

FOLIATIONS IN SUPERGRAVITY

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Abstract

Following upon a previous paper [1] on the existence of chiral transformations in a foliated version of the Cremmer, Julia and Scherk model, we deduce a couple of interesting properties of the model. These are:

- (i) TM_4 is isomorphic to a quotient Lie pseudoalgebra on the algebra of basic functions in M_{11} ;
- (ii) There is a locally trivial fibration which exhibits M_{11} as $M_7 \times \mathcal{U}$, $\mathcal{U} \subset W$ and W is the basic manifold of the foliation [5];
- (iii) The chiral group of the model is identified as $Cl^x(L, g_L) \times Cl^x(Q, g_Q)$, the factors are respectively the multiplication groups of units in the Clifford algebras $Cl^x(L, g_L)$ and $Cl^x(Q, g_Q)$ and matching of this group with phenomenology is briefly discussed.

1. Introduction

It is generally accepted that one of the main problems with the Cremmer, Julia and Scherk Model (CJS) [2] is its inability to support chiral transformations [8]. In a previous paper [1], we show that this point can be remedied in such a way that none of the essential features of the 11-dimensional Kaluza-Klein mechanism in the original model needs to be changed. The starting point of the analysis is the observation that the metric tensor used in the original work for M_{11} is bundle-line [6] and therefore suggests that M_{11} has a foliation structure of dimension 7 and codimension 4. The nature of the foliation is, of course, open. As a first step, we suggest that this foliation is a simple foliation in the sense of Molino [5] and is brought about by a global submersion of M_{11} into M_4 with leaves diffeomorphic to M_7 ,

$$f : M_{11} \rightarrow M_4. \quad (1)$$

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The restriction of f to a leaf gives rise to a map $\varphi : M_7 \rightarrow M_4$ which serves to relate happenings in the internal manifold to the physical world. It is important to note that the existence of this map is, by and large, implied by the bundle-like nature of the metric for M_{11} . In the second section, we show that the submersion (1) leads to some consequences on the tangent space of M_4 . In the third section, we show that there is a locally trivial fibration $\pi_b : M_{11} \rightarrow W$, where W is the basic manifold of the foliation. The fiber of this fibration is exactly M_7 and hence there is a foliation induced on it by (1). This foliation is a Lie foliation and its relation to the one assumed on M_7 in order to bring about chiral transformation is intriguing. At the present, we have not been able to resolve this problem. In the third section, we follow up on the analysis of [1] and identify the internal chiral group as $Cl^x(L, g_L) \times Cl^x(Q, g_Q)$, where the factors may be identified as automorphism groups of $Cl^x(L, g_L)$ and $Cl^x(Q, g_Q)$, where L is the involutive subbundle of TM_7 which brings about the foliation in M_7 , and Q is the normal bundle. In the fourth section we conclude our analysis.

2. Structural properties of the CJS model

In the 11-dimensional Kaluza-Klein theory [8], one assumes that the ground state is not an 11-dimensional Minkowski space but a product space $M_7 \times M_4$, where M_7 is a compact manifold of dimension 7 and M_4 is the physical world. Symmetries on M_7 will be observed as gauge symmetries in the 4-dimensional world M_4 . If $(x^m; m = 1, \dots, 7)$ are the coordinates of M_7 and $(T^a; a = 1, \dots, N)$ are the generators of the symmetry group G of M_7 , then the action of T^a on x^m may be represented by $x^m \rightarrow x^m + K^{am}(x)$, where $K^{am}(x)$ is the Killing vector associated with the symmetry T^a . The massless excitation of $M_7 \times M_4$ corresponds to the following ansatz for the metric tensor for M_{11} (known normally as the DeWitt metric):

$$g_{AB} dz^A dz^B = g_{uv}(y) dy^u dy^v + g_{mn}(x, y)(dx^m - K^{am}(x)B_u^a(y)dy^u)(dx^n - K^{bn}(x)B_v^b(y)dy^v). \quad (2)$$

Here $A, B = 1, \dots, 11$, $u, v = 1, \dots, 4$, $a, b = 1, \dots, N$; $m, n = 1, \dots, 7$. The metric tensor of the 4-dimensional world M_4 is denoted by g_{uv} and $B_u^a(y)$ is the massless gauge field of the group G . This metric is bundle-like [6] and therefore suggests that M_{11} carries a foliation of dimension 7 and codimension 4. The inherent nature of this foliation is, at this point, open to model builders. As a first step, we suggest that it is a simple foliation [5] in the sense that the involutive subbundle that defines the foliation is the vertical tangent bundle of some surjective submersion

$$f : M_{11} \rightarrow M_4. \quad (3)$$

As in [1], this simple foliation is required to have leaves diffeomorphic to M_7 . This gives rise to a map $\varphi = f|_{M_7}$, mapping M_7 to M_4 .

