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A concave–convex elliptic problem involving the fractional Laplacian

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We study a nonlinear elliptic problem defined in a bounded domain involving fractional powers of the Laplacian operator together with a concave–convex term. We completely characterize the range of parameters for which solutions of the problem exist and prove a multiplicity result. We also prove an associated trace inequality and some Liouville-type results.

1. Introduction

Over the last few decades the problem

$$\begin{aligned} -\Delta u &= f(u) \quad \text{in } \Omega \subset \mathbb{R}^N, \\ u &= 0 \qquad\ \ \text{on } \partial\Omega, \end{aligned}$$

has been widely investigated (for a survey see [3], and for more specific problems see, for example, [5, 13, 36], where different nonlinearities and different classes of domains, bounded and unbounded, are considered). Other different diffusion operators, like the p-Laplacian, fully nonlinear operators, etc., have also been studied (see, for example, [9, 16, 27] and the references therein). We deal with a non-local version of the above problem for a particular type of nonlinearities, i.e. we study a concave–convex problem involving the fractional Laplacian operator

$$\left.\begin{aligned} (-\Delta)^{\alpha/2}u = \lambda u^q + u^p, \quad u > 0 \text{ in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{aligned}\right\} \tag{1.1}$$

with $0 < \alpha < 2$, $0 < q < 1 < p < (N+\alpha)/(N-\alpha)$, $N > \alpha$, $\lambda > 0$ and where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain. The critical case $p = (N+\alpha)/(N-\alpha)$ is studied in [7].

The non-local operator $(-\Delta)^{\alpha/2}$ in $\mathbb{R}^N$ is defined on the Schwartz class of functions $g \in \mathcal{S}$ through the Fourier transform

$$[(-\Delta)^{\alpha/2}g]^\wedge(\xi) = (2\pi|\xi|)^\alpha \hat{g}(\xi), \tag{1.2}$$

or via the Riesz potential (see, for example, [33,40]). Observe that $\alpha = 2$ corresponds to the standard local Laplacian.

This type of diffusion operators arises in several areas such as physics, probability and finance (see, for example, [6, 8, 23, 44]). In particular, the fractional Laplacian can be understood as the infinitesimal generator of a stable Lévy process [8].

There is another way of defining this operator. In fact, in the case $\alpha = 1$, there is an explicit means of calculating the half-Laplacian acting on a function u in the whole space $\mathbb{R}^N$ as the normal derivative on the boundary of its harmonic extension to the upper half-space $\mathbb{R}^{N+1}_+$: the so-called Dirichlet-to-Neumann operator. The 'α derivative' $(-\Delta)^{\alpha/2}$ can be characterized in a similar way, defining the α-harmonic extension to the upper half-space (see [17] and § 2 for details). This extension is commonly used in the recent literature since it allows non-local problems to be written in a local way and this permits the use of variational techniques for these kind of problems.

In cases where the operator is defined in bounded domains Ω, the above characterization has to be adapted. The fractional powers of a linear positive operator in Ω are defined by means of the spectral decomposition. In [15], the fractional operator $(-\Delta)^{1/2}$ is considered, defined using the mentioned Dirichlet-to-Neumann operator, but restricted to the cylinder $\Omega \times \mathbb{R}_+ \subset \mathbb{R}^{N+1}_+$, and it is shown that this definition is coherent with the spectral one (see also [41] for the case where $\alpha \neq 1$). We recall that this is not the unique possibility of defining a non-local operator related to the fractional Laplacian in a bounded domain (see, for example, the definition of the so-called *regional fractional Laplacian* in [10, 31], where the authors consider the Riesz integral restricted to the domain Ω). This leads to a different operator related to a Neumann problem.

There are many results on the subject of concave–convex nonlinearity involving different (local) operators (see, for example, [1, 5, 9, 18, 22, 27]). Some ideas in the present paper are taken from [5]. In most of the problems considered in the aforementioned works, a critical exponent appears, which separates generically the ranges where compactness results can or cannot be applied (in the fully nonlinear case, the situation is slightly different, but a critical exponent still appears [18]). In our case, the critical exponent with respect to the corresponding Sobolev embedding is given by $2^*_\alpha = 2N/(N-\alpha)$. This is a reason why problem (1.1) is studied in the subcritical case $p < 2^*_\alpha - 1 = (N+\alpha)/(N-\alpha)$ (see also the non-existence result for supercritical nonlinearities in corollary 5.6).

The main results proven here characterize the existence of solutions of (1.1) in terms of the parameter λ. The competition between the sublinear and superlinear terms plays a role, which leads to different results concerning the existence and multiplicity of solutions, among other things. By 'solution' we mean an energy solution (see the precise definition in § 5).

THEOREM 1.1. *There exists $\Lambda > 0$ such that, for problem (1.1), the following hold.*

(i) *If $0 < \lambda < \Lambda$, there is a minimal solution. Moreover, the family of minimal solutions is increasing with respect to λ.*

(ii) *If $\lambda = \Lambda$, there is at least one solution.*

(iii) *If $\lambda > \Lambda$, there is no solution.*

(iv) *For any $0 < \lambda < \Lambda$, there exist at least two solutions.*

For $\alpha \in [1, 2)$ and p subcritical, we also prove that there exists a universal L^∞-bound for every solution to problem (1.1) independently of λ.

THEOREM 1.2. *Let $\alpha \geqslant 1$. Then there exists a constant $C > 0$ such that, for any $0 \leqslant \lambda \leqslant \Lambda$, every solution to problem (1.1) satisfies*

$$\|u\|_\infty \leqslant C.$$

The proof of this last result relies on the classical argument of rescaling introduced in [28], which yields to problems on unbounded domains. Therefore, some Liouville-type results are required, and this is the point where the restriction $\alpha \geqslant 1$ appears.

The paper is organized as follows. We devote §§ 2 and 3 to studying the fractional Laplacian in the whole space by means of the α-harmonic extension and obtain a trace inequality and two non-existence results. In § 4 we consider the α-harmonic extension in cylinders in order to define the fractional Laplacian in a bounded domain. We also study an associated linear equation in the local version. Finally, § 5 contains the results related to the non-local nonlinear problem (1.1), where we prove theorems 1.1 and 1.2.

2. The fractional Laplacian in the whole space

In order to study problem (1.1), we need to properly define the fractional Laplacian in bounded domains. This is done in § 4. Prior to this, we recall the definition of the fractional Laplacian in the whole space and some of its properties. More specifically, we obtain a trace inequality that is relevant in what follows and which also has interest in itself.

2.1. Preliminaries

The fractional Laplacian is defined in the whole space through the Fourier transform (see (1.2)). Recently, it was shown in [17] that this operator can be realized in a local way by using one more variable and the so-called α-harmonic extension. This is a main tool in our investigation.

More precisely, if u is a regular function in $\mathbb{R}^N$, we say that $w = E_\alpha(u)$ is its α-harmonic extension to the upper half-space, $\mathbb{R}^{N+1}_+$, if w is a solution to the problem

$$\begin{aligned} -\operatorname{div}(y^{1-\alpha}\nabla w) = 0 &\quad \text{in } \mathbb{R}^{N+1}_+, \\ w = u &\quad \text{on } \mathbb{R}^N \times \{y = 0\}. \end{aligned}$$

In [17] it is proved that

$$\lim_{y\to 0^+} y^{1-\alpha}\frac{\partial w}{\partial y}(x, y) = -\kappa_\alpha(-\Delta)^{\alpha/2}u(x), \tag{2.1}$$

where

$$\kappa_\alpha = \frac{2^{1-\alpha}\Gamma(1-\frac{1}{2}\alpha)}{\Gamma(\frac{1}{2}\alpha)}.$$

The appropriate functional spaces to work with are

$$X^\alpha(\mathbb{R}^{N+1}_+) \quad \text{and} \quad \dot{H}^{\alpha/2}(\mathbb{R}^N),$$

defined as the completion of $\mathcal{C}_0^\infty(\overline{\mathbb{R}_+^{N+1}})$ and $\mathcal{C}_0^\infty(\mathbb{R}^N)$, respectively, under the norms

$$\begin{aligned}\|\phi\|_{X^\alpha}^2 &= \int_{\mathbb{R}_+^{N+1}} y^{1-\alpha}|\nabla\phi(x,y)|^2\,\mathrm{d}x\,\mathrm{d}y,\\ \|\psi\|_{\dot H^{\alpha/2}}^2 &= \int_{\mathbb{R}^N} |2\pi\xi|^\alpha|\hat\psi(\xi)|^2\,\mathrm{d}\xi\\ &= \int_{\mathbb{R}^N} |(-\Delta)^{\alpha/4}\psi(x)|^2\,\mathrm{d}x.\end{aligned}$$

The extension operator is well defined for smooth functions through a Poisson kernel, whose explicit expression is given in [17]. It can also be defined in the space $\dot H^{\alpha/2}(\mathbb{R}^N)$, and, in fact,

$$\|E_\alpha(\psi)\|_{X^\alpha} = c_\alpha\|\psi\|_{\dot H^{\alpha/2}} \quad \text{for all } \psi\in\dot H^{\alpha/2}(\mathbb{R}^N), \tag{2.2}$$

where $c_\alpha = \sqrt{\kappa_\alpha}$ (see lemma 2.2). On the other hand, for a function $\phi\in X^\alpha(\mathbb{R}_+^{N+1})$, we shall denote its trace on $\mathbb{R}^N\times\{y=0\}$ as $\mathrm{tr}(\phi)$. This trace operator is also well defined and it satisfies

$$\|\mathrm{tr}(\phi)\|_{\dot H^{\alpha/2}} \leqslant c_\alpha^{-1}\|\phi\|_{X^\alpha}. \tag{2.3}$$

2.2. A trace inequality

From expression (2.3), the Sobolev embedding implies that the trace also belongs to $L^{2_\alpha^*}(\mathbb{R}^N)$, where $2_\alpha^* = 2N/(N-\alpha)$. Even the best constant associated to this inclusion is attained and can be characterized. Although most of the results used in order to prove the following theorem are known, we have collected them here for the reader's convenience.

Theorem 2.1. *For every $z\in X^\alpha(\mathbb{R}_+^{N+1})$, it holds that*

$$\left(\int_{\mathbb{R}^N}|v(x)|^{2N/(N-\alpha)}\,\mathrm{d}x\right)^{(N-\alpha)/N} \leqslant S(\alpha,N)\int_{\mathbb{R}_+^{N+1}} y^{1-\alpha}|\nabla z(x,y)|^2\,\mathrm{d}x\,\mathrm{d}y, \tag{2.4}$$

where $v=\mathrm{tr}(z)$. The best constant takes the exact value

$$S(\alpha,N) = \frac{\varGamma(\frac12\alpha)\varGamma(\frac12(N-\alpha))(\varGamma(N))^{\alpha/N}}{2\pi^{\alpha/2}\varGamma(\frac12(2-\alpha))\varGamma(\frac12(N+\alpha))(\varGamma(\frac12 N))^{\alpha/N}} \tag{2.5}$$

and it is achieved when v takes the form

$$v(x) = \tau^{(N-\alpha)/2}(|x-x_0|^2+\tau^2)^{-(N-\alpha)/2} \tag{2.6}$$

for some $x_0\in\mathbb{R}^N$, $\tau>0$ and $z=E_\alpha(v)$.

The analogous results for the classical Laplace operator can be found in [25, 36].

Lemma 2.2. *Let $v\in\dot H^{\alpha/2}(\mathbb{R}^N)$, and let $z=E_\beta(v)$ be its β-harmonic extension, $\beta\in(\frac12\alpha,2)$. Then $z\in X^\alpha(\mathbb{R}_+^{N+1})$ and, moreover, there exists a positive universal constant $c(\alpha,\beta)$ such that*

$$\|v\|_{\dot H^{\alpha/2}} = c(\alpha,\beta)\|z\|_{X^\alpha}. \tag{2.7}$$

In particular, if $\beta=\alpha$, we have $c(\alpha,\alpha)=1/\sqrt{\kappa_\alpha}$.

Inequality (2.4) only needs the case $\beta=\alpha$, which is deduced directly from the proof of the local characterization of $(-\Delta)^{\alpha/2}$ in [17]. The calculations performed in [17] can be extended to cover the range $\frac{1}{2}\alpha<\beta<2$, including, in particular, the case $\beta=1$ proved in [45].

