A Planetary Theory with Elliptic Functions and Elliptic Integrals Exhibiting No Small Divisors

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Abstract. This paper develops a planetary theory in three dimensions with elliptic functions and elliptic integrals. In an earlier treatment, (Williams, Van Flandern, and Wright, 1987) presented a two dimensional planetary theory to the first order of a Picard iteration. The theory did avoid expansions in powers of the ratio of the semi-major axes and it contained only two explicit small divisors, $n - n'$ and $2n - n'$. These advantages are retained in the new theory and in fact no small divisors appear explicitly. Secular terms are removed by adopting an averaging technique rather than continuing the Picard iteration. The Lie series method of (Deprit, 1969) is chosen for the averaging. In order to simplify the Lie operator, the framework for the problem is chosen to be the circular restricted three body problem written in the polar-nodal coordinates of Whittaker. The algorithm is described and a few representative terms are discussed.

Key words:
Planetary Theory, Elliptic Functions, Restricted Problem.

1. Introduction

A method of simplifying planetary theories by averaging the reciprocal of the distance (Δ) between two planets with elliptic functions was presented by (Richardson, 1982). He established a generator for a Lie transformation together with an averaged Hamiltonian to first order for a problem with an arbitrary number of planets, but for only one degree of freedom per planet.

A first order treatment for two degrees of freedom appeared in (Williams, Van Flandern, and Wright, 1987). The basic problem in going to two dimensions is the incorporation of a second degree of freedom into the integration of the Lie operator to obtain an averaged Hamiltonian. If a linear function of the synodic angle, φ, is the first frequency and the mean longitude of a planet, λ, the second, then a typical term exhibiting both angles is $\cos\lambda/\Delta(\varphi)$. This problem was handled in (Williams, Van Flandern, and Wright, 1987) by expanding trigonometric functions of λ as Fourier series in integral multiples of φ, by writing

$$\lambda = \omega\varphi + \varphi_0, \tag{1}$$

$$\cos\lambda = \sum_{n=0}^{\infty} a_n \cos n\varphi. \tag{2}$$

To first order, both λ and φ are linear functions of time and ω can be treated as a constant. The method of handling the second frequency took advantage of the fact that ω was constant. In fact, this allowed an exact representation for $\cos\lambda$ with a finite number of terms when ω was an integer. This case corresponded to mean motion commensurabilities of the type $p/(p+1)$ and $(2p-1)/(2p+1)$. If this method were carried to higher orders, when ω can no longer be treated as a constant, mixed secular terms would most likely arise.

S. Ferraz-Mello (ed.), Chaos, Resonance and Collective Dynamical Phenomena in the Solar System, 43–48.

One way to avoid adopting Eq.(1) is to set the planetary theory in the context of the restricted three body problem. In a rotating coordinate system, in two dimensions, the synodic angle is the polar angle of the massless particle. For the right choice of coordinates, this feature will allow us to apply an averaging method, using this angle as the basis for the averaging. The addition of the third dimension will not destroy this property, and so a three dimensional model is adopted for this work.

In the previous paper, the Fourier series associated with Eq.(1) gave rise to secular perturbations from $\cos 2n\varphi/\Delta(\varphi)$. The adoption of a method of averaging allows for the removal of the secular terms; that of (Deprit, 1969) is chosen here.

2. Development of the Hamiltonian

The three dimensional circular restricted three body problem is considered in a coordinate system rotating about the z axis with angular frequency ω and with origin at m_1, one of the primaries. The second primary m_2 is fixed on the x axis at $x_2 = \alpha$. The Hamiltonian of this system is given by

$$\mathcal{H} = \frac{1}{2}(X^2 + Y^2 + Z^2) - \omega(xY - yX) + \frac{\omega\alpha m_2}{m_1 + m_2}Y - \frac{k^2 m_1}{r} - \frac{k^2 m_2}{\Delta}, \tag{3}$$

where

$$\begin{aligned} r^2 &= x^2 + y^2 + z^2, \\ \Delta^2 &= (x-\alpha)^2 + y^2 + z^2, \\ \omega^2\alpha^3 &= k^2(m_1 + m_2). \end{aligned}$$

The momenta (X, Y, Z) are conjugate to the rectangular coordinates (x, y, z) of the particle in this system. Polar-nodal coordinates are introduced relative to this frame. These coordinates are (r, θ, ν), the radius vector of the particle orbit, the argument of the latitude of the particle, and the longitude of the ascending node of the particle's orbit measured in the x-y plane from the rotating x axis. The conjugate momenta are R, Θ, and N, where N is related to the inclination of the orbit through $N = \Theta\cos I$. In these variables, the Hamiltonian of Eq.(3) becomes

$$\begin{aligned} \mathcal{H} = {} & \frac{1}{2}(R^2 + \frac{\Theta^2}{r^2}) - \frac{k^2 m_1}{r} - \omega N - \frac{k^2 m_2}{\Delta} \\ & + \frac{\omega\alpha m_2}{2(m_1+m_2)}\left[\left(1+\frac{N}{\Theta}\right)\left(R\sin(\theta+\nu) + \frac{\Theta}{r}\cos(\theta+\nu)\right)\right. \\ & \left. - \left(1-\frac{N}{\Theta}\right)\left(R\sin(\theta-\nu) + \frac{\Theta}{r}\cos(\theta-\nu)\right)\right]. \end{aligned} \tag{4}$$