Now we are ready to state the first consequence of this simple foliation F [4]. Let Δ be the involutive subbundle that defines the foliation F . Call a function $u \in C(M_{11})$ basic if $X(u) = 0$ for all $X \in \Gamma\Delta$ and denote the algebra of basic functions by $C(M_{11})^\Delta$. Let

$$L(M_{11}, \Delta) = \{X \in \Gamma TM_{11} \mid Y \in \Gamma\Delta \Rightarrow [X, Y] \in \Gamma\Delta\} \tag{4}$$

be the normalizer [7] of $\Gamma\Delta$ in ΓTM_{11} . Elements of $L(M_{11}, \Delta)$ are called foliate vector fields [5]. Since F is simple, the basic functions are precisely the pull-backs of functions on M_4 and the foliate vector fields are precisely those which project to vector fields in M_4 . In terms of local coordinates this means the following. Let the metric in M_{11} be expressed in the form:

$$ds^2 = \sum_{i,j=1}^7 g_{ij}(x, f)\omega^i\omega^j + \sum_{\alpha,\beta=1}^4 g_{\alpha\beta}(f)df^\alpha df^\beta, \tag{5}$$

where the 1-forms ω^i are of the form

$$\omega^i = dx^i = \sum_{\alpha=1}^4 a_\alpha^i df^\alpha \tag{6}$$

and $(\omega^1, \dots, \omega^7, df^1, \dots, df^4)$ is a basis of the cotangent space T^*M_{11} . The tangent space TM_{11} is spanned by $(\partial/\partial x^1, \dots, \partial/\partial x^7, \nu_1, \dots, \nu_4)$ with

$$\nu_\alpha = \frac{\partial}{\partial f^\alpha} - \sum_{i=1}^4 a_\alpha^i \frac{\partial}{\partial x^i}. \tag{7}$$

Thus a basic function $u \in C(M_{11})$ is one that does not depend on x^1, \dots, x^7 and is of the form $u \sim \tilde{u} \circ f$ where $\tilde{u} \in C(M_4)$. A foliate vector field is one whose last four components do not depend on x^1, \dots, x^7 and under f , such a vector field is projected into a vector field in M_4 .

$L(M_{11}, \Delta)$ is a Lie pseudoalgebra over $C(M_{11})^\Delta$ and $\Gamma\Delta$ is an ideal. Let $l(M_{11}, \Delta) = L(M, \Delta)/\Gamma\Delta$ be the quotient Lie pseudoalgebra, then $l(M_{11}, \Delta)$ is canonically isomorphic to TM_4 :

$$l(M_{11}, \Delta) \cong TM_4. \tag{8}$$

This result depicts the tangent space of the physical world as a quotient Lie pseudoalgebra over the algebra of basic functions in M_{11} . In our opinion, it is interesting and may have consequences which may further prove or disprove the model.

The next structural property of the foliated model may be explained as follows. If we assume that the foliate vector fields on M_{11} span TM_{11} , then F is known as a transversally complete foliation [5]. As explained in [5], \forall neighbourhood θ of $x \in M_{11}$, one can find basic functions f_1, \dots, f_{q_b} whose differentials are linearly independent (in the neighbourhood under consideration). Consequently the system of equations

$$df_1 = 0, \quad \dots, \quad df_{q_b} = 0 \tag{9}$$

is completely integrable. This system defines a foliation F_b which is called a basic foliation of (M_{11}, F) . It has codimension q_b and its leaves are unions of the leaves of F . The leaf space M_{11}/F is Hausdorff and has the structure of a q_b -dimensional manifold, called W , the basic manifold of the foliation F . When F is transversally complete, the closures of the leaves (these are just copies of M_7 , since M_7 is closed) are fibers of a locally trivial fibration $\pi_b : M_{11} \rightarrow W$ onto the basic manifold W . On each fiber of π_b , the foliation induced by F is a Lie foliation.