Proof. Since $z=E_\beta(v)$, by definition z solves $\operatorname{div}(y^{1-\beta}\nabla z)=0$, which is equivalent to

$$\Delta_x z+\frac{1-\beta}{y}\frac{\partial z}{\partial y}+\frac{\partial^2 z}{\partial y^2}=0.$$

Taking the Fourier transform in $x\in\mathbb{R}^N$ for fixed $y>0$, we have

$$-4\pi^2|\xi|^2\hat{z}+\frac{1-\beta}{y}\frac{\partial\hat{z}}{\partial y}+\frac{\partial^2\hat{z}}{\partial y^2}=0$$

and $\hat{z}(\xi,0)=\hat{v}(\xi)$. Therefore, $\hat{z}(\xi,y)=\hat{v}(\xi)\phi_\beta(2\pi|\xi|y)$, where ϕ_β solves the problem

$$-\phi+\frac{1-\beta}{s}\phi'+\phi''=0,\qquad \phi(0)=1,\qquad \lim_{s\to\infty}\phi(s)=0. \tag{2.8}$$

In fact, ϕ_β minimizes the functional

$$H_\beta(\phi)=\int_0^\infty(|\phi(s)|^2+|\phi'(s)|^2)s^{1-\beta}\,\mathrm{d}s$$

and it can be shown that it is a combination of Bessel functions [34]. More precisely, ϕ_β satisfies the following asymptotic behaviour:

$$\phi_\beta(s)\sim\begin{cases}1-c_1s^\beta & \text{for } s\to0,\\ c_2s^{(\beta-1)/2}\mathrm{e}^{-s} & \text{for } s\to\infty,\end{cases} \tag{2.9}$$

where

$$c_1(\beta)=\frac{2^{1-\beta}\Gamma(1-\frac{1}{2}\beta)}{\beta\Gamma(\frac{1}{2}\beta)},\qquad c_2(\beta)=\frac{2^{(1-\beta)/2}\pi^{1/2}}{\Gamma(\frac{1}{2}\beta)}.$$

Now we observe that

$$\begin{aligned}\int_{\mathbb{R}^N}|\nabla z(x,y)|^2\,\mathrm{d}x&=\int_{\mathbb{R}^N}\left(|\nabla_x z(x,y)|^2+\left|\frac{\partial z}{\partial y}(x,y)\right|^2\right)\mathrm{d}x\\&=\int_{\mathbb{R}^N}\left(4\pi^2|\xi|^2|\hat{z}(\xi,y)|^2+\left|\frac{\partial\hat{z}}{\partial y}(\xi,y)\right|^2\right)\mathrm{d}\xi.\end{aligned}$$

Then, multiplying by $y^{1-\alpha}$ and integrating in y,

$$\begin{aligned}&\int_0^\infty\int_{\mathbb{R}^N}y^{1-\alpha}|\nabla z(x,y)|^2\,\mathrm{d}x\,\mathrm{d}y\\&\quad=\int_0^\infty\int_{\mathbb{R}^N}4\pi^2|\xi|^2|\hat{v}(\xi)|^2(|\phi_\beta(2\pi|\xi|y)|^2+|\phi_\beta'(2\pi|\xi|y)|^2)y^{1-\alpha}\,\mathrm{d}\xi\,\mathrm{d}y\\&\quad=\int_0^\infty(|\phi_\beta(s)|^2+|\phi_\beta'(s)|^2)s^{1-\alpha}\,\mathrm{d}s\int_{\mathbb{R}^N}|2\pi\xi|^\alpha|\hat{v}(\xi)|^2\,\mathrm{d}\xi.\end{aligned}$$

Using (2.9), we see that the integral

$$\int_0^\infty (|\phi_\beta|^2 + |\phi_\beta'|^2) s^{1-\alpha}\,\mathrm{d}s$$

is convergent provided $\beta > \frac{1}{2}\alpha$. This proves (2.7) with $c(\alpha,\beta) = (H_\alpha(\phi_\beta))^{-1/2}$. □

REMARK 2.3. If $\beta = 1$, we have $\phi_1(s) = \mathrm{e}^{-s}$, and $H_\alpha(\phi_1) = 2^{\alpha-1}\Gamma(2-\alpha)$ [45]. Moreover, when $\beta = \alpha$, integrating by parts and using the equation in (2.8), with (2.9), we obtain

$$\begin{aligned} H_\alpha(\phi_\alpha) &= \int_0^\infty [\phi_\alpha^2(s) + (\phi_\alpha')^2(s)] s^{1-\alpha}\,\mathrm{d}s \\ &= -\lim_{s\to 0} s^{1-\alpha}\phi_\alpha'(s) \\ &= \alpha c_1(\alpha) \\ &= \kappa_\alpha. \end{aligned} \tag{2.10}$$

LEMMA 2.4. *Let* $z \in X^\alpha(\mathbb{R}_+^{N+1})$ *and let* $w = E_\alpha(\mathrm{tr}(z))$ *be its* α*-harmonic associated function (the extension of the trace). Then*

$$\|z\|_{X^\alpha}^2 = \|w\|_{X^\alpha}^2 + \|z-w\|_{X^\alpha}^2.$$

Proof. Observe that, for $h = z - w$, we have

$$\|z\|_{X^\alpha}^2 = \int_{\mathbb{R}_+^{N+1}} y^{1-\alpha}(|\nabla w|^2 + |\nabla h|^2 + 2\langle\nabla w, \nabla h\rangle).$$

But, since $\mathrm{tr}(h) = 0$, we have

$$\int_{\mathbb{R}_+^{N+1}} y^{1-\alpha}\langle\nabla w, \nabla h\rangle\,\mathrm{d}x\,\mathrm{d}y = 0.$$

□

LEMMA 2.5. *If* $g \in L^{2N/(N+\alpha)}(\mathbb{R}^N)$*, and* $f \in \dot{H}^{\alpha/2}(\mathbb{R}^N)$*, then there exists a constant* $\ell(\alpha,N) > 0$ *such that*

$$\left|\int f(x)g(x)\,\mathrm{d}x\right| \leqslant \ell(\alpha,N)\|f\|_{\dot{H}^{\alpha/2}}\|g\|_{2N/(N+\alpha)}. \tag{2.11}$$

Moreover, the equality in (2.11) with the best constant holds when f and g take the form (2.6).

The proof follows by a standard argument that can be found, for example, in [24, 45].

Proof. By Parçeval's identity and the Cauchy–Schwarz inequality, we have

$$\begin{aligned} \left(\int_{\mathbb{R}^N} f(x)g(x)\,\mathrm{d}x\right)^2 &= \left(\int_{\mathbb{R}^N} \hat{f}(\xi)\hat{g}(\xi)\,\mathrm{d}\xi\right)^2 \\ &\leqslant \left(\int_{\mathbb{R}^N} |2\pi\xi|^\alpha |\hat{f}(\xi)|^2\,\mathrm{d}\xi\right)\left(\int_{\mathbb{R}^N} |2\pi\xi|^{-\alpha}|\hat{g}(\xi)|^2\,\mathrm{d}\xi\right). \end{aligned}$$

The second term can be written using the results of [35] as

$$\int_{\mathbb{R}^N} |2\pi\xi|^{-\alpha}|\hat{g}(\xi)|^2 \,\mathrm{d}\xi = b(\alpha,N)\int_{\mathbb{R}^{2N}} \frac{g(x)g(x')}{|x-x'|^{N-\alpha}}\,\mathrm{d}x\,\mathrm{d}x',$$

where

$$b(\alpha,N) = \frac{\Gamma(\frac{1}{2}(N-\alpha))}{2^\alpha \pi^{N/2}\Gamma(\frac{1}{2}\alpha)}.$$

We now use the following Hardy–Littlewood–Sobolev inequality [35],

$$\int_{\mathbb{R}^{2N}} \frac{g(x)g(x')}{|x-x'|^{N-\alpha}}\,\mathrm{d}x\,\mathrm{d}x' \leqslant d(\alpha,N)\|g\|^2_{2N/(N+\alpha)},$$

where

$$d(\alpha,N) = \frac{\pi^{(N-\alpha)/2}\Gamma(\frac{1}{2}\alpha)(\Gamma(N))^{\alpha/N}}{\Gamma(\frac{1}{2}(N+\alpha))(\Gamma(\frac{1}{2}N))^{\alpha/N}},$$

with equality if g takes the form (2.6). From this, we obtain the desired estimate (2.11) with the constant $\ell(\alpha,N)=\sqrt{b(\alpha,N)d(\alpha,N)}$.

When applying the Cauchy–Schwarz inequality, we obtain an identity provided the functions $|\xi|^{\alpha/2}\hat{f}(\xi)$ and $|\xi|^{-\alpha/2}\hat{g}(\xi)$ are proportional. This means

$$\hat{g}(\xi) = c|\xi|^\alpha \hat{f}(\xi) = c[(-\Delta)^{\alpha/2}f]^\wedge(\xi).$$

We conclude by observing that if g takes the form (2.6) and $g=c(-\Delta)^{\alpha/2}f$, then f also takes the form (2.6). □

Proof of theorem 2.1. We apply lemma 2.5 with $g=|f|^{(N+\alpha)/(N-\alpha)-1}f$, then use lemma 2.2 and conclude using lemma 2.4. The best constant is

$$S(\alpha,N) = \frac{\ell^2(\alpha,N)}{\kappa_\alpha}.$$

□

Related to this result, we cite [20], where it is proved that the only positive regular solutions to $(-\Delta)^{\alpha/2}f = cf^{(N+\alpha)/(N-\alpha)}$ take the form (2.6).

REMARK 2.6. If we let α tend to 2, when $N>2$, we recover the classical Sobolev inequality for a function in $H^1(\mathbb{R}^N)$, with the same constant [43]. In order to pass to the limit on the right-hand side of (2.4), at least formally, we observe that $(2-\alpha)y^{1-\alpha}\,\mathrm{d}y$ is a measure on compact sets of $\mathbb{R}_+$ converging (in the weak* sense) to a Dirac delta. Hence,

$$\lim_{\alpha\to 2^-}\int_0^1 \biggl(\int_{\mathbb{R}^N} |\nabla z(x,y)|^2\,\mathrm{d}x\biggr)(2-\alpha)y^{1-\alpha}\,\mathrm{d}y = \int_{\mathbb{R}^N} |\nabla v(x)|^2\,\mathrm{d}x.$$

We then obtain

$$\biggl(\int_{\mathbb{R}^N} |v(x)|^{2N/(N-2)}\,\mathrm{d}x\biggr)^{(N-2)/N} \leqslant S(N)\int_{\mathbb{R}^N} |\nabla v(x)|^2\,\mathrm{d}x,$$

with the best constant

$$\begin{aligned} S(N) &= \lim_{\alpha\to 2^-} \frac{S(\alpha,N)}{2-\alpha} \\ &= \frac{1}{\pi N(N-2)}\left(\frac{\Gamma(N)}{\Gamma(\frac{1}{2}N)}\right)^{2/N}. \end{aligned}$$

It is achieved when v takes the form (2.6) with α replaced by 2.

3. Some non-existence results in unbounded domains

In this section we prove two Liouville-type results in the half-space $\mathbb{R}^{N+1}_+$ and the quarter-space $\mathbb{R}^{N+1}_{++}$ that will be useful in § 5.4 in order to obtain uniform *a priori* bounds for the solutions to problem (1.1). These results have a corresponding formulation for the fractional Laplacian operator.

3.1. A problem in the half-space

THEOREM 3.1. *Let $1 \leqslant \alpha < 2$. Then the problem in the half-space $\mathbb{R}^{N+1}_+$,*

$$\left.\begin{aligned} -\operatorname{div}(y^{1-\alpha}\nabla w) &= 0 && \text{in } \mathbb{R}^{N+1}_+, \\ \frac{\partial w}{\partial \nu^\alpha}(x) &= w^p(x,0) && \text{on } \partial\mathbb{R}^{N+1}_+ = \mathbb{R}^N, \end{aligned}\right\} \tag{3.1}$$

has no positive bounded solution provided $p < (N+\alpha)/(N-\alpha)$.

Theorem 3.1 is proved in the case $\alpha = 1$ in [32]. On the other hand, Chen *et al*. [19] consider the corresponding non-local problem and perform a different proof of the same result, using integral estimates, which is valid in the whole interval $0 < \alpha < 2$.

The proof that we present here is based on the well-known *method of moving planes*, introduced by Alexandrov and first used in the context of partial differential equations by [29, 39], among others.

We begin by establishing some useful notation in order to apply the moving planes method. The points of the upper half-space $\mathbb{R}^{N+1}_+$ are denoted by $X = (x, y)$, where $x = (x_1, \dots, x_N)$ and $y > 0$. Fix $\rho > 0$ and consider the sets

$$\Sigma_\rho = \{X \in \mathbb{R}^{N+1}_+;\ x_N > \rho\}, \qquad T_\rho = \{X \in \overline{\mathbb{R}^{N+1}_+};\ x_N = \rho\}. \tag{3.2}$$

For every $X = (x, y) \in \mathbb{R}^{N+1}_+$ we define the reflection across the hyperplane T_ρ by

$$X^\rho = (x^\rho, y) = X + 2(\rho - x_N)e_N = (x_1, \dots, 2\rho - x_N, y).$$

Let us also consider the point $P_\rho = (0, \dots, 0, 2\rho, 0) \in \Sigma_\rho$, whose reflection is the origin, and the set $\tilde{\Sigma}_\rho = \bar{\Sigma}_\rho \setminus \{P_\rho\}$. Let B_r^+ denote the half-ball $B_r^+ = \{|X| < r,\ y > 0\}$ ($B_r^+(X_0)$ when the centre $X_0 = (x_0, 0)$ is not the origin), and let its non-flat part of the boundary be denoted by $S_r^+ = \{|X| = r,\ y > 0\}$ (respectively, $S_r^+(X_0)$).

