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ON THE COMPLETE RING OF QUOTIENTS

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In [2: p. 415], P. Gabriel proves that if R is a ring with 1 and S is a non-empty multiplicative set such that $0 \notin S$, then $S^{-1}R$ exists if and only if for every pair $(a, s) \in R \times S$, there is a pair $(b, t) \in R \times S$ such that $at = sb$ and if $s_1 a = 0$ for some s_1 in S then $as_2 = 0$ for some s_2 in S. The purpose of this note is to give a self contained elementary proof of Gabriel's result.

THEOREM 1. *Let R be a ring (not necessarily with 1) and S be a non-empty multiplicative set such that $0 \notin S$. Then the following statements are equivalent:*

(1) *For every pair $(a, s) \in R \times S$, there is a pair $(b, t) \in R \times S$ such that $at = sb$ and if a is an element of R such that $s_1 a = 0$ for some s_1 in S then $as_2 = 0$ for some s_2 is S,*

(2) *There is a non-zero ring $S^{-1}R$ with an identity element and a ring homomorphism θ from R into $S^{-1}R$ such that* (i) *For every $s \in S$, $\theta(s)$ is a unit in $S^{-1}R$,* (ii) *$\theta(a) = 0$ implies that $as = 0$ for some $s \in S$,* (iii) *Every element of $S^{-1}R$ is of the form $\theta(a)\theta(s)^{-1}$ for some $a \in R$ and $s \in S$,* (iv) *If there is a ring homomorphism g from R into a ring B such that $g(s)$ is a unit in B for every s in S and every element of B is of the form $g(a)g(s)^{-1}$ for some $a \in R$, $s \in S$, then there is a unique homomorphism h from $S^{-1}R$ into B such that $h \circ \theta = g$.*

The fact that (1) is a consequence of (2) is fairly easy to see. For if $(a, s) \in R \times S$ then $\theta(s)^{-1}\theta(a) = \theta(b)\theta(t)^{-1}$ for some $(b, t) \in R \times S$. Since every element of $S^{-1}R$ is of the form $\theta(b)\theta(t)^{-1}$ for some $(b, t) \in R \times S$. Hence $\theta(a)\theta(t) = \theta(s)\theta(b)$ and $\theta(at - sb) = 0$. Thus $(at - sb)s_1 = 0$ for some s_1 in S and $ats_1 = abs_1$. Since $\{t, s_1\} \subseteq S$ and S is a multiplicative set, ts_1 is an element of S.

In order to prove that (1) implies (2), we use a concept of *partial homomorphism* which is introduced in [1] and [3]. Recall that if R is a ring, and B and A are right R-modules, and if D is any R-submodule of B, then a R-homomorphism of D into A is called a *partial homomorphism* from B into A.

LEMMA 1. *Let R_R be the regular right R-module (i.e. the module operation is the given ring multiplication). Let H be the set of all partial R-homomorphisms from R_R into R_R. Define $H(S) = \{f \in H \mid \operatorname{dom} f \cap S \neq \phi\}$. For every f, g in $H(S)$, define $(f+g)(x) = f(x) + g(x)$ for every $x \in \operatorname{dom} f \cap \operatorname{dom} g$ and $(f \circ g)(x) = f(g(x))$ for every $x \in g^{-1}(\operatorname{dom} f)$. Then $(H(S), +)$ is an abelian group and $(H(S), 0)$ is a semigroup.*

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Proof. Let $s \in S \cap \operatorname{dom} f$ and $t \in S \cap \operatorname{dom} g$. Then there exist $s_1 \in S$ and $b \in R$ such that $ss_1 = tb$. Hence $S \cap (\operatorname{dom} f \cap \operatorname{dom} g) \neq \phi$ and $f+g \in H(S)$. Since $g(t)$ is an element of R, there is s_2 in S and a in R such that $g(t)s_2 = sa$. Therefore $ts_2 \in S \cap g^{-1}(\operatorname{dom} f)$ and $f \circ g \in H(S)$. It is clear that $(H(S), +)$ is an abelian group and $(H(S), 0)$ is a semigroup.

