties for Dirac structures that will appear at many places in this book:

$$\bar{\psi}\gamma_\mu\psi = \bar{\psi}_L\gamma_\mu\psi_L + \bar{\psi}_R\gamma_\mu\psi_R \qquad \text{(vector)}, \tag{1.90}$$
$$\bar{\psi}\psi = \bar{\psi}_L\psi_R + \bar{\psi}_R\psi_L \qquad \text{(scalar)}, \tag{1.91}$$
$$\bar{\psi}\gamma_\mu\gamma_5\psi = -\bar{\psi}_L\gamma_\mu\psi_L + \bar{\psi}_R\gamma_\mu\psi_R \qquad \text{(axial vector)}, \tag{1.92}$$
$$\bar{\psi}\gamma_5\psi = \bar{\psi}_L\psi_R - \bar{\psi}_R\psi_L \qquad \text{(pseudo scalar)}. \tag{1.93}$$

We note that vector-type terms with γ_μ and $\gamma_\mu\gamma_5$ connect only fields of the same helicity, whereas the scalar-type terms connect only fields of opposite helicity. The latter case is often called *helicity flip*. With this information at hand we can present Lagrangians involving fermions that possess global and local, abelian and nonabelian symmetries.

1.9.2 Abelian Global Symmetry

Our starting point is the free Lagrangian for a fermion with mass m. It can be found by demanding that through Euler–Lagrangian equations the Dirac equation follows:

$$\left(i\gamma_\mu\partial^\mu - m\right)\psi = 0. \tag{1.94}$$

This turns out to be

$$\mathscr{L}_{\text{free}} = \bar{\psi}\left(i\gamma_\mu\partial^\mu\right)\psi - m\bar{\psi}\psi. \tag{1.95}$$

We note the appearance of a vector and a scalar structure discussed earlier. However, for the time being we will not decompose ψ into ψ_L and ψ_R and present first Lagrangians and their symmetries for the full ψ. That is, we assume first that ψ_L and ψ_R transform identically under symmetries considered by us. We will see that this works for QCD and QED but fails for weak interactions that break within the SM parity P maximally. There are also important global (flavor) symmetries under which ψ_L and ψ_R transform differently. These will also be discussed in our book.

Evidently the Lagrangian in (1.95) is invariant under a global U(1) transformation

$$\psi \to e^{i\theta r}\psi, \qquad \bar{\psi} \to \bar{\psi}e^{-i\theta r}. \tag{1.96}$$

1.9.3 Nonabelian Global Symmetry

If several fermionic fields are present, so that

$$\psi = \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_N \end{pmatrix}, \qquad \bar{\psi} = \left(\bar{\psi}_1, \ldots, \bar{\psi}_N\right), \tag{1.97}$$

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the Lagrangian in (1.95) is invariant under SU(N) or U(N) global symmetry:

$$\psi \to \exp\left(i\theta^a T^a\right)\psi, \qquad \bar{\psi} \to \bar{\psi}\exp\left(-i\theta^a T^a\right), \tag{1.98}$$

where again summation over repeated indices is understood. One of the implications of nonabelian symmetries is the equality of masses of $\psi_1, \ldots \psi_N$ belonging to a multiplet.

1.9.4 Abelian Local Symmetry

The Lagrangian having a local U(1) symmetry can be constructed following the steps of Section 1.5 with the result

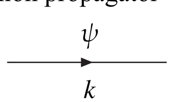
$$\mathscr{L}_{\text{gauged}} = \bar{\psi}\left(i\gamma^\mu D_\mu\right)\psi - m\bar{\psi}\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \tag{1.99}$$

where

$$D_\mu = \partial_\mu - irgA_\mu \tag{1.100}$$

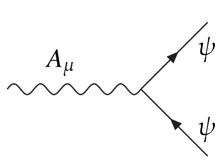
is the covariant derivative, g the gauge coupling, and r the charge of ψ under the U(1) symmetry. $F_{\mu\nu}$ is given in (1.26). The local U(1) transformation is given by (1.98) with θ^a replaced by $\theta^a(x)$. The transformation on A_μ is as in (1.25), and also the covariant derivative in (1.100) is identical to the scalar case in (1.24). However as we deal now with the interaction of a vector particle A_μ with a fermion and not a scalar, the structure of the first two terms in (1.99), the so-called kinetic terms, differs from the one in (1.27). In particular the fermionic Lagrangian is linear in the covariant derivative, while the scalar Lagrangian involves a product of D_μ and D^μ necessary to obtain a Lorentz invariant Lagrangian. This, of course, has direct implications on the structure of interactions.

Similar to the scalar case, we can derive the Feynman rules for this theory. For the fermion propagator we just have



$$\frac{i}{\not{k} - m + i\varepsilon} = i\frac{\not{k} + m}{k^2 - m^2 + i\varepsilon},$$

where $\not{k} = \gamma_\mu k^\mu$. The interaction vertex is then



$$= igr\gamma_\mu$$

where as commonly done we do not show explicitly external fields, $A_\mu, \bar{\psi}, \psi$ on the right-hand side of this rule.