We also define the operator

$$-L_\alpha w = y^{\alpha-1}\operatorname{div}(y^{1-\alpha}\nabla w) = \Delta w + \frac{1-\alpha}{y}\frac{\partial w}{\partial y} \tag{3.3}$$

in such a way that problem (3.1) can be written in the form

$$\left.\begin{aligned} L_\alpha w = 0 \quad & \text{in } \mathbb{R}^{N+1}_+, \\ \frac{\partial w}{\partial \nu^\alpha} = w^p \quad & \text{for } y = 0. \end{aligned}\right\} \tag{3.4}$$

Two easy properties of the operator L_α are used in the following:

$$\left.\begin{aligned} L_\alpha(fg) &= fL_\alpha g + gL_\alpha f + 2\langle \nabla f, \nabla g\rangle, \\ L_\alpha(|X|^\gamma) &= \gamma(\gamma + N - \alpha)|X|^{\gamma-2}. \end{aligned}\right\} \tag{3.5}$$

Finally, the so-called *fractional Kelvin transform* will also be useful. We consider, for a function f defined in $\mathbb{R}^N$, its fractional Kelvin transform as $K_\alpha(f)(x) = |x|^{\alpha-N} f(x/|x|^2)$. It is well known that this transform behaves under the action of the fractional Laplacian in a similar way to the standard Kelvin transform with the Laplacian

$$(-\Delta)^{\alpha/2} K_\alpha(f)(x) = |x|^{-\alpha-N}(-\Delta)^{\alpha/2} f(x/|x|^2).$$

We are interested in defining the analogous fractional Kelvin transform for the function w and the operator L_α. Let $z(X) = |X|^\gamma w(\xi)$, $\xi = X/|X|^2$. It is a matter of calculus to see that

$$L_\alpha z(X) = |X|^{\gamma-4}(L_\alpha w(\xi) + (\gamma + N - \alpha)|X|^2(\gamma w(\xi) - 2\langle \xi, \nabla w(\xi)\rangle)).$$

Therefore, if we choose $\gamma = \alpha - N$ and w is α-harmonic, we obtain that z is also α-harmonic, and so it turns out to be the α-harmonic extension of $K_\alpha(f)$ if w is the α-harmonic extension of f. In other words, $E \circ K_\alpha = K_\alpha \circ E$.

Now let w be any solution to problem (3.4), and set $\mu = \sup_{B_1^+} w$. Then there exists $\varepsilon > 0$ such that $w(X) \geqslant \varepsilon |X|^{\alpha-N}$ for $|X| \geqslant 1$, $y > 0$. To see this, observe that, by the Harnack inequality [14, lemma 4.8], we have $\varepsilon = \inf_{S_1^+} w \geqslant c\mu > 0$. By comparison, we conclude using (3.5) and [14, proposition 4.10]. Let $v = K_\alpha(w)$. We have that v satisfies analogous properties to w, but for the inversion variable

$$\left.\begin{aligned} v(X) \geqslant \varepsilon \quad & \text{in } B_1^+, \\ v(X) \leqslant \mu |X|^{\alpha-N} \quad & \text{in } \mathbb{R}^{N+1}_+ \setminus B_1^+, \end{aligned}\right\} \tag{3.6}$$

as well as being a solution to the problem

$$\left.\begin{aligned} -L_\alpha v = 0 \quad & \text{in } \mathbb{R}^{N+1}_+, \\ \frac{\partial v}{\partial \nu^\alpha} = |x|^{-\gamma} v^p \quad & \text{for } y = 0,\ |x| \neq 0, \end{aligned}\right\} \tag{3.7}$$

where $\gamma = (N + \alpha) - (N - \alpha)p > 0$.

We now proceed with the reflection. Let

$$\psi_\rho(X) = v(X^\rho) - v(X). \tag{3.8}$$

Clearly, $L_\alpha(\psi_\rho) = 0$ in $\mathbb{R}^{N+1}_+$. We want to prove that $\psi_\rho \geqslant 0$ in $\tilde{\Sigma}_\rho$. Recall that v may have a singularity at the origin, and therefore ψ_ρ may have a singularity at P_ρ. We begin with the following result.

Lemma 3.2. *With the above notation, we have* $\psi_\rho \geqslant 0$ *in* $\tilde{\Sigma}_\rho$, *provided* $\rho > 0$ *is sufficiently large.*

Proof. Let $\beta > 0$ be some constant to be chosen later, and let

$$\varphi_\rho(X) = |Z|^\beta \psi_\rho(X), \qquad Z = X + e_{N+1} = (x, y+1). \tag{3.9}$$

From equation (3.7), we obtain

$$L_\alpha(\varphi_\rho) - \beta y^{1-\alpha}|Z|^{-2}[(-\beta + N - \alpha)\varphi_\rho + 2\langle Z, \nabla\varphi_\rho\rangle] = 0. \tag{3.10}$$

Assume by contradiction that there exists $\delta > 0$ such that

$$\inf_{\tilde{\Sigma}_\rho} \varphi_\rho = -\delta < 0. \tag{3.11}$$

First of all, we observe that (3.6) implies

$$|\varphi_\rho| \leqslant c|X|^{\beta+\alpha-N} \to 0 \quad \text{for } |X| \to \infty,$$

if we take $\beta < N - \alpha$. On the other hand, close to the possible singularity P_ρ, we have $\varphi_\rho > 0$. In fact, if $X \in B_1^+(P_\rho)$, then $X^\rho \in B_1^+$, and then $v(X^\rho) \geqslant \varepsilon$. Since $v(X) \leqslant \mu|X|^{\alpha-N} \leqslant \mu\rho^{\alpha-N}$, we obtain

$$\varphi_\rho(X) \geqslant |Z|^\beta(\varepsilon - \mu|\rho|^{\alpha-N}) > 0 \quad \text{in } B_1^+(P_\rho),$$

provided ρ is sufficiently large. Therefore, the infimum in (3.11) is achieved in a point of regularity of φ_ρ. As for the interior points, the above choice of β gives that equation (3.10) does not allow for interior minima to exist. Finally, the fact that $\varphi_\rho = 0$ on T_ρ leads to the only possibility of the infimum being achieved, namely, on the part of the boundary $\Sigma_\rho \cap \{y = 0\}$. Then let $(x_0, 0) \in \Sigma_\rho \cap \{y = 0\}$ be such that $\varphi_\rho(x_0, 0) = -\delta$.

We claim that the boundary condition in (3.7) implies

$$\frac{\partial\varphi_\rho}{\partial\nu^\alpha}(x_0) > 0, \tag{3.12}$$

which will give the desired contradiction. It is at this point that the condition $\alpha \geqslant 1$ enters.

By Leibniz's rule, we have

$$\frac{\partial\varphi_\rho}{\partial\nu^\alpha}(x_0) = |(x_0, 1)|^\beta \frac{\partial\psi_\rho}{\partial\nu^\alpha}(x_0) + \psi_\rho(x_0, 0)\frac{\partial|Z|^\beta}{\partial\nu^\alpha}(x_0).$$

The first term is bounded below, since, by using (3.6), (3.7) and the mean value theorem, we obtain

$$\begin{aligned}\frac{\partial\psi_\rho}{\partial\nu^\alpha}(x_0) &= |x_0^\rho|^{-\gamma}v^p(x_0^\rho, 0) - |x_0|^{-\gamma}v^p(x_0, 0)\\ &\geqslant |x_0|^{-\gamma}(v^p(x_0^\rho, 0) - v^p(x_0, 0))\\ &\geqslant p|x_0|^{-\gamma}v^{p-1}(x_0, 0)\psi_\rho(x_0, 0),\end{aligned} \tag{3.13}$$

and thus

$$|(x_0, 1)|^\beta \frac{\partial\psi_\rho}{\partial\nu^\alpha}(x_0) \geqslant -p\delta|x_0|^{-\gamma-(p-1)(N-\alpha)} \geqslant -c\rho^{-2}.$$

For the second term,

$$\frac{\partial |Z|^{\beta}}{\partial \nu^{\alpha}}(x_0) = \begin{cases} 0 & \text{if } \alpha < 1, \\ -\beta |(x_0, 1)|^{\beta-2} & \text{if } \alpha = 1, \\ -\infty & \text{if } \alpha > 1. \end{cases}$$

For the case where $\alpha > 1$, we conclude that $\partial \varphi_{\rho} / \partial \nu^{\alpha}(x_0) = +\infty$. In the case where $\alpha = 1$, a sharp control of the above terms gives (3.12); this is done in [32]. In the case where $\alpha < 1$, the condition (3.12) is not necessarily true. □

The moving planes method begins with a plane in which we find some kind of symmetry and then we see how far this plane can be moved while keeping that symmetry. The above lemma, instrumental in unbounded domains, provides a 'starting plane'. The following lemma establishes that we can move that plane up to the origin.

Lemma 3.3. *Let* ρ_0 *be defined as*

$$\rho_0 = \inf\{\rho > 0;\ \varphi_{\mu} \geqslant 0 \textit{ in } \tilde{\Sigma}_{\mu} \textit{ for all } \rho < \mu < \infty\}. \tag{3.14}$$

Then $\rho_0 = 0$.

Proof. By lemma 3.2, ρ_0 is finite. Suppose that $\rho_0 > 0$. By continuity, we have $\varphi_{\rho_0} = |z|^{\beta} \psi_{\rho_0} \geqslant 0$ in $\tilde{\Sigma}_{\rho_0}$. Since $\gamma > 0$ and $\rho_0 > 0$, we have by the boundary condition that $\psi_{\rho_0} \not\equiv 0$ in $\tilde{\Sigma}_{\rho_0}$. Also, by (3.13), $\partial \psi_{\rho_0} / \partial \nu^{\alpha} \geqslant 0$ on $\{y = 0\} \cap \bar{\Sigma}_{\rho_0}$. Clearly, $L_{\alpha}(\psi_{\rho_0}) = 0$ in $\mathbb{R}^{N+1}_{+}$ and, in particular, in the set

$$R_0 = \{|X - P_{\rho_0}| = \tfrac{1}{2}|\rho_0|,\ y \geqslant 0\}.$$

Therefore, by proposition 4.10 of [14], we have $\psi_{\rho_0} > 0$ in R_0. Let $\delta = \inf_{R_0} \psi_{\rho_0} > 0$. The function ψ_{ρ_0} may have a singularity at P_{ρ_0}, so we construct the following auxiliary function. Let h_{ε} be the solution to the problem

$$\left.\begin{aligned} L_{\alpha}(h_{\varepsilon})(X) &= 0, & \varepsilon < |X - P_{\rho_0}| &< \tfrac{1}{2}|\rho_0|, & y &> 0, \\ h_{\varepsilon}(X) &= \delta, & |X - P_{\rho_0}| &= \tfrac{1}{2}|\rho_0|, & y &\geqslant 0, \\ h_{\varepsilon}(X) &= 0, & |X - P_{\rho_0}| &= \varepsilon, & y &\geqslant 0, \\ \frac{\partial h_{\varepsilon}}{\partial \nu^{\alpha}}(X) &= 0, & \varepsilon < |X - P_{\rho_0}| &< \tfrac{1}{2}|\rho_0|, & y &= 0. \end{aligned}\right\} \tag{3.15}$$

Then lemma 4.11 of [14] implies

$$\psi_{\rho_0} \geqslant h_{\varepsilon} \quad \text{in } \varepsilon \leqslant |X - P_{\rho_0}| \leqslant \tfrac{1}{2}|\rho_0|, \quad y \geqslant 0. \tag{3.16}$$

Letting $\varepsilon \to 0^{+}$ we have $\lim_{\varepsilon \to 0^{+}} h_{\varepsilon} \equiv \delta$ by the uniqueness of the α-harmonic extension. Therefore,

$$\psi_{\rho_0} \geqslant \delta \quad \text{in } 0 < |X - P_{\rho_0}| \leqslant \tfrac{1}{2}|\rho_0|, \quad y \geqslant 0. \tag{3.17}$$

Since $\varphi_{\rho_0} \geqslant \psi_{\rho_0}$ in $\tilde{\Sigma}_{\rho_0}$, we have

$$\lim_{\rho \to \rho_0} \inf_{R_0} \varphi_{\rho} \geqslant \inf_{R_0} \varphi_{\rho_0} \geqslant \delta. \tag{3.18}$$

If ρ_0 is the infimum, there exists a sequence $\rho_k \nearrow \rho_0$ such that

$$\inf_{\bar{\Sigma}_{\rho_k}} \varphi_{\rho_k} < 0. \tag{3.19}$$

Clearly, $\lim_{|X|\to\infty} \varphi_{\rho_k} = 0$. Recalling (3.18) the infimum in (3.19) must be attained at some finite point $X^k \in \bar{\Sigma}_{\rho_k} \setminus B_{|\rho_0|/2}(P_{\rho_0})$ with sufficiently small $|\rho_k - \rho_0|$. On the other hand, $X^k \notin T_{\rho_k}$, since $\varphi_{\rho_k} \equiv 0$ in T_{ρ_k}. Therefore, X^k must belong to the set

$$\{X \in \mathbb{R}^{N+1};\ y = 0,\ x_N > 0,\ |X - P_{\rho_0}|^2 \geqslant \tfrac{1}{4}|\rho_0|^2\}. \tag{3.20}$$

Reasoning as in lemma 3.2, this leads to the desired contradiction. □

Now we can deal with the proof of the main theorem in this subsection.