The distance between the particle and m_2 is

$$\Delta = [r^2 + \alpha^2 - 2\alpha r\cos(\theta+\nu) - r^2\sin^2 I\sin^2\theta - 4\alpha r\sin^2\frac{I}{2}\sin\theta\sin\nu]^{\frac{1}{2}}$$

. With the choice of the Keplerian part of Eq.(4) as the Hamiltonian of order zero, we may define the Lie derivative in polar-nodal variables as

$$\mathcal{L}_0 = R\frac{\partial}{\partial r} + \left(\frac{\Theta}{r^3} - \frac{k^2 m_1}{r^2}\right)\frac{\partial}{\partial R} + \frac{\Theta}{r^2}\frac{\partial}{\partial \theta}. \tag{5}$$

One way to simplify this operator is, of course, to treat the problem in the Delaunay chart where $\mathcal{L}_0 = n\partial/\partial\ell$. However it is possible to retain the polar-nodal variables and yet simplify the operator given in Eq.(5). The simplification follows a method used by (Deprit, 1981) in the elimination of the parallax. Define two variables

$$\begin{aligned} S &:= \left(\frac{\Theta}{r} - \frac{\Theta}{p}\right)\sin(\theta+\nu) - R\cos(\theta+\nu) = \frac{\Theta}{p}e\sin\varpi \\ C &:= \left(\frac{\Theta}{r} - \frac{\Theta}{p}\right)\cos(\theta+\nu) + R\sin(\theta+\nu) = \frac{\Theta}{p}e\cos\varpi, \end{aligned} \tag{6}$$

and $p = \Theta^2/k^2 m_1 = a(1-e^2)$.

If it is agreed to replace r and R in the Hamiltonian and in the generating function for the Lie transformation by the use of Eq.(6), then the Lie operator reduces to

$$\mathcal{L}_0 = \frac{\Theta}{r^2}\frac{\partial}{\partial\theta}. \tag{7}$$

One reason for this success is that S and C are in the kernel of $\mathcal{L}_0$, *i.e.* $\mathcal{L}_0(S) = \mathcal{L}_0(C) = 0$. In the variables $(S, C, \theta, \Theta, \nu, N)$, $1/\Delta$ is given by,

$$\frac{1}{\Delta} = \frac{1}{(1+\frac{\alpha}{p})r\delta}\left[1 + \frac{\Delta_1 + \Delta_2}{(1+\frac{\alpha}{p})^2\delta^2}\right]^{-1/2} \tag{8}$$

where

$$\begin{aligned} \varphi &= \frac{1}{2}(\theta+\nu+\pi), \\ \kappa^2 &= \frac{4\alpha/p}{1+\alpha/p)^2}, \\ \delta &= \sqrt{1-\kappa^2\sin^2\varphi}, \\ \gamma^2 &= \sin^2\frac{I}{2}, \\ P_1 &= \frac{p}{r} - 1 = \frac{p}{\Theta}\left[S\sin(\theta+\nu) + C\cos(\theta+\nu)\right] = e\cos(\theta+\nu-\varpi), \\ \Delta_1 &= 2\frac{\alpha}{p}P_1\left[\frac{\alpha}{p} - \cos(\theta+\nu)\right], \\ \Delta_2 &= \frac{\alpha^2}{p^2}P_1^2 - 4\gamma^2\sin^2\theta\left[(1+\gamma^2)\sin\theta + \frac{\alpha}{p}\sin\nu(1+P_1)\right]. \end{aligned}$$

3. Solution at First Order

From Eq.(4) and Eq.(8), we obtain the Hamiltonian to first order

$$\mathcal{H}_1 = -\omega N + \frac{\omega \alpha m_2}{m_1 + m_2} \frac{\Theta}{p} \cos(\theta + \nu) - \frac{k^2 m_2}{(1 + \frac{\alpha}{p})} \frac{1}{r\delta}. \tag{9}$$

We choose the new first order Hamiltonian, $\mathcal{K}$, so that secular terms are removed from the generating function, W. Formally, we choose

$$\mathcal{K}_1 = -\omega N + \frac{\omega \alpha m_2}{m_1 + m_2} \frac{A}{p} - \frac{k^2 m_2}{1 + \frac{\alpha}{p}} \frac{B}{r^2 \delta}, \tag{10}$$

where A and B are to be determined when W_1 is found. We write W_1 as the sum $W_{1A} + W_{1B}$, obviously connected to the two terms in Eq.(10). In fact,

$$\begin{aligned} \frac{\Theta}{r^2} \frac{\partial W_{1A}}{\partial \theta} &= \frac{\omega \alpha m_2}{m_1 + m_2} \frac{\Theta}{p} (A - \cos(\theta + \nu)), \\ \frac{\Theta}{r^2} \frac{\partial W_{1B}}{\partial \theta} &= -\frac{k^2 m_1}{(1 + \frac{\alpha}{p})} \left(\frac{1}{r\delta} - \frac{B}{r^2 \delta} \right). \end{aligned} \tag{11}$$