LEMMA 2. *For every f, g in $H(S)$, define $f \sim g$ if and only if $f(s) = g(s)$ for some s in S. Then $\sim$ is a congruence relation with respect to the addition and multiplication of $H(S)$ and $H(S)/\sim$ is a ring with an identity.*

Proof. Clearly, the relation is reflexive and symmetric. Suppose $f \sim g$ and $g \sim h$ for some f, g and h in $H(S)$. There exist s and t in S such that $f(s) = g(s)$ and $g(t) = h(t)$. There exist $s_1 \in S$ and $a \in S$ such that $ss_1 = ta$. Hence $f(ss_1) = g(ss_1) = g(ta) = h(ta) = h(ss_1)$ and $f \sim h$. Thus the relation is an equivalence relation on $H(S)$. Now suppose $f \sim g$ and $f' \sim g'$ for some f, f', g and g' in $H(S)$. Then there exist s, t in S such that $f(s) = g(s)$ and $f'(t) = g'(t)$. Let $a \in R$ and $s_1 \in S$ such that $ss_1 = ta$. Then

$$(f+f')(ss_1) = f(ss_1) + f'(ss_1) = g(ss_1) + g'(ta) = g(ss_1) + g'(ss_1) = (g+g')(ss_1)$$

and

$$f+f' \sim g+g'.$$

Now, there exist $s_1 \in S$ and $a \in R$ such that $f'(t)s_1 = sa$. Hence

$$f \circ f'(ts_1) = f(f'(ts_1)) = f(sa) = g(sa) = g(f'(ts_1)) = g(g'(ts_1)) = g \circ g'(ts_1).$$

Thus $f \circ f' \sim g \circ g'$. If $f \in H(S)$, let $[f]$ be the equivalence class represented by f. Define $[f] + [g] = [f+g]$ and $[f] \cdot [g] = [f \circ g]$ for every f, g in $H(S)$. For every f, g and h in $H(S)$ we claim that $[f] \cdot ([g] + [h]) = [f] \cdot [g] + [f] \cdot [h]$ and $([g] + [h]) \cdot [f] = [g] \cdot [f] + [h] \cdot [f]$. Recall that if $s_0 \in S \cap \operatorname{dom} f$, $s \in S \cap \operatorname{dom} g$ and $t \in S \cap \operatorname{dom} h$, then $ss_1 \in \operatorname{dom} g+h$ for some s_1 in S and $ss_1s_2 = s_0a$ for some s_2 in S and $a \in R$ such that $ss_1s_2 \in S \cap (g+h)^{-1}(\operatorname{dom} f)$ (refer a proof of Lemma 1). Hence $f \circ (g+h) \sim f \circ g + f \circ h$. Similarly, $(g+h) \circ f \sim g \circ f + h \circ f$. Thus $H(S)/\sim$ is a ring with an identity.

DEFINITION. For each a in R, let $t_a(x) = ax$ for every x in R.

LEMMA 3. *Let $\Gamma(R, S) = H(S)/\sim$. Then there is a ring homomorphism η from R into $\Gamma(R, S)$ such that $\eta(s)$ is a regular element for every s in S. If every element of S is regular in R, then η is a monomorphism and every element of $\Gamma(R, S)$ is of the form $\eta(a)\eta(s)^{-1}$ for some a in R and s in S.*

Proof. Define $\eta(a) = [t_a]$ for every a in R. Clearly, η is a ring homomorphism of R into $\Gamma(R, S)$. If $[t_i] \cdot [f] = 0$, for some s in S and $f \in H(S)$, then $sf(s') = 0$ for some $s' \in S \cap \operatorname{dom} f$. Hence, $f(s') = s'' = 0$ for some s'' in S and $[f] = 0$. If $[f] \cdot [t_s] = 0$, then $f(ss_1) = 0$ for some s_1 in S and $[f] = 0$. Thus $\eta(s)$ is a regular element of $\Gamma(R, S)$ for every $s \in S$. Now, suppose every element of S is regular in R Then

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clearly, the kernel of η is zero and for every t_s, there exist f, g in $H(S)$ such that $t_s \circ f(sx)=sx$ and $g \circ t_s(x)=x$ for every x in R. Hence $f(s^2)=g(fs^2)$ and $[f]=[g]=[t_s]^{-1}$. Now, let $[f]$ be an arbitrary element of $\Gamma(R, S)$. Then $f(s)=a$ for some $s \in S$ and $a \in R$ and $[f]\cdot[t_s]=[t_a]$. Thus $[f]=[t_a]\cdot[t_s]^{-1}$.