Proof of theorem 3.1. Let w be any solution to problem (3.1) and consider its fractional Kelvin transform $v = K_\alpha(w)$. By lemma (3.3) we have

$$v(x_1, \dots, x_N, y) \geqslant v(x_1, \dots, -x_N, y) \quad \text{for } x_N > 0.$$

The same argument fits for negative x_N giving the reverse inequality. Therefore, $v(X)$ is symmetric with respect to the x_N-axis. Obviously, we can apply this argument in every direction perpendicular to y-axis. Hence, $v(X)$ is a two-variable function, as is $w(X)$. Indeed,

$$w(X) = \phi(|x|, y) \tag{3.21}$$

for some function ϕ. Hence, w is independent of $(x_1, \dots, x_N)$, and therefore $w(X) = w(y)$.

To complete the proof we consider the problem in one dimension.

$$\left.\begin{aligned} -(y^{1-\alpha} w')' &= 0 \quad \text{for } y > 0, \\ \lim_{y\to 0^+} y^{1-\alpha} w'(y) &= w^p(0). \end{aligned}\right\} \tag{3.22}$$

The solutions of this problem are of the form $w(y) = c - c^p/\alpha y^\alpha$ with $c \geqslant 0$, which implies that the only non-negative solution is $w \equiv 0$. □

3.2. A problem in a quarter-space

THEOREM 3.4. *Let $1 \leqslant \alpha < 2$. Then the problem in the first quarter*

$$\mathbb{R}^{N+1}_{++} = \{X = (x', x_N, y) \mid x' \in \mathbb{R}^{N-1},\ x_N > 0,\ y > 0\}$$

and

$$\left.\begin{aligned} -\operatorname{div}(y^{1-\alpha}\nabla w) &= 0, \quad x_N > 0,\ y > 0, \\ \frac{\partial w}{\partial \nu^\alpha}(x', x_N) &= w^p(x', x_N, 0), \\ w(x', 0, y) &= 0, \end{aligned}\right\} \tag{3.23}$$

has no positive bounded solution, provided that $p < (N+\alpha)/(N-\alpha)$.

Theorem 3.4 is proved in the case $\alpha = 1$ in [15]. We begin with a generalization of proposition 6.1 of [21].

LEMMA 3.5. *Suppose w is a solution of the following problem:*

$$\left.\begin{aligned} -L_\alpha w \geqslant 0, \quad w \geqslant 0 \quad &\textit{in } \mathbb{R}^2_+, \\ \frac{\partial w}{\partial \nu^\alpha} \geqslant 0 \quad &\textit{for } y = 0, \end{aligned}\right\} \tag{3.24}$$

then w is a constant.

Proof. Let $X_0 = (x_0, y_0) \in \overline{\mathbb{R}^2_+}$. Given ε, $\delta \in (0,1)$, we define the function

$$\psi(X) = \varepsilon w(X_0) \log\left(\frac{|X - X_0|^2}{\delta^2}\right) + C_\delta, \tag{3.25}$$

where

$$C_\delta = \max_{\overline{S^+_\delta(X_0)}} (w(X_0) - w(X)),$$

where

$$\overline{S^+_\delta(X_0)} = \{|X - X_0| = \delta,\ y \geqslant 0\}.$$

It is clear that

$$\psi(X) \equiv C_\delta \quad \text{on } \overline{S^+_\delta(X_0)}$$

and, taking sufficiently small δ, we have

$$\psi(X) \geqslant w(X_0) \geqslant w(X_0) - w(X) \quad \text{on } \overline{S^+_{\mathrm{e}^{1/\varepsilon}}(X_0)}. \tag{3.26}$$

A direct calculation shows that if $\alpha \in (1,2)$, then

$$\begin{aligned} -L_\alpha \psi \leqslant 0 \quad &\text{in } \mathbb{R}^2_+, \\ \frac{\partial \psi}{\partial \nu^\alpha} = \infty \quad &\text{for } y = 0. \end{aligned}$$

Thus, by the maximum principle,

$$\psi(X) \geqslant w(X_0) - w(X) \quad \text{for } X \in \overline{\mathbb{R}^2_+}, \quad \delta < |X - X_0| < \mathrm{e}^{1/\varepsilon}.$$

Letting $\varepsilon \to 0$ and then $\delta \to 0$, we have $w(X_0) - w(X) \leqslant 0$ for any $X_0, X \in \overline{\mathbb{R}^2_+}$. □

LEMMA 3.6. *Let $p \geqslant 0$ and let C be a positive constant. Then there is no solution to the problem*

$$\left.\begin{aligned} -L_\alpha w = 0, \quad & 0 < w \leqslant C \textit{ in } \mathbb{R}^2_{++} = \{x > 0,\ y > 0\}, \\ \frac{\partial w}{\partial \nu^\alpha} = w^p \quad & \textit{on } \{x > 0,\ y = 0\}, \\ w = 0 \quad & \textit{on } \{x = 0,\ y \geqslant 0\}. \end{aligned}\right\} \tag{3.27}$$

Proof. First, we show that $w(x,0) \to 0$ as $x \to \infty$. Suppose, by contradiction, that there exists a sequence $\eta_m \to \infty$ as $m \to \infty$ and such that $w(\eta_m, 0) \to K > 0$. Let us define $w_m(x,y) = w(x + \eta_m, y)$. It clearly holds that

$$\left.\begin{aligned} -L_\alpha w_m = 0, \quad & 0 < w_m \leqslant C \text{ in } R_m = \{x > \eta_m,\ y > 0\}, \\ \frac{\partial w_m}{\partial \nu^\alpha} = w^p \quad & \text{on } \{x > \eta_m,\ y = 0\}, \\ w_m = 0 \quad & \text{on } \{x = \eta_m,\ y \geqslant 0\}. \end{aligned}\right\} \tag{3.28}$$

Moreover, $w_m(0,0) \to K$. So that, taking a subsequence of w_m if necessary, we have $w_m \to \tilde{w}$ with

$$\left.\begin{aligned} -L_\alpha \tilde{w} &= 0, && 0 \leqslant \tilde{w} \leqslant C \text{ in } \mathbb{R}^2_+, \\ \frac{\partial \tilde{w}}{\partial \nu^\alpha} &= \tilde{w}^p \geqslant 0 && \text{for } y = 0. \end{aligned}\right\} \tag{3.29}$$

Since $\tilde{w}(0,0) = K$, lemma 3.5 implies $\tilde{w} \equiv K$, but by the boundary condition we have that

$$\frac{\partial \tilde{w}}{\partial \nu^\alpha}(0,0) = \tilde{w}^p(0,0) = K^p > 0,$$

which leads to a contradiction. Therefore, $w(x,0) \to 0$ as $x \to \infty$.

Following [14], we define the function

$$\varPhi(x) = \frac{1}{2}\int_0^\infty y^{1-\alpha}(|w_x(x,y)|^2 - |w_y(x,y)|^2)\,\mathrm{d}y,$$

see also [15] for the case where $\alpha = 1$. Differentiating inside the integral, we have

$$\frac{1}{2}\int_0^\infty \frac{\partial}{\partial x}[y^{1-\alpha}(|w_x|^2 - |w_y|^2)]\,\mathrm{d}y = \int_0^\infty y^{1-\alpha}(w_{xx}w_x - w_y w_{xy})\,\mathrm{d}y.$$

We want to see that this integral converges. By lemma 4.3 of [14], we know that there exists some $\beta \in (0,1)$ such that $w \in C^{2,\beta}$. Moreover, by proposition 4.6 of [14],

$$\begin{aligned} &\int_0^\infty y^{1-\alpha}(|w_{xx}w_x| + |w_y w_{xy}|)\,\mathrm{d}y \\ &\qquad \leqslant M_1\biggl(\int_0^1 y^{1-\alpha}(|w_x| + |w_y|)\,\mathrm{d}y + \int_1^\infty y^{1-\alpha}(|w_x| + |w_y|)\,\mathrm{d}y\biggr) \\ &\qquad \leqslant M_2\biggl(M_3 + \int_1^\infty \frac{y^{1-\alpha}}{y+1}\,\mathrm{d}y\biggr) \\ &\qquad < \infty, \end{aligned}$$

for some constants $M_1, M_2, M_3 > 0$. Note that the last integral is convergent provided $1 < \alpha < 2$. We recall that, in the case where $\alpha = 1$, a sharper estimate is used in [15]. Now let

$$G(w) = \int_0^w f(s)\,\mathrm{d}s.$$

By dominated convergence, and since $|\nabla w(x,y)| \to 0$ as $y \to \infty$, integrating by parts yields

$$\begin{aligned} [\varPhi(x) + G(w(x,0))]_x &= \int_0^\infty y^{1-\alpha}[w_{xx}w_x - w_y w_{xy}](x,y)\,\mathrm{d}y + [f(w)w_x](x,0) \\ &= \lim_{y\to 0}[y^{1-\alpha}w_y w_x + f(w)w_x](x,y) \\ &= \lim_{y\to 0}[y^{1-\alpha}w_y w_x - y^{1-\alpha}w_y w_x](x,y) \\ &= 0. \end{aligned}$$

Therefore, $\Phi(x) + G(w(x,0))$ is constant. The rest of the proof is exactly the same as in [15]. Using that $w(x,0) \to 0$ as $x \to \infty$ and lemma 5.1 of [14], we obtain

$$\Phi(x) + G(w(x,0)) \equiv 0.$$

Since $w \equiv 0$ in $\{x = 0, y \geqslant 0\}$, it follows that

$$0 = 2\Phi(0) = \int_0^\infty |w_x|^2(0,y)\,\mathrm{d}y,$$

which implies $w_x = 0$ on $\{x = 0, y > \varepsilon\}$ for every $\varepsilon > 0$. Since L_α is a non-degenerated elliptic operator in $\{x = 0, y > \varepsilon\}$, by Hopf's lemma this leads to a contradiction. □

With these two results, a standard argument completes the proof.

Proof of theorem 3.4. By an analogous argument to the proof of theorem 3.4 for the $(x_1, \dots, x_{N-1})$ variables (with the analogous lemmas 3.2 and 3.3), it is easy to see that any positive solution of (3.23) depends only on two variables: x_N and y. Therefore, applying proposition 3.6, the proof is complete. □

4. The fractional Laplacian in a bounded domain

4.1. The α-harmonic extension

To define the fractional Laplacian in a bounded domain we follow [15] (see also [41]). The idea is to use the α-harmonic extension introduced in [17] to define the same operator in the whole space, but restricted to our bounded domain. To this end, we consider the cylinder

$$C_\Omega = \{(x,y) \colon x \in \Omega,\ y \in \mathbb{R}_+\} \subset \mathbb{R}_+^{N+1},$$

and denote by $\partial_{\mathrm{L}} C_\Omega$ its lateral boundary.

We first define the extension operator and fractional Laplacian for smooth functions.

Definition 4.1. Given a regular function u, we define its α-harmonic extension $w = E_\alpha(u)$ to the cylinder C_Ω as the solution to the problem

$$\left.\begin{aligned} -\operatorname{div}(y^{1-\alpha}\nabla w) &= 0 \quad \text{in } C_\Omega, \\ w &= 0 \quad \text{on } \partial_{\mathrm{L}} C_\Omega, \\ w &= u \quad \text{on } \Omega. \end{aligned}\right\} \tag{4.1}$$

As in the whole space, there is also a Poisson formula for the extension operator in a bounded domain, defined through the Laplace transform and the heat semigroup generator $\mathrm{e}^{t\Delta}$ (see [41] for details).

Definition 4.2. The fractional operator $(-\Delta)^{\alpha/2}$ in Ω, acting on a regular function u, is defined by

$$(-\Delta)^{\alpha/2} u(x) = -\frac{1}{\kappa_\alpha} \lim_{y \to 0^+} y^{1-\alpha} \frac{\partial w}{\partial y}(x,y), \tag{4.2}$$

where $w = E_\alpha(u)$ and κ_α is given as in (2.1).

This operator can be extended by density to a fractional Sobolev space.

4.2. Spectral decomposition

It is classical that the powers of a positive operator in a bounded domain (or in an unbounded domain, provided the spectrum is discrete) are defined through the spectral decomposition using the powers of the eigenvalues of the original operator. We show next that, in this case, this is coherent with the Dirichlet–Neumann operator defined above. Let (φ_j, λ_j) be the eigenfunctions and eigenvectors of $-\Delta$ in Ω with Dirichlet boundary data. Define the space of functions defined in our domain Ω as

$$H_0^{\alpha/2}(\Omega) = \Big\{ u = \sum a_j\varphi_j \in L^2(\Omega) \colon \|u\|_{H_0^{\alpha/2}} = \Big(\sum a_j^2\lambda_j^{\alpha/2}\Big)^{1/2} < \infty \Big\},$$

its topological dual as $H^{-\alpha/2}(\Omega)$, and the energy space $X_0^\alpha(C_\Omega)$ of functions defined in the cylinder C_Ω as the completion of $C_0^\infty(\Omega\times[0,\infty))$ with the norm

$$\|z\|_{X_0^\alpha} = \Big(\int_{C_\Omega} y^{1-\alpha}|\nabla z|^2\Big)^{1/2}.$$

Lemma 4.3.