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1.9.5 Nonabelian Local Symmetry

Proceeding as in Section 1.8 we find the Lagrangian

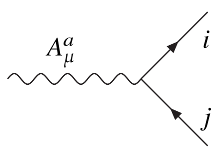
$$\mathscr{L} = \bar{\psi}\left(i\gamma_\mu D^\mu \psi\right) - m\bar{\psi}\psi - \frac{1}{4}F^a_{\mu\nu}F^{\mu\nu,a}, \tag{1.101}$$

where ψ and $\bar{\psi}$ are given in (1.97) and

$$D_\mu \psi = \left(\partial_\mu - igA^a_\mu T^a\right)\psi, \tag{1.102}$$

with g being the gauge coupling and A^a_μ the gauge bosons corresponding to the generators T^a. This Lagrangian is invariant under the transformation (1.98) for fermions with θ^a replaced by $\theta^a(x)$ and (1.65) for gauge bosons. The pure gauge sector, in particular interactions among different gauge bosons, did not change relative to the scalar case.

The interaction of a given gauge boson with fermions belonging to a given multiplet takes the form



$$= ig\gamma_\mu \left(T^a\right)_{ij}\,.$$

We again note that all these interactions are given entirely in terms of a single-gauge coupling g and various "charges" $(T^a)_{ij}$, which are fully fixed by the symmetry.

1.9.6 Important Properties

Let us end the first discussion of fermionic Lagrangians by stressing several properties of both the interaction and the mass term. To this end it is sufficient to rewrite the interaction and mass terms in (1.99) in terms of ψ_L and ψ_R:

$$\mathscr{L}_{\text{int}} = +gr\left(\bar{\psi}_L\gamma_\mu A^\mu \psi_L + \bar{\psi}_R\gamma_\mu A^\mu \psi_R\right) \tag{1.103}$$

$$\mathscr{L}_{\text{mass}} = -m\left(\bar{\psi}_R\psi_L + \bar{\psi}_L\psi_R\right). \tag{1.104}$$

We observe

- The gauge interactions connect only fields of the same helicity. There is no helicity flip.
- A mass term connects fields of opposite helicity. Thus, in order to dynamically generate a mass term, a helicity flip is needed. An exception is the so-called Majorana mass term in the case of neutrinos or generally neutral fermions.
- The two properties imply that if the interactions in a given theory are just gauge interactions and there is no mass term, there is no way to generate masses of fermions radiatively through interactions.

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- The strength of gauge interactions of ψ_L and ψ_R is the same if ψ_L and ψ_R transform identically under the gauge group. This is the case of vectorial gauge theories (only γ_μ matters). Prominent examples are QED and QCD.
- However, quite often ψ_L and ψ_R transform differently under global and gauge symmetry groups. Such symmetries are called chiral. The $SU(3)_L \times SU(3)_R$ chiral global symmetry of QCD and the $SU(2)_L$ gauge symmetry of the SM are well-known examples.
- If ψ_L and ψ_R transform differently under a given symmetry group, local or global, fermion mass terms are forbidden by the symmetry. This is also the case for gauge bosons in general, as we have already seen on previous pages.

As we will see in the next chapter, both the masses of weak gauge bosons $W^\pm$ and Z^0 as well as of all quarks and leptons are forbidden by $SU(2)_L$. This is clearly a disaster, and one could ask whether one could simply introduce some explicit terms in the Lagrangian that would break $SU(2)_L$ so that masses of weak gauge bosons and of quarks and leptons would become nonzero. Fortunately it turns out that this is not possible without destroying other properties of the SM related in particular to its renormalizability, a topic of Section 4.2. It is fortunate because otherwise the theory would not be as predictive as it is. Yet, the problem of the generation of gauge boson and fermion masses in the presence of exact symmetry in a given Lagrangian must be solved somehow. This is what we will do next.

1.10 Spontaneous Symmetry Breakdown (SSB)

1.10.1 Preliminaries

If the Lagrangian possesses a certain symmetry $\mathcal{S}$ but the ground state (vacuum) is not invariant under such symmetry transformation, then we call $\mathcal{S}$ a spontaneously broken symmetry. Both discrete and continuous symmetries can be spontaneously broken. In Section 1.10.2 we will first discuss SSB of a discrete symmetry and in Section 1.10.3 of a continuous symmetry. But first we want to outline here the connection between vacuum expectation values (vev) and SSB.