Proof of Theorem. We have already shown that (2) implies (1). So assume (1). Then in the ring $\Gamma(R, S)$, $\eta(s)$ is a regular element for every s in S. Let $\bar{S}=\eta(S)$ and let $\bar{R}=\eta(R)$. Then clearly $\bar{S}$ is a multiplicative set in the ring $\bar{R}$, every element of $\bar{S}$ is a regular element in $\bar{R}$ and furthermore, if $[t_s] \in \bar{S}$ and $[t_a] \in \bar{R}$ for some s in S and a in R, then $[t_a]\cdot[t_{s_1}]=[t_s]\cdot[t_b]$ for some s_1 in S and b in R. Thus by Lemma 3, there is a monomorphism, say ϕ from $\bar{R}$ into the ring $\Gamma(\bar{R}, \bar{S})$ such that $\phi([t_s])$ is a unit element for every $[t_s]$ in $\bar{S}$ and every element of $\Gamma(\bar{R}, \bar{S})$ is of the form $\phi([t_a])\phi([t_s])^{-1}$ for some $[t_a]$ in $\bar{R}$ and $[t_s]$ in $\bar{S}$. Let $S^{-1}R=\Gamma(\bar{R}, \bar{S})$ and define $\theta=\phi \circ \eta$. Then θ is a ring homomorphism of R into $S^{-1}R$ and if $\theta(a)=0$, then $\phi(\eta(a))=0$ and $\eta(a)=0$ since ϕ is a monomorphism. Hence $[t_a]=0$ and $as=0$ for some $s \in S$. If $as=0$ for some a in R and s in S, then $\eta(a)\eta(s)=0$ and $\eta(a)=0$ since $\eta(s)$ is a unit and therefore $\theta(a)=0$. By Lemma 3 and by the definition of θ, if $s \in S$, then $\theta(s)$ is a unit element in $S^{-1}R$ and every element of $S^{-1}R$ is of the form $\theta(a)\theta(s)^{-1}$ for some $a \in R$ and $s \in S$. Suppose that g is a ring homomorphism from R into a ring B such that $g(s)$ is a unit in B for every s in S and if $b \in B$, then $b=g(a)g(s)^{-1}$ for some $a \in R$ and $s \in S$. Define h from $S^{-1}R$ into B by $h(\theta(a)\theta(s)^{-1})=g(a)g(s)^{-1}$ for every $a \in R$ and $s \in S$. If $\theta(a)\theta(s)^{-1}=0$, then $\theta(a)=0$ and $as'=0$ for some s' in S. Therefore, $g(a)g(s')=0$ and $g(a)=0$. Hence $h(0)=0$. Consider $h(\theta(a)\theta(s)^{-1}+\theta(b)\theta(t)^{-1})$ for some a, b in R and s, t in S. There exist s_1 in S and c in R such that $ss_1=tc$. Hence

$$\begin{aligned}\theta(a)\theta(s)^{-1}+\theta(b)\theta(t)^{-1} &= (\theta(a)\theta(s)^{-1}+\theta(b)\theta(t)^{-1})\theta(s)\theta(s_1)\theta(s_1)^{-1}\theta(s)^{-1}\\ &= (\theta(a)\theta(s_1)+\theta(b)\theta(t)^{-1}\theta(s)\theta(s_1))\theta(s_1)^{-1}\theta(s)^{-1}\\ &= (\theta(a)\theta(s_1)+\theta(b)\theta(c))[\theta(s)\theta(s_1)]^{-1}\end{aligned}$$

since

$$\theta(t)^{-1} = \theta(c)\theta(ss_1)^{-1}.$$

Thus

$$\begin{aligned}h(\theta(a)\theta(s)^{-1}+\theta(b)\theta(t)^{-1}) &= [g(as_1)+g(bc)]g(ss_1)^{-1}\\ &= g(a)g(s)^{-1}+g(b)g(c)g(s_1)^{-1}g(s)^{-1}\\ &= g(a)g(s)^{-1}+g(b)g(t)^{-1}\\ &= h(\theta(a)\theta(s)^{-1})+h(\theta(b)\theta(t)^{-1}). \text{ Since } g(t)^{-1} = g(c)g(ss_1)^{-1}.\end{aligned}$$

Now consider $h[\theta(a)\theta(s)^{-1}\theta(b)\theta(t)^{-1}]$. There exist s_1 in S and a_1 in R such that $bs_1=sa_1$. Hence $\theta(b)\theta(s_1)=\theta(s)\theta(a_1)$ and

$$\begin{aligned}\theta(a)\theta(s)^{-1}\theta(b)\theta(t)^{-1} &= \theta(a)\theta(s)^{-1}\theta(s)\theta(a_1)\theta(s_1)^{-1}\theta(t)^{-1}\\ &= \theta(a)\theta(a_1)\theta(s_1)^{-1}\theta(t)^{-1} = \theta(aa_1)\theta(ts_1)^{-1}.\end{aligned}$$

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Thus

$$\begin{aligned} h[\theta(a)\theta(s)^{-1}\theta(b)\theta(t)^{-1}] &= g(aa_1)g(ts_1)^{-1} \\ &= g(a)g(a_1)g(s_1)^{-1}g(t)^{-1} \\ &= g(a)g(s)^{-1}g(b)g(t)^{-1} \\ &= g(a)g(s)^{-1}g(b)g(t)^{-1} = h(\theta(a)\theta(s)^{-1})h(\theta(b)\theta(t)^{-1}) \end{aligned}$$

since $g(s)^{-1}g(b)=g(a_1)g(s_1)^{-1}$. Clearly, $h \circ \theta = g$ and h is unique.

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