This result says that if F is transversally complete then there are q_b functions in $C(M_4)$ which, when pulled back to M_{11} , define level surfaces of F_b . Along directions transverse to these level surfaces (which are defined by physical objects) M_{11} is locally trivial, that is, is of the form $M_4 \times \mathcal{U}$ for some $\mathcal{U} \subset W$. This comes close to explaining mathematically what we mean by saying that in the ground state M_{11} is of the form $M_7 \times M_4$. It would be interesting to see what happens if we take the fibration π instead of the product as the starting structure of M_{11} .

3. The chiral group $Cl_p^k \times Cl_q^k$

In [1], we show that if the G -action of Kaluza-Klein has the property that for every point $p \in M_7$ the tangent space to the orbit of the G -action has fixed dimension, then the orbits themselves define a foliation on M_7 . The tangent space and the Riemannian metric of M_7 acquire the following orthogonal splitting;

$$TM_7 = L \oplus Q, \tag{10}$$

$$g_{M_7} = g_L \oplus g_Q. \tag{11}$$

The 7-dimensional Clifford algebra $Cl(TM_7, g_{M_7})$ is then isomorphic to the Z_2 -graded tensor product $Cl(L, g_L) \otimes^\wedge Cl(Q, g_Q)$ [3]. It has been shown in [1] that if M_7 is spin then both L and Q are spin. It is then meaningful to talk about the spinor bundles $S(Q)$ and $S(L)$. Along with the orthogonal splittings (10) and (11), the spinor bundle on M_7 , becomes a Z_2 -graded tensor product $S(L) \otimes^\wedge S(Q)$ and spinors with definite helicities are of the form

$$\psi_{\text{int}} = (1 \pm \Omega_p)\psi \otimes (1 \pm \Omega_q)\sigma, \tag{12}$$

where p and q are respectively the dimension and the codimension of the foliation and Ω_p and Ω_q are the volume elements in the respective Clifford algebras and $\psi \in S(L)$ and $\sigma \in S(Q)$. Chiral transformations are generated by matrices of the form

$$\Gamma_{ij} = 2\gamma_i \otimes \lambda_j. \quad (13)$$

Here $\{\gamma_1, \dots, \gamma_p\}$ forms a basis for $Cl(L, g_L)$ and $\{\lambda_1, \dots, \lambda_q\}$ forms a basis for $Cl(Q, g_Q)$. This set of matrices clearly generates a Lie group which is of the form $Cl^x(L, g_L) \times Cl^x(Q, g_Q)$. Here the group $Cl^x(V, q)$ is the multiplicative group of units [3] in the Clifford algebra of the vector space V and bilinear form q and is defined to be the subset

$$Cl^x(V, q) \equiv \{\varphi \in Cl(V, q) \mid \exists \varphi^{-1} \text{ with } \varphi^{-1}\varphi = \varphi\varphi^{-1} = 1\}. \quad (14)$$

When the vector space is of finite dimension, say n , it is a Lie group of dimension 2^n . In general there is an associated Lie algebra $cl^x(V, q) = Cl(V, q)$ with Lie bracket given by

$$[x, y] = xy - yx. \quad (15)$$

The group of units acts naturally as automorphisms of the algebra, that is, there is a homomorphism

$$\text{Ad} : Cl^x(V, q) \rightarrow \text{Aut}(Cl(V, q)) \quad (16)$$

called the adjoint representation given by

$$\text{Ad}_\phi(x) = \phi x \phi^{-1}. \quad (17)$$

There is also a natural exponential mapping given by $\exp : cl^x(V, q) \rightarrow Cl^x(V, q)$ defined by setting

$$\exp(v) = \sum_{m=0}^{\infty} v^m / m!. \quad (18)$$

Thus we conclude that the chiral group of the model is $Cl(L, g_L) \times Cl(Q, g_Q)$. The dimension of this group is $2^p + 2^q$ with $p + q = 7$.

Just on the basis of simple minded dimension counting and assuming that $SU(3) \times SU(2) \times U(1)$ is all the symmetry that we have got, this is clearly too large a number. Maybe there is a case to reduce the dimension of the internal manifold to, say, $p = 3$, $q = 3$ because then an internal symmetry group $U(3) \times U(2) \times U(1)$ would fit just nicely.

4. Conclusion

In this paper, we have pointed out that the DeWitt metric for M_{11} suggests that the 11-dimensional manifold in the CJS model is foliated. If one assumes that this foliation is simple in the sense of Molino, one arrives at interesting structural properties on top of the existence of chiral transformations. It would be extremely interesting to study these properties further in detail and to correlate them with observed phenomenology.

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