Let us consider an infinitesimal transformation of a continuous symmetry as in (1.40) by expanding the exponential and keeping only the term linear in θ^a

$$\varphi_i \to \varphi_i' = \varphi_i + i\theta^a T^a_{ij}\varphi_j. \tag{1.105}$$

For an unbroken symmetry we have

$$\langle 0|\varphi_i|0\rangle \stackrel{!}{=} \langle 0|\varphi_i'|0\rangle = \langle 0|\varphi_i|0\rangle + i\theta^a T^a_{ij}\langle 0|\varphi_j|0\rangle. \tag{1.106}$$

For an irreducible representation T^a this implies

$$\langle 0|\varphi_j|0\rangle = 0. \tag{1.107}$$

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According to the Noether theorem there is a conserved charge Q^a that corresponds to the generator T^a of the symmetry. As shown in many books on quantum field theory, these charges also generate the symmetry transformation in the following manner

$$\varphi_i' = e^{i\theta^a Q^a} \varphi_i e^{-i\theta^a Q^a}. \tag{1.108}$$

Considering only infinitesimal transformations and comparing with (1.106) this means

$$i\theta^a T^a_{ij} \varphi_j = i\theta^a \left[Q^a, \varphi_i \right]. \tag{1.109}$$

Now for an unbroken symmetry the vacuum is neutral, i.e., $Q^a|0\rangle = 0$. Together with (1.109) we conclude that for an uncharged vacuum the vacuum expectation value has to vanish:

$$Q^a|0\rangle = 0 \qquad \Rightarrow \qquad \langle 0|\varphi_j|0\rangle = 0. \tag{1.110}$$

If, on the other hand, at least one component φ_j has a nonvanishing vacuum expectation value, we have

$$v_j := \langle 0|\varphi_j|0\rangle \neq 0, \qquad Q^a|0\rangle \neq 0. \tag{1.111}$$

Consequently the vacuum is charged under the symmetry, and all symmetries under which φ_j transforms nontrivially are spontaneously broken while the symmetries under which φ_j transforms as a singlet are unbroken. Because we do not want to break Lorentz invariance φ_j can only be a scalar field and, for example, not a vector field.

1.10.2 Spontaneous Breakdown of a Discrete Symmetry

In order to introduce the concept of spontaneous symmetry breakdown of a symmetry we consider the Lagrangian for a real scalar field

$$\mathscr{L} = \frac{1}{2}\left(\partial_\mu \varphi\right)\left(\partial^\mu \varphi\right) - V(\varphi), \tag{1.112}$$

where the potential is given by

$$V(\varphi) = \frac{1}{2}\mu^2\varphi^2 + \frac{1}{4}\lambda\varphi^4, \qquad \lambda > 0. \tag{1.113}$$

The parameter λ describes the strength of the scalar interactions with itself. The condition $\lambda > 0$ ensures that the potential is bounded from below. The parameter μ will play a crucial role in a moment. Evidently the Lagrangian (1.112) is invariant under the discrete symmetry

$$\varphi \to -\varphi. \tag{1.114}$$

We next look at the term quadratic in φ and consider two cases.

- $\mu^2 > 0$
  Comparing (1.113) with (1.1) we conclude that $\mu = m$ is just the mass of φ. In Fig. 1.2 we show $V(\varphi)$ for this case. We observe that $V(\varphi)$ has a unique minimum at $\varphi = 0$. This is the ground state of the theory, which as usually done will be called vacuum. Thus

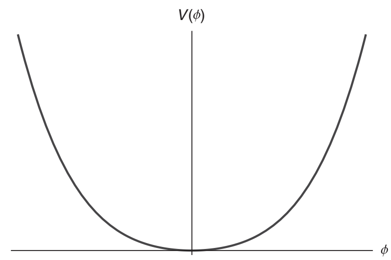

Figure 1.2 Potential for a real scalar field with $\mu^2 > 0$.

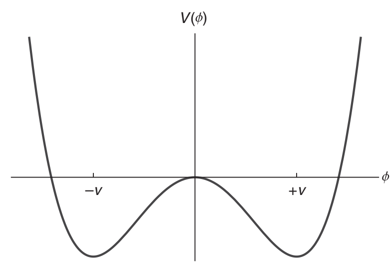

Figure 1.3 Potential for a real scalar field with $\mu^2 < 0$.

$$\varphi_{\text{vac}} = 0, \qquad \mu^2 > 0. \tag{1.115}$$

Moreover the vacuum is symmetric with respect to $\varphi \to -\varphi$. Nothing exciting so far.

- $\mu^2 < 0$

  This case is more interesting. Indeed, from

$$\frac{\partial V}{\partial \varphi} = \varphi\left(\mu^2 + \lambda\varphi^2\right) = 0, \tag{1.116}$$

we learn as seen in Fig. 1.3 that there are two minima so that

$$\varphi_{\text{vac}} = \pm v, \qquad v = \sqrt{-\frac{\mu^2}{\lambda}}. \tag{1.117}$$

The potential looks still symmetric under $\varphi \to -\varphi$ but in order to calculate predictions of the theory we have to choose the ground state. This breaks the symmetry. Indeed sitting in one of the two vacuua the world does not look symmetric anymore. The best proof of this is that flipping the sign of φ we move to a different world with φ_{vac} having opposite sign.