(i) *The eigenfunctions and eigenvectors of* $(-\Delta)^{\alpha/2}$ *in* Ω *with Dirichlet boundary data are given by* $(\varphi_j, \lambda_j^{\alpha/2})$.

(ii) *If*

$$u = \sum a_j\varphi_j \in H_0^{\alpha/2}(\Omega),$$

then $E_\alpha(u) \in X_0^\alpha(C_\Omega)$ *and*

$$E_\alpha(u)(x,y) = \sum a_j\varphi_j(x)\psi(\lambda_j^{1/2}y),$$

where $\psi(s)$ *solves the problem*

$$\begin{aligned} \psi'' + \frac{1-\alpha}{s}\psi' &= \psi, \qquad s>0,\\ -\lim_{s\to0^+} s^{1-\alpha}\psi'(s) &= \kappa_\alpha,\\ \psi(0) &= 1. \end{aligned}$$

(iii) *In the same hypotheses,* $(-\Delta)^{\alpha/2}u \in H^{-\alpha/2}(\Omega)$, *and*

$$(-\Delta)^{\alpha/2}u = \sum a_j\lambda_j^{\alpha/2}\varphi_j.$$

(iv) *It holds that*

$$\|(-\Delta)^{\alpha/2}u\|_{H^{-\alpha/2}} = \|(-\Delta)^{\alpha/4}u\|_{L^2} = \|u\|_{H_0^{\alpha/2}} = \kappa_\alpha^{-1/2}\|E_\alpha(u)\|_{X_0^\alpha}.$$

The proof of this result is straightforward. The function ψ coincides with the solution ϕ_α in problem (2.8). The calculation of the norms is also straightforward. Using the orthogonality of the family $\{\varphi_j\}$, together with $\int_\Omega \varphi_j^2 = 1$, $\int_\Omega |\nabla\varphi_j|^2 = \lambda_j$ and (2.10), we have

$$\begin{aligned}
&\int_{C_\Omega} y^{1-\alpha}|\nabla E_\alpha(u)(x,y)|^2\,\mathrm{d}x\,\mathrm{d}y \\
&\quad= \int_0^\infty y^{1-\alpha}\int_\Omega \Big(\sum a_j^2|\nabla\varphi_j(x)|^2\psi(\lambda_j^{1/2}y)^2 + a_j^2\lambda_j\varphi_j(x)^2(\psi'(\lambda_j^{1/2}y))^2\Big)\,\mathrm{d}x\,\mathrm{d}y \\
&\quad= \int_0^\infty y^{1-\alpha}\sum a_j^2\lambda_j(\psi(\lambda_j^{1/2}y)^2 + (\psi'(\lambda_j^{1/2}y))^2)\,\mathrm{d}y \\
&\quad= \sum a_j^2\lambda_j^{\alpha/2}\int_0^\infty s^{1-\alpha}(\psi(s)^2 + (\psi'(s))^2)\,\mathrm{d}s \\
&\quad= \kappa_\alpha\sum a_j^2\lambda_j^{\alpha/2} \\
&\quad= \kappa_\alpha\sum (a_j\lambda_j^{\alpha/4})^2.
\end{aligned}$$

4.3. The trace inequality

Using the trace inequality in the whole space (2.4), we obtain the corresponding inequality for bounded domains. To do this, we consider $v \in X_0^\alpha(C_\Omega)$. Its extension by zero outside the cylinder C_Ω can be approximated by functions with compact support in $\mathbb{R}_+^{N+1}$. Thus, the trace inequality (2.4), together with Hölder's inequality, gives a trace inequality for bounded domains.

Theorem 4.4. *For any* $1 \leqslant r \leqslant 2N/(N-\alpha)$, *and every* $z \in X_0^\alpha(C_\Omega)$, *it holds that*

$$\left(\int_\Omega |v(x)|^r\,\mathrm{d}x\right)^{2/r} \leqslant C(r,\alpha,N,|\Omega|)\int_{C_\Omega} y^{1-\alpha}|\nabla z(x,y)|^2\,\mathrm{d}x\,\mathrm{d}y, \tag{4.3}$$

where $v = \mathrm{tr}(z)$.

4.4. The linear problem

We now use the extension problem (4.1) and the expression (4.2) to reformulate the non-local problems in a local way. Let g be a regular function and consider the following problems: the non-local problem

$$\left.\begin{aligned}
(-\Delta)^{\alpha/2}u &= g(x) && \text{in } \Omega, \\
u &= 0 && \text{on } \partial\Omega,
\end{aligned}\right\} \tag{4.4}$$

and the corresponding local one

$$\left.\begin{aligned}
-\operatorname{div}(y^{1-\alpha}\nabla w) &= 0 && \text{in } C_\Omega, \\
w &= 0 && \text{on } \partial_{\mathrm{L}}C_\Omega, \\
-\frac{1}{\kappa_\alpha}\lim_{y\to 0^+} y^{1-\alpha}\frac{\partial w}{\partial y} &= g(x) && \text{on } \Omega.
\end{aligned}\right\} \tag{4.5}$$

We want to define the concept of the solution to (4.4), which is done in terms of the solution to problem (4.5).

Definition 4.5. We say that $w \in X_0^\alpha(C_\Omega)$ is an energy solution to problem (4.5) if, for every function $\varphi \in \mathcal{C}_0^1(C_\Omega)$, it holds that

$$\int_{C_\Omega} y^{1-\alpha}\langle \nabla w(x,y), \nabla\varphi(x,y)\rangle \,\mathrm{d}x\,\mathrm{d}y = \int_\Omega \kappa_\alpha g(x)\varphi(x,0)\,\mathrm{d}x. \tag{4.6}$$

In fact, more general test functions can be used in the above formula whenever the integrals make sense. A supersolution (respectively, subsolution) is a function that verifies (4.6) with equality replaced by '$\geqslant$' (respectively, '$\leqslant$') for every non-negative test function.

Definition 4.6. We say that $u \in H_0^{\alpha/2}(\Omega)$ is an energy solution to problem (4.4) if it is the trace on Ω of a function w that is an energy solution to problem (4.5).

It is clear that a solution exists, for example, for every $g \in H^{-\alpha/2}(\Omega)$. In order to deal with problem (4.5) we shall assume, without loss of generality, that $\kappa_\alpha = 1$, by changing the function g.

This linear problem is also mentioned in [14]. There, some results are obtained using the theory of degenerate elliptic equations developed in [26]. In particular, a regularity result for bounded solutions to this problem is obtained in [14]. We prove here that the solutions are in fact bounded if g satisfies a minimal integrability condition.

Theorem 4.7. *Let w be a solution to problem (4.5). If $g \in L^r(\Omega)$, $r > N/\alpha$, then $w \in L^\infty(C_\Omega)$.*

Proof. The proof follows from the well-known Moser iterative technique, which we take from [30, theorem 8.15], and uses the trace inequality (4.3). Without loss of generality, we may assume $w \geqslant 0$, and this simplifies notation. The general case is obtained in a similar way.

We define, for $\beta \geqslant 1$ and $K \geqslant k$ (k to be chosen later), a $\mathcal{C}^1([k,\infty))$ function H as follows:

$$H(z) = \begin{cases} z^\beta - k^\beta, & z \in [k,K], \\ \text{linear}, & z > K. \end{cases}$$

Let us also define $v = w + k$, $\nu = \operatorname{tr}(v)$, and choose as test function φ:

$$\begin{aligned} \varphi = G(v) &= \int_k^v |H'(s)|^2\,\mathrm{d}s, \\ \nabla\varphi &= |H'(v)|^2\nabla v. \end{aligned}$$

Observe that it is an admissible test function, although it is not $\mathcal{C}^1$. Substituting this test function into the definition of the energy solution, we obtain, on the one hand

$$\begin{aligned}\int_{C_\Omega} y^{1-\alpha}\langle\nabla w,\nabla\varphi\rangle\,\mathrm{d}x\,\mathrm{d}y &= \int_{C_\Omega} y^{1-\alpha}|\nabla v|^2|H'(v)|^2\,\mathrm{d}x\,\mathrm{d}y\\ &= \int_{C_\Omega} y^{1-\alpha}|\nabla H(v)|^2\,\mathrm{d}x\,\mathrm{d}y\\ &\geqslant \biggl(\int_\Omega |H(\nu)|^{2N/(N-\alpha)}\,\mathrm{d}x\biggr)^{(N-\alpha)/N}\\ &= \|H(\nu)\|^2_{2N/(N-\alpha)}, \qquad (4.7)\end{aligned}$$

where the last inequality follows by (4.3), and, on the other hand,

$$\begin{aligned}\int_\Omega g(x)\varphi(x,0)\,\mathrm{d}x &= \int_\Omega g(x)G(\nu)\,\mathrm{d}x\\ &\leqslant \int_\Omega g(x)\nu G'(\nu)\,\mathrm{d}x\\ &\leqslant \frac{1}{k}\int_\Omega g(x)\nu^2|H'(\nu)|^2\,\mathrm{d}x\\ &= \frac{1}{k}\int_\Omega g(x)|\nu H'(\nu)|^2\,\mathrm{d}x. \qquad (4.8)\end{aligned}$$

Inequality (4.7), together with (4.8), leads to

$$\|H(\nu)\|_{2N/(N-\alpha)} \leqslant \biggl(\frac{1}{k}\|g\|_r\biggr)^{1/2}\|(\nu H'(\nu))^2\|^{1/2}_{r/(r-1)} = \|\nu H'(\nu)\|_{2r/(r-1)}, \qquad (4.9)$$

by choosing $k=\|g\|_r$. Letting $K\to\infty$ in the definition of H, inequality (4.9) becomes

$$\|\nu\|_{2N\beta/(N-\alpha)} \leqslant \|\nu\|_{2r\beta/(r-1)}.$$

Hence, for all $\beta\geqslant 1$, the inclusion $\nu\in L^{2r\beta/(r-1)}(\Omega)$ implies the stronger inclusion $\nu\in L^{2N\beta/(N-\alpha)}(\Omega)$, since $2N\beta/(N-\alpha) > 2r\beta/(r-1)$ provided $r > N/\alpha$. The result now follows, as in [30], by an iteration argument, starting with

$$\beta = \frac{N(r-1)}{r(N-\alpha)} > 1 \quad\text{and}\quad \nu\in L^{2N/(N-\alpha)}(\Omega).$$

This gives $\nu\in L^\infty(\Omega)$, and then $w\in L^\infty(C_\Omega)$. In fact, we get the estimate

$$\|w\|_\infty \leqslant c(\|w\|_{X^\alpha} + \|g\|_r).$$

□

Corollary 4.8. *Let w be a solution to problem (4.5). If $g\in L^\infty(\Omega)$, then $w\in \mathcal{C}^\gamma(\bar C_\Omega)$ for some $\gamma\in(0,1)$.*

Proof. Using theorem 4.7, the result follows directly from [14, lemma 4.4], where it is proved that any bounded solution to problem (4.5) with a bounded g is $\mathcal{C}^\gamma$. □

5. The nonlinear non-local problem

5.1. The local realization

We deal now with the core of the paper, i.e. the study of the non-local problem (1.1). We write that problem in the local version as follows. A solution to problem (1.1) is a function $u = \operatorname{tr}(w) > 0$, the trace of w on $\Omega \times \{y = 0\}$, where w solves the local problem

$$\left.\begin{aligned} -\operatorname{div}(y^{1-\alpha}\nabla w) &= 0 && \text{in } C_\Omega, \\ w &= 0 && \text{on } \partial_{\mathrm{L}} C_\Omega, \\ \frac{\partial w}{\partial \nu^\alpha} &= f(w), && w > 0 \text{ in } \Omega, \end{aligned}\right\} \tag{5.1}$$

where

$$\frac{\partial w}{\partial \nu^\alpha}(x) = -\lim_{y\to 0^+} y^{1-\alpha}\frac{\partial w}{\partial y}(x, y). \tag{5.2}$$

Note. In order to simplify notation in what follows we shall denote, when no confusion arises, w for the function defined in the cylinder C_Ω as well as for its trace $\operatorname{tr}(w)$ on $\Omega \times \{y = 0\}$. Also, in the above definition we have neglected the constant κ_α appearing in (4.2) by a simple rescaling.

As we have said, we shall focus on the particular nonlinearity

$$f(s) = f_\lambda(s) = \lambda s^q + s^p. \tag{5.3}$$

However many auxiliary results are proved for more general reactions f satisfying the growth condition

$$0 \leqslant f(s) \leqslant c(1 + |s|^p) \quad \text{for some } p > 0. \tag{5.4}$$

Remark 5.1. According to the previous note, the results on the coefficient λ for the local problem (5.1)–(5.3) in this section are translated into problem (1.1) with λ multiplied by $\kappa_\alpha^{p(q-1)-1}$.

