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Remarks on theorems of Thompson and Freede

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Let A be a hermitian transformation on an n-dimensional unitary space E_n, with proper values $a_1 \geq \ldots \geq a_n$. Let M be a proper subspace of E_n. Suppose $b_1 \geq \ldots \geq b_h$ are the proper values of $A|M$ and $c_1 \geq \ldots \geq c_k$ are the proper values of $A|M^\perp$. Let $i_1 < \ldots < i_r$ and $j_1 < \ldots < j_r$ be sequences of positive integers, with $i_r \leq k$ and $j_r \leq h$. Then

$$\sum_{p=1}^{r} b_{i_p} + \sum_{p=1}^{r} c_{j_p} \leq \sum_{p=1}^{r} a_p + \sum_{p=1}^{r} a_{(i_p+j_p)} .$$

This is a special case of one of the Thompson-Freede theorems which is proved by use of certain invariants.

Some very interesting generalizations of an inequality of Aronszajn have been given by Thompson and Freede [4]. In this note we give a sample of expressing these theorems in terms of linear transformations and give a proof using some invariants.

1. Definitions and notations

An n-dimensional unitary space will be indicated by E_n. The inner product of two vectors α and β will be denoted by (α, β). An orthonormal set $\{\alpha_1, \ldots, \alpha_k\}$ will be indicated by $\{\alpha_p\}$ orthonormal.

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The subspace spanned by the set $\{\alpha_1, \ldots, \alpha_k\}$ will be denoted by $[\alpha_1, \ldots, \alpha_k]$. We write $\dim M = h$ if the dimension of the subspace M is h.

If A is a linear transformation on E_n and if M is a subspace of E_n, then we define a linear transformation $A|M$ as follows: if $\xi \in M$, let $[A|M]\xi = PA\xi$, where P is the orthogonal projection on M. We observe that if α and $\beta \in M$, then

$$([A|M]\alpha, \beta) = (PA\alpha, \beta) = (A\alpha, \beta) .$$

It follows that if A is hermitian, then $A|M$ is hermitian.

Given any sequence $i_1 \le \ldots \le i_k$ of positive integers such that $i_p \ge p$, for $p = 1, \ldots, k$, we define $(i_1', \ldots, i_k')$ recursively by $i_k' = i_k$ and $i_r' = \min(i_r, i_{r+1}'-1)$, for $r = k-1, \ldots, 1$, [1].

2. Some preliminary theorems

Let H be a hermitian transformation on E_n with proper values $m_1 \ge \ldots \ge m_n$. Then

$$(1) \quad m_1 + \ldots + m_k = \sup_{\{\xi_i\} \text{ orthonormal}} [(H\xi_1, \xi_1) + \ldots + (H\xi_k, \xi_k)] .$$

This theorem is due to Fan [2]. Further, if $i_1 \le \ldots \le i_k$ is a sequence of positive integers such that $i_p \le n$ and $i_p \ge p$, $p = 1, \ldots, k$, then

$$(2) \quad m_{i_1'} + \ldots + m_{i_k'} = \sup_{\substack{M_1 \subset \ldots \subset M_k \\ \dim M_p = i_p}} \; \inf_{\substack{\xi_p \in M \\ \{\xi_p\} \text{ orthonormal}}} [(H\xi_1, \xi_1) + \ldots + (H\xi_k, \xi_k)] ,$$

where M_p is a subspace of E_n, [1].

3.

THEOREM. *Let A be a hermitian transformation on E_n with proper*

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values $a_1 \geq \ldots \geq a_n$. *Let* $R_1, \ldots, R_s$ *be proper subspaces of* E_n *such that* $E_n = R_1 \oplus \ldots \oplus R_s$ *and* R_i *is orthogonal to* R_j *, for* $i \neq j$. *Let* $\dim R_q = h_q$, $q = 1, \ldots, s$. *Suppose the proper values of* $A|R_q$ *are* $b_{q1} \geq \ldots \geq b_{qh_q}$, $q = 1, \ldots, s$. *Let* $i_{q1} \leq \ldots \leq i_{qr}$, $q = 1, \ldots, s$, *be sequences of positive integers such that* $i_{qp} \leq h_q$ *and* $i_{qp} \geq p$ *, for* $p = 1, \ldots, r$ *and* $q = 1, \ldots, s$. *Then*

$$\sum_{q=1}^{s} \left(\sum_{p=1}^{r} b_{q,i'_{qp}} \right) \leq \sum_{j=1}^{r(s-1)} a_j + \sum_{p=1}^{r} a_{\left(\sum_{q=1}^{s} i_{qp}\right)'} . \tag{1}$$