Let us investigate the consequences of SSB. To this end we expand $\varphi(x)$ around the vacuum state $\varphi = v$

$$\varphi(x) = v + \eta(x), \qquad \eta_{\text{vac}} = 0, \tag{1.118}$$

with η describing fluctuations around this vacuum.

The Lagrangian (1.112) expressed in terms of η is given as follows

$$\mathscr{L} = \frac{1}{2}(\partial_\mu \eta)(\partial^\mu \eta) - \lambda v^2 \eta^2 - \lambda v \eta^3 - \frac{1}{4}\lambda \eta^4 + \text{const.}, \tag{1.119}$$

where the constant terms do not involve η and are physically irrelevant. From the second term we find the mass of η:

$$m_\eta = \sqrt{2\lambda} v = \sqrt{-2\mu^2}. \tag{1.120}$$

The same result can also be obtained from

$$\frac{\partial^2 V}{\partial \varphi^2} = \mu^2 + 3\lambda\varphi^2\Big|_{\varphi=v} = -2\mu^2 = m_\eta^2. \tag{1.121}$$

We collect a few lessons from this simple exercise:

- Useful formula for scalar masses

$$m_\eta^2 = \frac{\partial^2 V}{\partial \varphi^2}\bigg|_{\varphi=v}. \tag{1.122}$$

- There are two degenerate vacua connected by the original symmetry $\varphi \to -\varphi$.
- η has a nonvanishing mass given in (1.120).
- The Lagrangian in (1.119) is clearly not invariant under $\eta \to -\eta$ because of the η^3 term.

While these results are still not terribly exciting, we will see that they will turn out to be useful in the context of the spontaneous breakdown of continuous symmetries.

1.10.3 Spontaneous Breakdown of a Continuous Abelian Global Symmetry

We next consider the generalization of (1.112) to a complex scalar field φ:

$$\mathscr{L} = \frac{1}{2}\left(\partial_\mu \varphi^*\right)(\partial^\mu \varphi) - V(\varphi^*, \varphi), \tag{1.123}$$

where the potential is given by

$$V(\varphi^*, \varphi) = \mu^2 \varphi^* \varphi + \frac{1}{4}\lambda(\varphi^*\varphi)^2, \qquad \lambda > 0. \tag{1.124}$$

This Lagrangian is invariant under the U(1) symmetry

$$\varphi \to e^{i\theta}\varphi, \qquad \varphi^* \to \varphi^* e^{-i\theta}. \tag{1.125}$$

This symmetry is spontaneously broken for $\mu^2 < 0$. Indeed, we find now

$$|\varphi_{\text{vac}}|^2 = -\frac{2\mu^2}{\lambda} \equiv \frac{v^2}{2}, \qquad v = \sqrt{-4\frac{\mu^2}{\lambda}}. \tag{1.126}$$

The resulting potential is shown in Fig. 1.4. It looks like a Mexican hat.

The profound difference from the case of the discrete symmetry is the full circle of degenerate minima connected by the original symmetry as the minimum condition (1.126) does not fix the phase of φ_{vac}. We choose now one of this vacua as our ground state, and

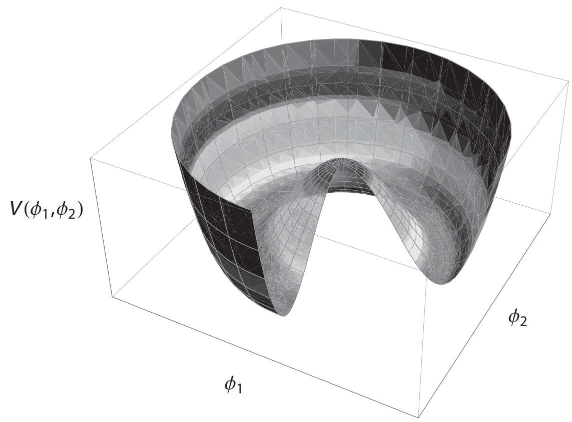

Figure 1.4 Potential for a complex scalar field with $\mu^2 < 0$.

this breaks the U(1) symmetry spontaneously. Let us find the mass spectrum after SSB. To this end we write

$$\varphi(x) = \frac{1}{\sqrt{2}}\left(\varphi_1(x) + i\varphi_2(x)\right), \tag{1.127}$$

with φ_1 and φ_2 being real. The vacuum condition now reads

$$\varphi_1^2 + \varphi_2^2\Big|_{\text{vac}} = v^2. \tag{1.128}$$

We next choose the vacuum to be

$$(\varphi_1, \varphi_2)_{\text{vac}} = (v, 0), \tag{1.129}$$

and having two degrees of freedom we introduce two fields η and ξ, which describe fluctuations around the vacuum (1.129)

$$\varphi(x) = \frac{1}{\sqrt{2}}\left(v + \eta(x) + i\xi(x)\right). \tag{1.130}$$