The integration of Eqs.(11) require that r and r^2 be written as functions of θ. From Eq.(6), we obtain

$$r = p/[1 + e\cos(\theta + \nu - \varpi)], \tag{12}$$

and expand r and r^2 in powers of $e\cos(\theta + \nu - \varpi)$. Since ϖ and ν are in the kernel of $(\mathcal{L}_0)$, integration of W_{1A} is immediate and A is chosen to cancel the secular term. To the first order in the eccentricity

$$W_{1A} = \frac{1}{2} \frac{\omega \alpha m_2}{m_1 + m_2} \frac{p^2}{\Theta} [C \sin(2\theta + 2\nu) - S \cos(2\theta + 2\nu)] \tag{13}$$

where $A = -e\cos\varpi = -pC/\Theta$.

For the integration of W_{1B} we introduce the variable u by

$$u = \int_0^{\varphi} \sqrt{1 - \kappa^2 \sin^2 x}\, dx, \tag{14}$$

so that

$$\begin{aligned} \sin\varphi &= sn\, u \\ \cos\varphi &= cn\, u \\ \delta &= dn\, u = \sqrt{1 - \kappa^2 \sin^2 \varphi}. \end{aligned}$$

For the purposes of integration, we also need the differential relation

$$d\theta = 2d\varphi = 2dn\, u\, du. \tag{15}$$

Then, choosing B so that secular terms in u are removed from W_{1B}, we obtain

$$W_{1B} = \frac{-2m_2}{m_1(1+\frac{\alpha}{p})\kappa^2}\left(\frac{C}{\Theta}Z(u) - \frac{S}{\Theta}dn\ u\right), \tag{16}$$

where $B = C/\Theta - 1\ - [2C/(\kappa^2\Theta)](E/K - \kappa'^2)$. E and K are the complete elliptic integrals of the first and second kind respectively and κ' is the complementary modulus.

4. Remarks Concerning the Second Order

The equation for the generating function and $\mathcal{K}_2$ at second order is

$$\mathcal{L}_0(W_2) = \mathcal{K}_2 - \mathcal{H}_2 - ((\mathcal{H}_1 + \mathcal{K}_1); W_1). \tag{17}$$

The choice for $\mathcal{H}_2$ depends on the assumptions made concerning the size of the eccentricity and inclination. Some terms involving Δ_1/δ^3 or Δ_2/δ^3 will certainly be involved; they present no difficulty in the averaging. The Poisson bracket for W_1, is more complicated. There are many terms present in this expression. Those involving merely $\sin(\theta+\nu), \cos(\theta+\nu)$, and δ can be treated in a manner similar to what has been given in the previous section. This is also true for $\mathcal{H}_2$. The terms we need to mention are those that contain $Z(u)$. There are three types of integrals arising for the determination of W_2 from Eq.(17). These are:

$$\begin{aligned} &\int sn^i u\ cn^j u\ Z(u)\, du, \\ &\int sn^i u\ cn^j u\ dn\ u\ Z(u)\, du, \\ &\int \frac{sn^i u\ cn^j u}{dn^2 u} Z(u)\, du, \end{aligned} \tag{18}$$

where i and j are non-negative integers and $i+j$ is even.

Some of these terms, if they are not retained in $\mathcal{K}_2$, will introduce the quantity $\ln[\Theta(u)/\Theta(0)]$ in the generator. With the exclusion of the theta function, $\Theta(u)$, from W_2 it would appear that higher order partial differential equations for the generator will require only the Jacobian elliptic functions and the Lagrange elliptic integrals of the first and second kind. However, if the theta functions are allowed, the Hamiltonian will contain very few terms that have not been averaged and no other functions will be required at higher orders.

Further algebraic study is required before one can be certain of the terms that the new Hamiltonian, $\mathcal{K}$, will contain. It appears that r is introduced at higher orders with negative powers (a feature that aids the development) and that the absence of small divisors will continue.

5. Conclusion

The recent development of computer codes to integrate rational functions of the Jacobi elliptic functions in symbolic form have encouraged this exploration. Computer codes have been developed by Abad, V. Coppola, and A. Deprit in *Mathematica* and B. Miller, in MACSYMA. It is clear that before the full power of algebraic manipulators can be brought to bear on these problems, codes will be needed to integrate quantities containing the elliptic integrals of the first, second, and third kind, as well as theta functions.

Our understanding of the classical averaging methods, especially with the presence of small divisors, is not complete. It is hoped, that by developing perturbations with elliptic functions, since they contain no small divisors explicitly, we may come to reach a better understanding of this question.

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