Proof. By §2 (2), there exist subspaces $M_{q1} \subset \ldots \subset M_{qr} \subset R_q$, $q = 1, \ldots, s$, with $\dim M_{qp} = i_{qp}$, for $p = 1, \ldots, r$ and $q = 1, \ldots, s$, such that

$$\sum_{p=1}^{r} b_{q,i'_{qp}} = \inf_{\substack{\eta_{qp} \in M_{qp} \\ \{\eta_{qp}\} \text{ orthonormal}}} \sum_{p=1}^{r} \left([A|R_q]\eta_{qp}, \eta_{qp}\right) \tag{2}$$

$$= \inf_{\substack{\eta_{qp} \in M_{qp} \\ \{\eta_{qp}\} \text{ orthonormal}}} \sum_{p=1}^{r} \left(A\eta_{qp}, \eta_{qp}\right) ,$$

for $q = 1, \ldots, s$.

Let $L_p = M_{1p} \oplus \ldots \oplus M_{sp}$, $p = 1, \ldots, r$. We observe that $L_1 \subset \ldots \subset L_r \subset E_n$ and $\dim L_p = \sum_{q=1}^{s} i_{qp}$, $p = 1, \ldots, r$. Let $\{\zeta_1, \ldots, \zeta_r\}$ be an orthonormal set in E_n such that $\zeta_p \in L_p$, $p = 1, \ldots, r$. Now, for each $p = 1, \ldots, r$, it is clear that there exists an orthonormal set $\{\eta_{1p}, \ldots, \eta_{sp}\}$ such that $\zeta_p \in [\eta_{1p}, \ldots, \eta_{sp}]$ and $\eta_{qp} \in M_{qp}$, $q = 1, \ldots, s$. Here the set $\{\eta_{11}, \ldots, \eta_{1r}, \eta_{21}, \ldots, \eta_{2r}, \ldots, \eta_{s1}, \ldots, \eta_{sr}\}$ may not be linearly independent. But, it is clear, there exists an orthonormal set

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$\{\eta'_{11}, \ldots, \eta'_{1r}, \ldots, \eta'_{s1}, \ldots, \eta'_{sr}\}$ such that $[\eta_{11}, \ldots, \eta_{1r}, \ldots, \eta_{s1}, \ldots, \eta_{sr}] \subset [\eta'_{11}, \ldots, \eta'_{1r}, \ldots, \eta'_{s1}, \ldots, \eta'_{sr}]$ and $\eta'_{qp} \in M_{qp}$, for $q = 1, \ldots, s$ and $p = 1, \ldots, r$. It is clear that $\{\zeta_1, \ldots, \zeta_r\}$ can be extended to an orthonormal set $\{\zeta_1, \ldots, \zeta_{sr}\}$ in such a way that $L = [\eta'_{11}, \ldots, \eta'_{1r}, \ldots, \eta'_{s1}, \ldots, \eta'_{sr}] = [\zeta_1, \ldots, \zeta_{sr}]$. Thus

$$(3) \qquad \sum_{q=1}^{s} \left(\sum_{p=1}^{r} (A\eta'_{qp}, \eta'_{qp}) \right) = \mathrm{trace}(A|L) = \sum_{i=1}^{sr} (A\zeta_i, \zeta_i) .$$

By §2 (1), it follows that

$$(4) \qquad \sum_{p=1}^{r(s-1)} a_p \geq \sum_{i=r+1}^{sr} (A\zeta_i, \zeta_i) .$$

Combining (3) and (4) we obtain

$$(5) \qquad \sum_{q=1}^{s} \left(\sum_{p=1}^{r} (A\eta'_{qp}, \eta'_{qp}) \right) \leq \sum_{i=1}^{r} (A\zeta_i, \zeta_i) + \sum_{j=1}^{r(s-1)} a_j .$$

Using (2) and (5) we obtain

$$(6) \qquad \sum_{q=1}^{s} \left(\sum_{p=1}^{r} b_{q,i'_{qp}} \right) \leq \sum_{j=1}^{r(s-1)} a_j + \sup_{\substack{K_1 \subset \ldots \subset K_r \\ \dim K_p = \sum_{q=1}^{s} i_{qp}}} \inf_{\substack{\delta_p \in K_p \\ \{\delta_p\} \text{ orthonormal}}} \sum_{p=1}^{r} (A\delta_p, \delta_p) .$$

Applying §2 (2) to the right side of (6) yields (1); thus the proof is complete.

4. Remark

We observe that the other results of Thompson and Freede can be proved in a manner similar to §3. These results may also be obtained as corollaries to §3 as was done in the Thompson and Freede paper.

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