Inserting this expression into (1.123) we find after some algebra

$$\mathscr{L} = \left[\frac{1}{2}(\partial_\mu\eta)(\partial^\mu\eta) - \frac{1}{2}m_\eta^2\eta^2\right] + \left[\frac{1}{2}(\partial_\mu\xi)(\partial^\mu\xi)\right] + \text{interactions}, \tag{1.131}$$

with

$$m_\eta^2 = \frac{\partial^2 V}{\partial\varphi_1^2}\bigg|_{(v,0)} = -2\mu^2, \tag{1.132}$$

$$m_\xi^2 = \frac{\partial^2 V}{\partial\varphi_2^2}\bigg|_{(v,0)} = 0. \tag{1.133}$$

The striking difference from the breakdown of a discrete symmetry is the appearance of a massless particle in addition to a massive one. The appearance of a massless particle can easily be understood by noting that the potential V is flat in the φ_2 direction. It does

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not cost any energy to move along this direction and this is only possible for a massless particle. This massless particle is called Goldstone boson. In the φ_1 direction the potential is not flat, and it costs some energy to move along it: the particle η has a mass.

This important result of the appearance of a massless particle as the consequence of a spontaneous breakdown of a continuous global symmetry is a special case of the Goldstone theorem, which states that for each broken symmetry there is one Goldstone boson. As the U(1) symmetry has only one generator, we have only one massless boson in our example.

1.10.4 Generalization to the Nonabelian Case

In order to better understand the implications of the Goldstone theorem, we will now generalize our considerations to a nonabelian global symmetry. In principle we could consider the breakdown of an SU(N) group, but it is easier to consider in this case first the breakdown of an orthogonal group, simply because in this case the fields are real. This was also the strategy in the book of Bailin and Love [16] from which we benefited a lot when presenting the following material and where further details can be found.[3] SU(2) and SU(3) groups will be worked out in detail in the context of the SM and its extensions later in our book.

We consider then n real fields, which form an n-dimensional representation described by a column vector

$$\vec{\varphi}(x) = \begin{pmatrix} \varphi_1(x) \\ \vdots \\ \varphi_n(x) \end{pmatrix}. \tag{1.134}$$

The relevant Lagrangian

$$\mathscr{L} = \frac{1}{2}\left(\partial_\mu \vec{\varphi}^\top\right)(\partial^\mu \vec{\varphi}) - V(\vec{\varphi}^\top \vec{\varphi}) \tag{1.135}$$

is invariant under infinitesimal global transformation

$$\vec{\varphi} \to \vec{\varphi} + \delta\vec{\varphi}, \qquad \delta\vec{\varphi} = i\theta^a T^a \vec{\varphi}, \tag{1.136}$$

where $a = 1, \dots N$. In component form we have

$$\delta\varphi_i = i\,(\theta^a T^a)_{ij}\,\varphi_j. \tag{1.137}$$

T^a are $n \times n$ Hermitian matrices, but iT^a must be real to keep the real character of the fields φ_i, and consequently T^a must be antisymmetric, precisely what the generators of an orthogonal group are. We have seen in the previous example that the invariance of $\mathscr{L}$ under a given symmetry still played an important role after SSB. Let us then investigate the implications of the invariance in this more complicated case.

From the invariance of V we have

$$\delta V = \frac{\partial V}{\partial \varphi_i}\delta\varphi_i = i\frac{\partial V}{\partial \varphi_i}\,(\theta^a T^a)_{ij}\,\varphi_j = 0. \tag{1.138}$$

[3] See chapter 13 in [16].

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But the θ^a are arbitrary, and consequently for every generator T^a we have

$$\frac{\partial V}{\partial \varphi_i}(T^a)_{ij}\varphi_j = 0 \qquad a = 1, \ldots N. \tag{1.139}$$

We next consider SSB, that is, in the vacuum

$$\langle 0|\vec{\varphi}|0\rangle = \vec{v} \qquad \text{or} \qquad \langle 0|\varphi_i|0\rangle = v_i, \tag{1.140}$$

and

$$\left.\frac{\partial V}{\partial \vec{\varphi}}\right|_{\vec{\varphi}=\vec{v}} = \vec{0} \qquad \text{or} \qquad \left.\frac{\partial V}{\partial \varphi_i}\right|_{\varphi_i = v_i} = 0. \tag{1.141}$$

Differentiating (1.139) with φ_k we find

$$\frac{\partial^2 V}{\partial \varphi_k \partial \varphi_i}(T^a)_{ij}\varphi_j + \frac{\partial V}{\partial \varphi_i}(T^a)_{ik} = 0, \tag{1.142}$$

and evaluating it at the minimum (1.141), we find

$$\left(\frac{\partial^2 V}{\partial \varphi_k \partial \varphi_i}\right)_{\vec{\varphi}=\vec{v}}(T^a)_{ij}v_j = 0, \tag{1.143}$$

which summarizes the implications of a global invariance around the ground state.