Following definition 4.5, we say that $w \in X_0^\alpha(\mathcal{C}_\Omega)$ is an energy solution of (5.1) if the following identity holds

$$\int_{C_\Omega} y^{1-\alpha}\langle\nabla w, \nabla\varphi\rangle \,\mathrm{d}x\,\mathrm{d}y = \int_\Omega f(w)\varphi \,\mathrm{d}x$$

for every regular test function φ. In an analogous way we define the sub- and supersolution.

We now consider the functional

$$J(w) = \frac{1}{2}\int_{C_\Omega} y^{1-\alpha}|\nabla w|^2 \,\mathrm{d}x\,\mathrm{d}y - \int_\Omega F(w)\,\mathrm{d}x,$$

where

$$F(s) = \int_0^s f(\tau)\,\mathrm{d}\tau.$$

For simplicity of notation, we define $f(s) = 0$ for $s \leqslant 0$. Recall that the trace satisfies $w \in L^r(\Omega)$ (again, this means $\operatorname{tr}(w) \in L^r(\Omega)$) for every $1 \leqslant r \leqslant 2N/(N-\alpha)$ if

$N > \alpha$ and $1 < r \leqslant \infty$ if $N \leqslant \alpha$. In particular, if $p \leqslant (N+\alpha)/(N-\alpha)$ and f verifies (5.4), then $F(w) \in L^1(\Omega)$, and the functional is well defined and bounded from below.

It is well known that critical points of J are solutions to (5.1) with a general reaction f. We also consider the minimization problem

$$I = \inf\biggl\{\int_{C_\Omega} y^{1-\alpha}|\nabla w|^2 \,\mathrm{d}x\,\mathrm{d}y \colon w \in X_0^\alpha(C_\Omega), \int_\Omega F(w)\,\mathrm{d}x = 1\biggr\},$$

for which, by classical variational techniques, below the critical exponent the infimum I is achieved. This gives a non-negative solution in a standard way. Later on in the paper, we shall see that this infimum is positive provided $\lambda > 0$ is sufficiently small. On the contrary, for sufficiently large λ, the infimum is the trivial solution.

We now establish two preliminary results. The first result is a classical procedure of sub- and supersolutions to obtain a solution. We omit its proof.

Lemma 5.2. *Assume that there exist a subsolution w_1 and a supersolution w_2 to problem (5.1) verifying $w_1 \leqslant w_2$. Then there also exists a solution w satisfying $w_1 \leqslant w \leqslant w_2$ in C_Ω.*

The second result is a comparison result for concave nonlinearities. The proof follows the lines of the corresponding one for the Laplacian performed in [11].

Lemma 5.3. *Assume that the function $f(t)/t$ is decreasing for $t > 0$ and consider $w_1, w_2 \in X_0^\alpha(C_\Omega)$ as positive subsolution and supersolution, respectively, to problem (5.1). Then $w_1 \leqslant w_2$ in $\bar{C}_\Omega$.*

Proof. By definition we have, for the non-negative test functions φ_1 and φ_2 to be chosen in an appropriate way,

$$\int_{C_\Omega} y^{1-\alpha}\langle\nabla w_1, \nabla\varphi_1\rangle \,\mathrm{d}x\,\mathrm{d}y \leqslant \int_\Omega f(w_1)\varphi_1\,\mathrm{d}x,$$
$$\int_{C_\Omega} y^{1-\alpha}\langle\nabla w_2, \nabla\varphi_2\rangle \,\mathrm{d}x\,\mathrm{d}y \geqslant \int_\Omega f(w_2)\varphi_2\,\mathrm{d}x.$$

Now let $\theta(t)$ be a smooth non-decreasing function such that $\theta(t) = 0$ for $t \leqslant 0$, $\theta(t) = 1$ for $t \geqslant 1$, and set $\theta_\varepsilon(t) = \theta(t/\varepsilon)$. If, in the above inequalities, we set

$$\varphi_1 = w_2\theta_\varepsilon(w_1 - w_2), \qquad \varphi_2 = w_1\theta_\varepsilon(w_1 - w_2),$$

we obtain

$$I_1 \geqslant \int_\Omega w_1 w_2\biggl(\frac{f(w_2)}{w_2} - \frac{f(w_1)}{w_1}\biggr)\theta_\varepsilon(w_1 - w_2)\,\mathrm{d}x,$$

where

$$I_1 := \int_{C_\Omega} y^{1-\alpha}\langle w_1\nabla w_2 - w_2\nabla w_1, \nabla(w_1 - w_2)\rangle\theta_\varepsilon'(w_1 - w_2)\,\mathrm{d}x\,\mathrm{d}y.$$

Now we estimate I_1 as follows:

$$\begin{aligned} I_1 &\leqslant \int_{\mathcal{C}_\Omega} y^{1-\alpha}\langle\nabla w_1,(w_1-w_2)\nabla(w_1-w_2)\rangle\theta'_\varepsilon(w_1-w_2)\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_{\mathcal{C}_\Omega} y^{1-\alpha}\langle\nabla w_1,\nabla\gamma_\varepsilon(w_1-w_2)\rangle\,\mathrm{d}x\,\mathrm{d}y, \end{aligned}$$

where $\gamma'_\varepsilon(t)=t\theta'_\varepsilon(t)$. Therefore, since $0\leqslant\gamma_\varepsilon\leqslant\varepsilon$, we have

$$I_1\leqslant\int_\Omega f(w_1)\gamma_\varepsilon(w_1-w_2)\,\mathrm{d}x\leqslant c\varepsilon.$$

We conclude as in [5]. Letting ε tend to zero, we obtain

$$\int_{\Omega\cap\{w_1>w_2\}} w_1w_2\left(\frac{f(w_2)}{w_2}-\frac{f(w_1)}{w_1}\right)\mathrm{d}x\leqslant 0,$$

which, together with the hypothesis on f, gives $w_1\leqslant w_2$ in Ω. Comparison in C_Ω follows easily by the maximum principle. ☐

Now we show that the solutions to problem (5.1)–(5.4) are bounded and Hölder continuous. In § 5.4 we obtain a uniform L^∞-estimate in the case where f is given by (5.3) and the convex power is subcritical.

Proposition 5.4. *Let f satisfy (5.4) with $p<(N+\alpha)/(N-\alpha)$, and let $w\in X_0^\alpha(C_\Omega)$ be an energy solution to problem (5.1). Then $w\in L^\infty(C_\Omega)\cap\mathcal{C}^\gamma(\bar{C}_\Omega)$ for some $0<\gamma<1$.*

Proof. The proof closely follows the technique of [12]. As in the proof of theorem 4.7, we assume $w\geqslant 0$. We consider, formally, the test function $\varphi=w^{\beta-p}$, for some $\beta>p+1$. The justification of the following calculations can be made substituting φ by some approximated truncation. We therefore proceed with the formal analysis. We obtain, using the trace immersion, the inequality

$$\left(\int_\Omega w^{(\beta-p+1)N/(N-\alpha)}\right)^{(N-\alpha)/N}\leqslant C(\beta,\alpha,N,\Omega)\int_\Omega w^\beta.$$

This estimate allows us to obtain the following iterative process

$$\|w\|_{\beta_{j+1}}\leqslant C\|w\|_{\beta_j}^{\beta_j/(\beta_j-p+1)},$$

with $\beta_{j+1}=N/(N-\alpha)(\beta_j+1-p)$. To have $\beta_{j+1}>\beta_j$ we need $\beta_j>(p-1)N/(\alpha)$. Since $w\in L^{2^*_\alpha}(\Omega)$, starting with $\beta_0=2N/(N-\alpha)$, we obtain the above restriction provided $p<(N+\alpha)/(N-\alpha)$. It is clear that, in a finite number of steps, we get, for $g(x)=f(w(x,0))$, the regularity $g\in L^r$ for some $r>N/\alpha$. As a consequence, we obtain the conclusion by applying theorem 4.7 and corollary 4.8. ☐

5.2. A non-existence result

The following result relies on the use of a classical Pohozaev-type multiplier.

PROPOSITION 5.5. *Assume that f is a continuous function with primitive F, and w is a bounded energy solution to problem (5.1). Then the following Pohozaev-type identity holds*

$$\frac{1}{2}\int_{\partial_{\mathrm{L}} C_\Omega} y^{1-\alpha}\langle x,\nu\rangle|\nabla w|^2\,\mathrm{d}\sigma - N\int_\Omega F(w)\,\mathrm{d}x + \tfrac{1}{2}(N-\alpha)\int_\Omega wf(w)\,\mathrm{d}x = 0,$$

where ν is the (exterior) normal vector to $\partial\Omega$.

Proof. Just use the identity

$$\begin{aligned}\langle x,\nu\rangle y^{\alpha-1}\operatorname{div}(y^{1-\alpha}\nabla w) + \operatorname{div}[y^{1-\alpha}(\langle (x,y),\nabla w\rangle\nabla w - \tfrac{1}{2}(x,y)|\nabla w|^2)]&\\ + (\tfrac{1}{2}(N+2-\alpha)-1)|\nabla w|^2 = 0.&\end{aligned}$$

This equality for $\mathcal{C}^2$ functions can be checked using calculus. For energy solutions we use a classical approximation procedure. □

As a consequence, we obtain a non-existence result in the supercritical case for domains with particular geometry.

THEOREM 5.6. *If Ω is star-shaped and the nonlinearity f, F are as in the previous proposition and satisfy the inequality $((N-\alpha)sf(s)-2NF(s))\geqslant 0$, then problem (5.1) has no bounded solution. In particular, in the case $f(s)=|s|^{p-1}s$, this means that there is no bounded solution for any $p\geqslant (N+\alpha)/(N-\alpha)$.*

The case $\alpha=1$ has been proved in [15]. The corresponding result for the Laplacian (problem (1.1) with $\alpha=2$) comes from [38].

5.3. Proof of theorem 1.1

Here we prove theorem 1.1 in terms of the solution of the local version (5.1). For the sake of readability, we split the proof into several lemmas. From now on, we define

$$\left.\begin{aligned} -\operatorname{div}(y^{1-\alpha}\nabla w) &= 0 && \text{in } C_\Omega,\\ w &= 0 && \text{on } \partial_{\mathrm{L}} C_\Omega,\\ \frac{\partial w}{\partial \nu^\alpha} &= \lambda w^q + w^p, && w>0 \text{ in } \Omega,\end{aligned}\right\} \qquad (P_\lambda)$$

and consider the associated energy functional

$$J_\lambda(w) = \frac{1}{2}\int_{C_\Omega} y^{1-\alpha}|\nabla w|^2\,\mathrm{d}x\,\mathrm{d}y - \int_\Omega F_\lambda(w)\,\mathrm{d}x,$$

where

$$F_\lambda(s) = \frac{\lambda}{q+1}s^{q+1} + \frac{1}{p+1}s^{p+1}.$$

LEMMA 5.7. *Let Λ be defined by*

$$\Lambda = \sup\{\lambda>0\colon \textit{problem } (P_\lambda) \textit{ has a solution}\}.$$

Then $0<\Lambda<\infty$.

Proof. Consider the eigenvalue problem associated to the first eigenvalue λ_1, and let $\varphi_1 > 0$ be an associated eigenfunction (see lemma 4.3). Then, using φ_1 as a test function in (P_λ), we have that

$$\int_\Omega (\lambda w^q + w^p)\varphi_1 \, \mathrm{d}x = \lambda_1 \int_\Omega w\varphi_1 \, \mathrm{d}x. \tag{5.5}$$

Since there exist positive constants c, δ such that $\lambda t^q + t^p > c\lambda^\delta t$, for any $t > 0$ we obtain from (5.5) (recall that $w > 0$) that $c\lambda^\delta < \lambda_1$, which implies $\Lambda < \infty$.

To prove $\Lambda > 0$, we use the sub- and supersolution technique to construct a solution for any small λ. In fact, a subsolution is obtained as $\underline{w} = \varepsilon\varphi_1$, $\varepsilon > 0$ small. A supersolution is a suitable multiple of the function g solution to

$$\begin{aligned} -\operatorname{div}(y^{1-\alpha}\nabla g) &= 0 \quad \text{in } C_\Omega, \\ g &= 0 \quad \text{on } \partial_{\mathrm{L}} C_\Omega, \\ \frac{\partial g}{\partial \nu^\alpha} &= 1 \quad \text{in } \Omega. \end{aligned}$$

This proves the third statement in theorem 1.1. □

Lemma 5.8. *Problem (P_λ) has at least a positive solution for every $0 < \lambda < \Lambda$. Moreover, the family $\{w_\lambda\}$ of minimal solutions is increasing with respect to λ.*

Remark 5.9. Although this Λ is not exactly the same as that of theorem 1.1 (see remark 5.1), we have not changed the notation for the sake of simplicity.