We next expand around the vacuum

$$\vec{\varphi} = \vec{v} + \vec{\tilde{\varphi}} \qquad \text{or} \qquad \varphi_i = v_i + \tilde{\varphi}_i, \tag{1.144}$$

with

$$\langle 0|\tilde{\varphi}_i|0\rangle = 0. \tag{1.145}$$

Expanding in small fluctuations $\tilde{\varphi}_i$ and using (1.141), we find

$$\mathscr{L} = \frac{1}{2}\left[(\partial_\mu \tilde{\varphi}_i)(\partial^\mu \tilde{\varphi}_i) - \tilde{\varphi}_i \tilde{\varphi}_j \left(\frac{\partial^2 V}{\partial \varphi_i \partial \varphi_j}\right)_{\vec{\varphi}=\vec{v}}\right] - V(\vec{v}) + O(\tilde{\varphi}^3). \tag{1.146}$$

Consequently, the mass spectrum after SSB is described on the basis of $(\tilde{\varphi}_1, \ldots, \tilde{\varphi}_n)$ by $n \times n$ mass matrix squared

$$\left(\mathsf{M}^2\right)_{ij} = \left(\frac{\partial^2 V}{\partial \varphi_i \partial \varphi_j}\right)_{\vec{\varphi}=\vec{v}}. \tag{1.147}$$

But according to (1.143), which followed from global invariance, this matrix satisfies the equations

$$\left(\mathsf{M}^2\right)_{ki}(T^a)_{ij}\, v_j = 0, \qquad a = 1, \ldots N \tag{1.148}$$

or more compactly

$$\mathsf{M}^2 T^a \vec{v} = 0, \qquad a = 1, \dots N. \tag{1.149}$$

We now denote the symmetry group by G and assume that it is broken spontaneously to its subgroup $H \subset G$. The generators T^a can now be divided into X^a and Y^a

$$Y^a \subset H\,; \qquad Y^a = T^a \qquad a = 1, \dots M \tag{1.150}$$

$$X^a \subset G/H\,; \qquad X^a = T^a \qquad a = M+1, \dots N, \tag{1.151}$$

with Y^a building the subgroup H and the broken generators X^a belonging to the so-called coset space. They are just the remaining generators, but by itself they do not build a subgroup of G.

Now in the case of the subgroup H the vacuum is invariant, and this means

$$Y^a \vec{v} = 0 \qquad \text{or} \qquad (Y^a)_{ij}\, v_j = 0. \tag{1.152}$$

This allows to satisfy (1.149) trivially and gives no constraint on M^2. On the other hand, for broken generators

$$X^a \vec{v} \neq 0 \qquad \text{or} \qquad (X^a)_{ij}\, v_j \neq 0, \tag{1.153}$$

and the conditions in (1.149) imply zero eigenvalues in the mass matrix M^2. In other words,

$$\vec{U}^a \equiv X^a \vec{v} \qquad \text{or} \qquad (U^a)_i = (X^a)_{ij} v_j \tag{1.154}$$

are the eigenvectors corresponding to zero masses.

In summary, for every broken generator X^a there is a massless Goldstone boson that is a linear combination of the fields $\tilde{\varphi}_i$ and given by

$$(\tilde{\varphi}_1, \dots \tilde{\varphi}_n)\, X^a \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = \left(\vec{\tilde{\varphi}}\right)^\top X^a \vec{v}. \tag{1.155}$$

1.10.5 Summary

We conclude that a spontaneous breakdown of a global symmetry generated another problem: new massless particles for each broken generator. Such massless particles, if they would exist in nature, would have been discovered already a long time ago. On the other hand, the lightest mesons like pions and kaons could be regarded as nearly Goldstone bosons of a broken global $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R$ symmetry, which would be exact at the level of Lagrangian if pions and kaons were massless. But as pions and kaons have masses, $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R$ has to be broken explicitly so that these mesons at the end obtain small masses. Consequently they are not true Goldstone bosons but the so-called pseudo-Goldstone bosons. A very nice article on such bosons in general terms is the one by Steven Weinberg [17].

An explicit breakdown of a global symmetry has no theoretical problems and combined with spontaneous symmetry turns out to be useful for the description of the physics of lightest mesons as we just mentioned.

On the other hand, explicit breakdown of a gauge symmetry is not allowed as it spoils the renormalization of such theories. Only spontaneous breakdown is admitted. From the Goldstone theorem it follows that if a global symmetry is spontaneously broken physical massless, spin-0 bosons, the Goldstone bosons, emerge. What happens now if one breaks a local symmetry spontaneously? We will discover soon that the "flat" directions $(T^a)_{ij}v_j$ of a local symmetry correspond to unphysical gauge degrees of freedom, i.e., the gauge symmetry is broken but we do not get physical Goldstone bosons. Yet at first sight the appearance of these Goldstone bosons (one for each broken generator) seems to be a new problem, but it turns out to be a way to generate the masses of gauge bosons without breaking explicitly the gauge symmetry of the Lagrangian. Indeed, a massless gauge boson, like the photon, has only two degrees of freedom corresponding to its two transverse polarizations. On the other hand, a massive gauge boson like $W^\pm$ and Z^0 has three degrees of freedom, the third one corresponding to its longitudinal polarization. It is the Goldstone boson of a spontaneously broken gauge symmetry that provides the third degree of freedom to every gauge boson corresponding to a broken generator. Thus at the end $W^\pm$ and Z^0 are massive, and Goldstone bosons do not appear in the particle spectrum. That's why the Goldstone bosons of a local symmetry are unphysical. One can say that they have been eaten by the gauge bosons. In the next section we will discuss this mechanism for generation of masses of gauge bosons in explicit terms.

1.11 Higgs Mechanism

1.11.1 U(1) Symmetry

Let us then gauge the Lagrangian in (1.123) so that

$$\mathscr{L}_{\text{gauged}} = \left(D_\mu\varphi\right)^* (D^\mu\varphi) - V(\varphi^*,\varphi) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}. \tag{1.156}$$

This Lagrangian is invariant under simultaneous U(1) transformations of φ and A_μ:

$$\varphi \to \varphi e^{i\theta(x)}, \qquad A_\mu \to A_\mu + \frac{1}{g}\partial_\mu\theta(x). \tag{1.157}$$

This invariance will be crucial for the removal of Goldstone bosons from the physical mass spectrum. We again write after SSB

$$\varphi(x) = \frac{1}{\sqrt{2}}\left(v + \eta(x) + i\xi(x)\right) \approx \frac{1}{\sqrt{2}}\exp\left(i\frac{\xi(x)}{v}\right)\left(v + \eta(x)\right), \tag{1.158}$$

but this time it will be useful to also have the last expression in (1.158).

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For the discussion of the mass generation for gauge bosons only the covariant derivative $D_\mu \varphi$ is of interest as only there the scalars and gauge bosons interact with each other. We have then

$$D_\mu \varphi = \frac{1}{\sqrt{2}} \left[\partial_\mu \eta - i \left(v g A_\mu - \partial_\mu \xi \right) - i g A_\mu \left(\eta + i \xi \right) \right] . \tag{1.159}$$

Of particular interest is the second term on the r.h.s. of this equation. We can rewrite it as follows

$$v g A_\mu - \partial_\mu \xi = g v \left(A_\mu - \frac{1}{g v} \partial_\mu \xi \right) \equiv g v A'_\mu . \tag{1.160}$$

Note that the original field A_μ and A'_μ are related by gauge transformation in (1.157) with

$$\theta(x) = -\frac{\xi(x)}{v} . \tag{1.161}$$

This transformation performed on the field φ in (1.158) results in

$$\varphi \to \varphi' = \frac{1}{\sqrt{2}} \left(v + \eta \right) . \tag{1.162}$$

But we know that such a gauge transformation leaves the Lagrangian invariant, and consequently this Lagrangian rewritten in terms of A'_μ and φ' describes the same physics as the original Lagrangian. In this new Lagrangian the Goldstone boson is not seen; it has been gauged away.

Dropping now the prime in (1.162) and denoting $\eta = H$ with H standing for a Higgs particle, this discussion shows that due to the gauge invariance of $\mathscr{L}$ we are allowed to write

$$\varphi = \frac{1}{\sqrt{2}} \left(v + H \right) , \tag{1.163}$$

$$D_\mu \varphi = \frac{1}{\sqrt{2}} \left(\partial_\mu H - i v g A_\mu - i g A_\mu H \right) . \tag{1.164}$$

Consequently we obtain

$$\mathscr{L}_{\text{gauged}} = \frac{1}{2} \partial_\mu H \partial^\mu H - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} m_H^2 H^2 + \frac{1}{2} M_A^2 A_\mu A^\mu + \cdots , \tag{1.165}$$

which implies

$$M_A^2 = v^2 g^2, \qquad m_H^2 = -2\mu^2 . \tag{1.166}$$

Indeed, the gauge boson is now massive. Its mass depends on the gauge coupling g and v. In a given theory M_A can be predicted if g and v have been determined somewhere else. On the other hand, m_H is rather arbitrary as μ is a parameter in the potential V.