Proof of lemma 5.8. We already proved in the previous lemma that problem (P_λ) has a solution for every small $\lambda > 0$. Another way of proving this result is to look at the associated functional J_λ. Using theorem 4.4, we have that this functional verifies

$$\begin{aligned} J_\lambda(w) &= \frac{1}{2}\int_{C_\Omega} y^{1-\alpha}|\nabla w|^2 \, \mathrm{d}x \, \mathrm{d}y - \int_\Omega F_\lambda(w) \, \mathrm{d}x \\ &\geqslant \frac{1}{2}\int_{C_\Omega} y^{1-\alpha}|\nabla w|^2 \, \mathrm{d}x \, \mathrm{d}y - \lambda C_1\biggl(\int_{C_\Omega} y^{1-\alpha}|\nabla w|^2 \, \mathrm{d}x \, \mathrm{d}y\biggr)^{(q+1)/2} \\ &\quad - C_2\biggl(\int_{C_\Omega} y^{1-\alpha}|\nabla w|^2 \, \mathrm{d}x \, \mathrm{d}y\biggr)^{(p+1)/2}, \end{aligned}$$

for some positive constants C_1 and C_2. Then, for sufficiently small λ, there exist two solutions of problem (P_λ), one given by minimization and another given by the mountain-pass theorem [4]. The proof is standard, based on the geometry of the function $g(t) = \frac{1}{2}t^2 - \lambda C_1 t^{q+1} - C_2 t^{p+1}$ (see, for example, [27] for more details). This in particular proves $\Lambda > 0$.

We now show that there exists a solution for every $\lambda \in (0, \Lambda)$. Later in the paper (see lemma 5.11), we shall prove that there are in fact at least two solutions in the whole interval $(0, \Lambda)$.

By definition of Λ, we know that there exists a solution corresponding to any value of λ close to Λ. Let us denote it by μ, and let w_μ be the associated solution. Now w_μ is a supersolution for all problems (P_λ) with $\lambda < \mu$. Take v_λ as the unique solution to problem (5.1) with $f(s) = \lambda s^q$. Obviously, v_λ is a subsolution to problem (P_λ). By lemma 5.3, $v_\lambda \leqslant w_\mu$. Therefore, by lemma 5.2, we conclude that there is a solution for all $\lambda \in (0, \mu)$ and, as a consequence, for the whole open interval $(0, \Lambda)$. Moreover, this solution is the minimal one. The monotonicity follows directly from the comparison lemma. ☐

This proves the first statement in theorem 1.1.

Lemma 5.10. *Problem (P_λ) has at least one solution if $\lambda = \Lambda$.*

Proof. Let $\{\lambda_n\}$ be a sequence such that $\lambda_n \nearrow \Lambda$. We denote by $w_n = w_{\lambda_n}$ the minimal solution to problem (P_{λ_n}). As in [5], we can prove that the linearized equation at the minimal solution has non-negative eigenvalues. Then it follows, as in [5], that $J_{\lambda_n}(w_n) < 0$. Since $J'_{\lambda_n}(w_n) = 0$, we easily obtain the bound $\|w_n\|_{X_0^\alpha(C_\Omega)} \leqslant k$. Hence, there exists a weakly convergent subsequence in $X_0^\alpha(C_\Omega)$ and, as a consequence, a weak solution of (P_λ) for $\lambda = \Lambda$. ☐

This proves the second statement in theorem 1.1.

To conclude the proof of theorem 1.1, next we show the existence of a second solution for every $0 < \lambda < \Lambda$. It is essential to have that the first solution is given as a local minimum of the associated functional, J_λ. To prove this last assertion we follow some ideas developed in [2].

Lemma 5.11. *Problem (P_λ) has at least two solutions for each $\lambda \in (0, \Lambda)$.*

Proof. Let $\lambda_0 \in (0, \Lambda)$ be fixed and consider $\lambda_0 < \bar{\lambda}_1 < \Lambda$. Take $\phi_0 = w_{\lambda_0}$, $\phi_1 = w_{\bar{\lambda}_1}$ as the two minimal solutions to problem (P_λ) with $\lambda = \lambda_0$ and $\lambda = \bar{\lambda}_1$, respectively, then, by comparison, $\phi_0 < \phi_1$. We define

$$M = \{w \in X_0^\alpha(C_\Omega) \colon 0 \leqslant w \leqslant \phi_1\}.$$

Note that M is a convex closed set of $X_0^\alpha(C_\Omega)$. Since J_{λ_0} is bounded from below in M and it is semicontinuous on M, we obtain the existence of $\underset{\sim}{\omega} \in M$ such that $J_{\lambda_0}(\underset{\sim}{\omega}) = \inf_{w \in M} J_{\lambda_0}(w)$. Let v_0 be the unique positive solution to problem

$$\left.\begin{aligned} -\operatorname{div}(y^{1-\alpha}\nabla v_0) &= 0 && \text{in } C_\Omega, \\ v_0 &= 0 && \text{on } \partial_{\mathrm{L}} C_\Omega, \\ \frac{\partial v_0}{\partial \nu^\alpha} &= v_0^q && \text{in } \Omega. \end{aligned}\right\} \tag{5.6}$$

The existence and uniqueness of this solution is clear (see lemma 5.3). Since, for $0 < \varepsilon \ll \lambda_0$ and $J_{\lambda_0}(\varepsilon v_0) < 0$, we have $\varepsilon v_0 \in M$, then $\underset{\sim}{\omega} \neq 0$. Therefore, $J_{\lambda_0}(\underset{\sim}{\omega}) < 0$. Using arguments similar to those in [42, theorem 2.4], we obtain that $\underset{\sim}{\omega}$ is a solution to problem (P_{λ_0}). There are two possibilities.

- If $\underset{\sim}{\omega} \not\equiv w_{\lambda_0}$, then the result follows.
- If $\underset{\sim}{\omega} \equiv w_{\lambda_0}$, we just have to prove that $\underset{\sim}{\omega}$ is a local minimum of J_{λ_0}. Assuming that this is true, the conclusion in part (iv) of theorem 1.1 follows by using a classical argument. The second solution is given by the mountain-pass theorem (see, for example, [4]).

We now prove that the minimal solution w_{λ_0} is in fact a local minimum of J_{λ_0}. We argue by contradiction.

Suppose that $\underline{\omega}$ is not a local minimum of J_{λ_0} in $X_0^\alpha(C_\Omega)$. Then there exists a sequence $\{v_n\} \subset X_0^\alpha(C_\Omega)$ such that $\|v_n - \underline{\omega}\|_{X_0^\alpha} \to 0$ and $J_{\lambda_0}(v_n) < J_{\lambda_0}(\underline{\omega})$.

Let $w_n = (v_n - \phi_1)^+$ and $z_n = \max\{0, \min\{v_n, \phi_1\}\}$. It is clear that $z_n \in M$ and

$$z_n(x,y) = \begin{cases} 0 & \text{if } v_n(x,y) \leqslant 0, \\ v_n(x,y) & \text{if } 0 \leqslant v_n(x,y) \leqslant \phi_1(x,y), \\ \phi_1(x,y) & \text{if } \phi_1(x,y) \leqslant v_n(x,y). \end{cases}$$

We set

$$\begin{aligned} T_n &\equiv \{(x,y) \in C_\Omega : z_n(x,y) = v_n(x,y)\}, & S_n &\equiv \operatorname{supp}(w_n), \\ \tilde{T}_n &= \bar{T}_n \cap \Omega, & \tilde{S}_n &= S_n \cap \Omega. \end{aligned}$$

Note that $\operatorname{supp}(v_n^+) = T_n \cup S_n$. We claim that

$$|\tilde{S}_n|_\Omega \to 0 \quad \text{as } n \to \infty, \tag{5.7}$$

where

$$|A|_\Omega \equiv \int_\Omega \chi_A(x)\,\mathrm{d}x.$$

By the definition of F_λ, we set

$$F_{\lambda_0}(s) = \frac{l_0}{q+1} s_+^{q+1} + \frac{1}{p+1} s_+^{p+1} \quad \text{for } s \in \mathbb{R},$$

and obtain

$$\begin{aligned} J_{\lambda_0}(v_n) &= \frac{1}{2} \int_{C_\Omega} y^{1-\alpha} |\nabla v_n|^2 \,\mathrm{d}x\,\mathrm{d}y - \int_\Omega F_{\lambda_0}(v_n)\,\mathrm{d}x \\ &= \frac{1}{2} \int_{T_n} y^{1-\alpha} |\nabla z_n|^2 \,\mathrm{d}x\,\mathrm{d}y - \int_{\tilde{T}_n} F_{\lambda_0}(z_n)\,\mathrm{d}x + \frac{1}{2} \int_{S_n} y^{1-\alpha} |\nabla v_n|^2 \,\mathrm{d}x\,\mathrm{d}y \\ &\quad - \int_{\tilde{S}_n} F_{\lambda_0}(v_n)\,\mathrm{d}x + \frac{1}{2} \int_{C_\Omega} y^{1-\alpha} |\nabla v_n^-|^2 \,\mathrm{d}x\,\mathrm{d}y \\ &= \frac{1}{2} \int_{T_n} y^{1-\alpha} |\nabla z_n|^2 \,\mathrm{d}x\,\mathrm{d}y - \int_{\tilde{T}_n} F_{\lambda_0}(z_n)\,\mathrm{d}x \\ &\quad + \frac{1}{2} \int_{S_n} y^{1-\alpha} |\nabla (w_n + \phi_1)|^2 \,\mathrm{d}x\,\mathrm{d}y - \int_{\tilde{S}_n} F_{\lambda_0}(w_n + \phi_1)\,\mathrm{d}x \\ &\quad + \frac{1}{2} \int_{C_\Omega} y^{1-\alpha} |\nabla v_n^-|^2 \,\mathrm{d}x\,\mathrm{d}y. \end{aligned}$$

Since

$$\int_{C_\Omega} y^{1-\alpha} |\nabla z_n|^2 \,\mathrm{d}x\,\mathrm{d}y = \int_{T_n} y^{1-\alpha} |\nabla v_n|^2 \,\mathrm{d}x\,\mathrm{d}y + \int_{S_n} y^{1-\alpha} |\nabla \phi_1|^2 \,\mathrm{d}x\,\mathrm{d}y$$

and

$$\int_\Omega F_{\lambda_0}(z_n)\,\mathrm{d}x = \int_{\tilde{T}_n} F_{\lambda_0}(v_n)\,\mathrm{d}x + \int_{\tilde{S}_n} F_{\lambda_0}(\phi_1)\,\mathrm{d}x,$$

by using the fact that ϕ_1 is a supersolution to (P_λ) with $l = \lambda_0$, we conclude that

$$\begin{aligned} J_{\lambda_0}(v_n) &= J_{\lambda_0}(z_n) + \frac{1}{2}\int_{S_n} y^{1-\alpha}(|\nabla(w_n+\phi_1)|^2 - |\nabla\phi_1|^2)\,\mathrm{d}x\,\mathrm{d}y \\ &\qquad - \int_{\tilde{S}_n} (F_{\lambda_0}(w_n+\phi_1) - F_{\lambda_0}(\phi_1))\,\mathrm{d}x + \frac{1}{2}\int_{C_\Omega} y^{1-\alpha}|\nabla v_n^-|^2\,\mathrm{d}x\,\mathrm{d}y \\ &\geqslant J_{\lambda_0}(z_n) + \tfrac{1}{2}\|w_n\|^2_{X_0^\alpha} + \tfrac{1}{2}\|v_n^-\|^2_{X_0^\alpha} \\ &\qquad - \int_\Omega \{F_{\lambda_0}(w_n+\phi_1) - F_{\lambda_0}(\phi_1) - (F_{\lambda_0})_u(\phi_1)w_n\}\,\mathrm{d}x \\ &\geqslant J_{\lambda_0}(\underline{\omega}) + \tfrac{1}{2}\|w_n\|^2_{X_0^\alpha} + \tfrac{1}{2}\|v_n^-\|^2_{X_0^\alpha} \\ &\qquad - \int_\Omega \{F_{\lambda_0}(w_n+\phi_1) - F_{\lambda_0}(\phi_1) - (F_{\lambda_0})_u(\phi_1)w_n\}\,\mathrm{d}x. \end{aligned}$$

On one hand, taking into account that $0 < q+1 < 2$, we obtain that

$$0 \leqslant \frac{1}{q+1}(w_n+\phi_1)^{q+1} - \frac{1}{q+1}\phi_1^{q+1} - \phi_1^q w_n \leqslant \tfrac{1}{2}q\frac{w_n^2}{\phi_1^{1-q}}.$$

The well-known Picone inequality [37] establishes

$$|\nabla u|^2 - \nabla\left(\frac{u^2}{v}\right)\cdot\nabla v \geqslant 0$$

for differentiable functions $v > 0$, $u \geqslant 0$. In our case, by an approximation argument, we obtain

$$\lambda_0 \int_\Omega \frac{w_n^2}{\phi_1^{1-q}}\,\mathrm{d}x \leqslant \|w_n\|^2_{X_0^\alpha}.$$

On the other hand, since $p+1 > 2$,

$$\begin{aligned} 0 &\leqslant \frac{1}{p+1}(w_n+\phi_1)^{p+1} - \frac{1}{p+1}\phi_1^{p+1} - \phi_1^p w_n \\ &\leqslant \tfrac{1}{2}rw_n^2(w_n+\phi_1)^{p-1} \\ &\leqslant C(p)(\phi_1^{p-1}w_n^2 + w_n^{p+1}). \end{aligned}$$

Hence, using that $p+1 < 2_\alpha^*$ and claim (5.7),

$$\int_\Omega \left\{\frac{1}{p+1}(w_n+\phi_1)^{p+1} - \frac{1}{p+1}\phi_1^{p+1} - \phi_1^p w_n\right\}\mathrm{d}x \leqslant o(1)\|w_n\|^2_{X_0^\alpha}.$$

As a consequence, we obtain that

$$\begin{aligned} J_{\lambda_0}(v_n) &\geqslant J_{\lambda_0}(\underline{\omega}) + \tfrac{1}{2}\|w_n\|^2_{X_0^\alpha}(1-q-o(1)) + \tfrac{1}{2}\|v_n^-\|^2_{X_0^\alpha} \\ &\equiv J_{\lambda_0}(\underline{\omega}) + \tfrac{1}{2}\|w_n\|^2_{X_0^\alpha}(1-q-o(1)) + o(1). \end{aligned}$$

Since $q < 1$, it results that

$$J_{\lambda_0}(\underline{\omega}) > J_{\lambda_0}(v_n) \geqslant J_{\lambda_0}(\underline{\omega}) \quad \text{for } n > n_0,$$

which is a contradiction with the main hypothesis. Hence, $\underline{\omega}$ is a minimum.