It should be remarked that the explicit disappearance of the Goldstone boson from the theory is only possible in the unitary gauge in which φ takes the form (1.162). This gauge is very useful for exhibiting the physical spectrum, but it is less convenient for Feynman diagram calculations. Therefore quite generally the latter calculations are done in other gauges, the so-called covariant gauges, in which Goldstone bosons appear in loop diagrams and one has to take these contributions into account in order to obtain a physical result that is gauge independent. The Feynman rules involving Goldstone bosons in the case of the SM can be found in Appendix B.

1.11.2 Nonabelian Gauge Symmetry

We next gauge the Lagrangian (1.135) to obtain

$$\mathscr{L} = \frac{1}{2}\left(D_\mu \vec{\varphi}^\top\right)(D^\mu \vec{\varphi}) - V(\vec{\varphi}^\top \vec{\varphi}) - \frac{1}{4} F^a_{\mu\nu} F^{\mu\nu,a}. \tag{1.167}$$

All these symbols have been defined earlier. We now consider as in Section 1.10.4 spontaneous symmetry breakdown $G \to H$ with unbroken generators denoted by Y^a and the broken ones by X^a. We have

$$Y^a \vec{v} = 0, \qquad a = 1, \dots M \tag{1.168}$$

$$X^a \vec{v} \neq 0, \qquad a = M+1, \dots N, \tag{1.169}$$

and the Goldstone bosons are given in (1.155). Expanding around the vacuum as in (1.144)–(1.146) we find

$$\begin{aligned}\left(D_\mu \vec{\varphi}\right)^\top (D^\mu \vec{\varphi}) = {} & \left(\partial_\mu \vec{\tilde{\varphi}}\right)^\top \left(\partial^\mu \vec{\tilde{\varphi}}\right) + g^2 A^a_\mu A^{\mu,b} \vec{v}^\top T^a T^b \vec{v} \\ & - ig\left(\partial_\mu \vec{\tilde{\varphi}}\right)^\top T^b \vec{v} A^{\mu,b} + ig \vec{v}^\top T^a \partial_\mu \vec{\tilde{\varphi}} A^{\mu,a} + \cdots \end{aligned} \tag{1.170}$$

The last two terms vanish for $T^a = Y^a$ but are nonvanishing for $T^a = X^a$. From (1.155) we know that these terms for $a = M+1, \dots N$ represent mixed terms involving Goldstone bosons and the gauge bosons corresponding to broken generators X^a. As in the case of abelian symmetry, they can be gauged away by going to the unitary gauge.

The mass spectrum of this theory after SSB is as follows

- The masses of gauge bosons can be read of from the second term on the r.h.s. of (1.170). The gauge boson mass matrix is simply given as follows

  $$\frac{1}{2}\left(M_A^2\right)^{ab} = g^2 \vec{v}^\top T^a T^b \vec{v}. \tag{1.171}$$

  As T^a includes both Y^a and X^a satisfying (1.168) and (1.169), this matrix has both vanishing and nonvanishing entries so that after diagonalization we will find massless gauge bosons corresponding to the generators of the unbroken subgroup H and the massive gauge bosons corresponding to the remaining generators of G.
- The masses of physical Higgs particles are the nonvanishing eigenvalues of the matrix in (1.147):

  $$\left(M^2\right)_{ij} = \left(\frac{\partial^2 V}{\partial \varphi_i \partial \varphi_j}\right)_{\vec{\varphi}=\vec{v}}, \qquad i,j = 1,\dots n. \tag{1.172}$$

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The vanishing entries correspond to the Goldstone bosons. The number of physical Higgs particles, N_H, is just given by

$$N_H = n - N_{\rm GB} = n - (N - M) \tag{1.173}$$

with n denoting the number of φ_i fields and $N - M$ the number of broken generators. As N_H is a positive number, it is evident that in order to break a given group G down to H, sufficient food in the form of φ_i has to be provided such that the gauge bosons corresponding to the broken generators become massive.

1.11.3 Summary

In this chapter we have presented the most important aspects of gauge theories that are necessary in order to follow the next chapters. In this context we have discussed also global symmetries, which, similar to local symmetries, play an important role in particle physics. In fact the first symmetries discussed in particle physics like SU(2), $\mathrm{SU(2)}_L \times \mathrm{SU(2)}_R$, $\mathrm{SU(3)}_L \times \mathrm{SU(3)}_R$, and SU(3) were all global symmetries. Even if in the 1970s gauge symmetries took over the leadership, due to significant development of flavor physics in the late 1980s and the following three decades, global symmetries play these days again a very important role, and we will discuss other important consequences of them in later chapters of this book.

With all this information at hand we are ready to move to Part II of our book in order to describe the SM of electroweak and strong interactions. This will allow us to begin to discuss the main topic of our book, namely weak decays of mesons and later leptons as well as other interesting rare processes.

WE ARE READY TO CLIMB TO THE BASE CAMP: THE STANDARD MODEL!