To complete the proof we have to prove claim (5.7). For small $\varepsilon > 0$, and for $\delta > 0$ (δ to be chosen later), we consider

$$\begin{aligned} E_n &= \{x \in \Omega \colon v_n(x) \geqslant \phi_1(x) \wedge \phi_1(x) > \underline{\omega}(x) + \delta\}, \\ F_n &= \{x \in \Omega \colon v_n(x) \geqslant \phi_1(x) \wedge \phi_1(x) \leqslant \underline{\omega}(x) + \delta\}. \end{aligned}$$

Using the fact that

$$\begin{aligned} 0 &= |\{x \in \Omega \colon \phi_1(x) < \underline{\omega}(x)\}| \\ &= \left| \bigcap_{j=1}^{\infty} \left\{ x \in \Omega \colon \phi_1(x) \leqslant \underline{\omega}(x) + \frac{1}{j} \right\} \right| \\ &= \lim_{j \to \infty} \left| \left\{ x \in \Omega \colon \phi_1(x) \leqslant \underline{\omega}(x) + \frac{1}{j} \right\} \right|, \end{aligned}$$

we obtain, for sufficiently large j_0, that if $\delta < 1/j_0$, then

$$|\{x \in \Omega \colon \phi_1(x) \leqslant \underline{\omega}(x) + \delta\}| \leqslant \tfrac{1}{2}\varepsilon.$$

Hence, we conclude that $|F_n|_\Omega \leqslant \frac{1}{2}\varepsilon$.

Since $\|v_n - \underline{\omega}\|_{X_0^\alpha} \to 0$ as $n \to \infty$, particularly by the trace embedding

$$\|v_n - \underline{\omega}\|_{L^2(\Omega)} \to 0,$$

we obtain that, for large $n \geqslant n_0$,

$$\tfrac{1}{2}\delta^2\varepsilon \geqslant \int_{C\Omega} |v_n - \underline{\omega}|^2 \,\mathrm{d}x \geqslant \int_{E_n} |v_n - \underline{\omega}|^2 \,\mathrm{d}x \geqslant \delta^2 |E_n|_\Omega.$$

Therefore, $|E_n|_\Omega \leqslant \frac{1}{2}\varepsilon$. Since $\tilde{S}_n \subset F_n \cup E_n$, we conclude that $|\tilde{S}_n|_\Omega \leqslant \varepsilon$ for $n \leqslant n_0$. Hence, $|\tilde{S}_n|_\Omega \to 0$ as $n \to \infty$, and the claim follows. ☐

5.4. Proof of theorem 1.2 and further results

We start with the uniform L^∞-estimates for solutions to problem (1.1) in its local version given by (P_λ).

THEOREM 5.12. *Assume* $\alpha \geqslant 1$, $p < (N+\alpha)/(N-\alpha)$ *and* $N \geqslant 2$. *Then there exists a constant* $C = C(p, \Omega) > 0$ *such that every solution to problem* (P_λ) *satisfies*

$$\|w\|_\infty \leqslant C$$

for every $0 \leqslant \lambda \leqslant \Lambda$.

The proof is based on a scaling method of [28] and two non-existence results (see theorems 3.1 and 3.4).

Proof of theorem 5.12. By contradiction, assume that there exists a sequence $\{w_n\} \subset X_0^\alpha(C_\Omega)$ of solutions to (P_λ) verifying that $M_n = \|w_n\|_\infty \to \infty$ as $n \to \infty$. By the maximum principle, which holds for our problem [26], the maximum of w_n is attained at a point $(x_n, 0)$, where $x_n \in \Omega$. We define $\Omega_n = 1/\mu_n(\Omega - x_n)$, with $\mu_n = M_n^{(1-p)/\alpha}$, i.e. we centre at x_n and dilate by $1/\mu_n \to \infty$ as $n \to \infty$.

We consider the scaled functions

$$v_n(x,y)=\frac{w_n(x_n+\mu_n x,\mu_n y)}{M_n}\quad\text{for } x\in\Omega_n,\ y\geqslant 0.$$

It is clear that $\|v_n\|\leqslant 1$, $v_n(0,0)=1$ and, moreover,

$$\left.\begin{aligned}-\operatorname{div}(y^{1-\alpha}\nabla v_n)&=0 &&\text{in } C_{\Omega_n},\\ v_n&=0 &&\text{on } \partial_{\mathrm{L}}C_{\Omega_n},\\ \frac{\partial v_n}{\partial\nu^\alpha}&=\lambda M_n^{q-p}v_n^q+v_n^p &&\text{in } \Omega_n\times\{0\}.\end{aligned}\right\}\tag{5.8}$$

By the Arzelà–Ascoli theorem (the solution is $\mathcal{C}^\gamma$; see proposition 5.4), there exists a subsequence, which we again denote by v_n, which converges to some function v as $n\to\infty$. In order to see the problem satisfied by v we pass to the limit in the weak formulation of (5.8). We observe that $\|v_n\|_\infty\leqslant 1$ implies $\|v_n\|_{X_0^\alpha(C_\Omega)}\leqslant C$, since

$$\int_{C_\Omega}y^{1-\alpha}|\nabla v_n|^2=\lambda M_n^{q-p}\int_\Omega v_n^{q+1}+\int_\Omega v_n^{p+1}\leqslant C.$$

Defining $d_n=\operatorname{dist}(x_n,\partial\Omega)$, there are two possibilities as $n\to\infty$ with regard to the behaviour of the ratio d_n/μ_n:

(i) $\{d_n/\mu_n\}_n$ is not bounded;

(ii) $\{d_n/\mu_n\}_n$ remains bounded.

In the first case, since $B_{d_n/\mu_n}(0)\subset\Omega_n$, and Ω_n is smooth, it is clear that Ω_n tends to $\mathbb{R}^N$ and v is a solution to

$$\begin{aligned}-\operatorname{div}(y^{1-\alpha}\nabla v)&=0 &&\text{in } \mathbb{R}^{N+1}_+,\\ \frac{\partial v}{\partial\nu^\alpha}&=v^p &&\text{on } \partial\mathbb{R}^{N+1}_+.\end{aligned}$$

Moreover, $v(0,0)=1$ and $v>0$, which is a contradiction with theorem 3.1.

In the second case, we may assume that $d_n/\mu_n\to s\geqslant 0$ as $n\to\infty$. As a consequence, passing to the limit, the domains Ω_n converge (up to a rotation) to some half-space $H_s=\{x\in\mathbb{R}^N: x_N>-s\}$. Here we obtain that v is a solution to

$$\begin{aligned}-\operatorname{div}(y^{1-\alpha}\nabla v)&=0 &&\text{in } H_s\times(0,\infty),\\ \frac{\partial v}{\partial\nu^\alpha}&=v^p &&\text{on } H_s\times\{0\},\end{aligned}$$

with $\|v\|_\infty=1$, $v(0,0)=1$. In the case where $s=0$, this is a contradiction with the continuity of v. If $s>0$, the contradiction comes from theorem 3.4. □

Next we prove a uniqueness result for solutions with a small norm.

THEOREM 5.13. *There exists at most one solution to problem* (P_λ) *with a small norm.*

We follow the arguments in [5] closely, thereby establishing the following result.

LEMMA 5.14. *Let z be the unique solution to problem (5.6). There exists a constant $\beta > 0$ such that*

$$\|\phi\|^2_{X_0^\alpha(C_\Omega)} - q\int_\Omega z^{q-1}\phi^2\,\mathrm{d}x \geqslant \beta\|\phi\|^2_{L^2(\Omega)} \quad \textit{for all } \phi \in X_0^\alpha(C_\Omega). \tag{5.9}$$

Proof. We recall that z can be obtained by minimization as follows:

$$\min\left\{\tfrac{1}{2}\|\omega\|^2_{X_0^\alpha(C_\Omega)} - \frac{1}{q+1}\|w\|^{q+1}_{L^{q+1}(\Omega)} : \omega \in X_0^\alpha(C_\Omega)\right\}.$$

As a consequence,

$$\|\phi\|^2_{X_0^\alpha(C_\Omega)} - q\int_\Omega z^{q-1}\phi^2\,\mathrm{d}x \geqslant 0 \quad \text{for all } \phi \in X_0^\alpha(C_\Omega).$$

This implies that the first eigenvalue a_1 of the linearized problem

$$\begin{aligned} -\operatorname{div}(y^{1-\alpha}\nabla\phi) &= 0 && \text{in } C_\Omega,\\ \phi &= 0 && \text{on } \partial_{\mathrm{L}}C_\Omega,\\ \frac{\partial\phi}{\partial\nu^\alpha} - qz^{q-1}\phi &= a\phi && \text{on } \Omega\times\{0\}, \end{aligned}$$

is non-negative.

Assume first that $a_1 = 0$ and let φ be a corresponding eigenfunction. Taking into account that z is the solution to (5.6), we obtain that

$$q\int_\Omega z^q\varphi\,\mathrm{d}x = \int_\Omega z^q\varphi\,\mathrm{d}x,$$

which is a contradiction.

Hence $a_1 > 0$, which proves (5.9). □

Proof of theorem 5.13. Consider $A > 0$ such that $pA^{p-1} < \beta$, where β is given in (5.9). Now we prove that problem (P_λ) has at most one solution with L^∞-norm less than A.

Assume by contradiction that (P_λ) has a second solution $w = w_\lambda + v$ verifying $\|w\|_\infty < A$. Since w_λ is the minimal solution, it follows that $v > 0$ in $\Omega\times[0,\infty)$. We now define $\eta = \lambda^{1/(1-q)}z$, where z is the solution to (5.6). Then it verifies $-\operatorname{div}(y^{1-\alpha}\nabla\eta) = 0$ with boundary condition $\lambda\eta^q$. Moreover, w_λ is a supersolution to the problem that η verifies. Then, by comparison, lemma 5.3, applied with $f(t) = \lambda t^q$, $v = \eta$ and $w = w_\lambda$, yields

$$w_\lambda \geqslant \lambda^{1/(1-q)}z \quad \text{on } \Omega\times\{0\}. \tag{5.10}$$

Since $w = w_\lambda + v$ is a solution to (P_λ), we have, on $\Omega\times\{0\}$,

$$\begin{aligned} \frac{\partial(w_\lambda + v)}{\partial\nu^\alpha} &= \lambda(w_\lambda + v)^q + (w_\lambda + v)^p\\ &\leqslant \lambda w_\lambda^q + \lambda q w_\lambda^{q-1}v + (w_\lambda + v)^p, \end{aligned}$$

where the inequality is a consequence of the concavity; hence,

$$\frac{\partial v}{\partial \nu^\alpha} \leqslant \lambda q w_\lambda^{q-1} v + (w_\lambda + v)^p - w_\lambda^p.$$

Moreover, (5.10) implies $w_\lambda^{q-1} \geqslant \lambda^{-1} z^{q-1}$. From the previous two inequalities we obtain

$$\frac{\partial v}{\partial \nu^\alpha} \leqslant q z^{q-1} v + (w_\lambda + v)^p - w_\lambda^p.$$

Using that $\|w_\lambda + v\|_\infty \leqslant A$, we obtain $(w_\lambda + v)^p - w_\lambda^p \leqslant pA^{p-1}v$. As a consequence,

$$\frac{\partial v}{\partial \nu^\alpha} - q z^{q-1} v \leqslant pA^{p-1} v.$$

Taking v as a test function and $\phi = v$ in (5.9), we arrive at

$$\beta \int_\Omega v^2 \,\mathrm{d}x \leqslant pA^{p-1} \int_\Omega v^2 \,\mathrm{d}x.$$

Since $pA^{p-1} < \beta$, we conclude that $v \equiv 0$, which gives the desired contradiction. ☐

Remark 5.15. This proof also provides the asymptotic behaviour of w_λ near $\lambda = 0$, namely, $w_\lambda \approx \lambda^{1/(1-q)} z$, where z is the unique solution to problem (5.6).

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