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THE LICHTENBAUM-QUILLEN CONJECTURE FOR FIELDS

J. F. JARDINE

1. **Preface.** I want to say immediately that, despite the authoritative-sounding title, I am not claiming a proof of anything like the Lichtenbaum-Quillen conjecture. My intent here is only to explain the conjecture in various special cases, but maybe from an idiosyncratic point of view.

This paper is based on the text of the Coxeter-James Lecture given at the Winter Meeting of the Canadian Mathematical Society at Montreal in December, 1992. I would like to thank the Society for awarding me the honour of giving this talk.

2. **K_0, K_1 and K_2.** Suppose that R is a commutative ring with 1. In the beginning, there was Grothendieck's group $K_0(R)$. This group is the group completion of the abelian monoid of isomorphism classes of finitely generated projective R-modules under direct sum. In more prosaic terms, this means that $K_0(R)$ is the free abelian group on the isomorphism classes $[P]$ of finitely generated projective R-modules P, modulo the relation

$$[P_1 \oplus P_2] = [P_1] + [P_2].$$

Another way of saying the same thing would be to assert a relation

$$[Q] = [P_1] + [P_2]$$

in the presence of an exact sequence

$$0 \longrightarrow P_1 \longrightarrow Q \longrightarrow P_2 \longrightarrow 0$$

of projective R-modules, because every such sequence splits.

Some examples are easy to compute: $K_0(\mathbb{Z}) \cong \mathbb{Z}$ since every finitely generated projective abelian group is free; similarly $K_0(F) \cong \mathbb{Z}$ for any field F. Others are more interesting: if A is the ring of integers in a number field F, then the kernel of the map

$$K_0(A) \xrightarrow{i^*} K_0(F) \cong \mathbb{Z}$$

induced by the inclusion $i: A \subset F$ can be identified up to isomorphism with the ideal class group $\mathrm{Cl}(A)$. We have explicitly used the functorial structure of $K_0(R)$ here. Any

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ring homomorphism $f\colon R \longrightarrow S$ induces a base change homomorphism $f^*\colon K_0(R) \longrightarrow K_0(S)$, which is defined by $[P] \longmapsto [S \otimes_R P]$. Tensor product also determines a ring structure on $K_0(R)$, which is respected by base change.

The notation f^* has been used here, along with the term "base change", because it's not a big secret that K_0 is a contravariant invariant on the category of schemes. Projective modules on a ring R are vector bundles on the associated scheme $\mathrm{Sp}(R)$, meaning that they're locally free finitely generated sheaves. This latter definition simultaneously makes sense for both schemes and topological spaces, as does the exact sequence description of the relation that is used to construct K_0, and so we get $K_0(X)$ for schemes X and topological spaces X.

In the topological setting, an isomorphism class $[\alpha]$ of complex vector bundles of rank n on a finite CW-complex Y has a homotopy classification in the sense that $[\alpha]$ may be identified with a homotopy class of maps $Y \longrightarrow BU_n$, where BU_n is the classifying space of the unitary group U_n, obtained by your favourite construction. The canonical inclusions $U_n \hookrightarrow U_{n+1}$ induce maps of classifying spaces $BU_n \longrightarrow BU_{n+1}$, and so we are entitled to a space

$$BU = \varinjlim_n BU_n.$$

There is a natural isomorphism

$$[Y, BU] \cong \tilde{K}_0(Y) = \ker\{rk\colon K_0(Y) \longrightarrow \mathbb{Z}\},$$

where rk is the rank homomorphism. The main calculational theorem in this area (at least for the purposes of this talk) is the complex Bott periodicity theorem:

THEOREM 1 (BOTT PERIODICITY). *There is a homotopy equivalence*

$$\Omega^2 BU \simeq BU \times \mathbb{Z}.$$

The space $\Omega^2 BU$ is the second loop space of BU, so that $\pi_{i+2}BU \cong \pi_i \Omega^2 BU$. The space $BU \times \mathbb{Z}$ is a disjoint union of copies of BU, indexed over the integers. Thus, there are isomorphisms

$$\pi_{i+2}BU \cong \pi_i \Omega^2 BU \cong \pi_i(BU \times \mathbb{Z})$$

for $i \geq 0$. In particular, low degree calculations of the homotopy groups of the unitary group itself imply that there are isomorphisms

$$\pi_i(BU \times \mathbb{Z}) \cong \begin{cases} 0 & \text{if } i \text{ is odd, and} \\ \mathbb{Z} & \text{if } i \text{ is even.} \end{cases}$$

This is an old result (late 1950's), and there are many proofs. The one I like currently is due to Bruno Harris [7]; his method is to show that the nerve $B(\sqcup_n BU_n)$ of the topological monoid $\sqcup_n BU_n$ (with monoidal structure given by direct sum) has the homotopy type of the unitary group U.

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The group $K_1(R)$ of a ring R was originally defined in a paper by Bass, Heller and Swan [1] as a type of universal determinant. Explicitly, they defined $K_1(R)$ to be the quotient of the K_0-group $K_0\big(\mathrm{Aut}\ P(R)\big)$ on the category of automorphisms of finitely generated projective R-modules, by the relation $[fg] = [f]+[g]$. An element of the general linear group $\mathrm{Gl}_n(R)$ is, by definition, an automorphism of the free R-module R^n, and so there is a canonical map

$$\phi_n\colon \mathrm{Gl}_n(R) \longrightarrow K_1(R).$$

Furthermore, the identity map goes to $0 \in K_1(R)$ by the extra condition that we have imposed, and so the ϕ_n's respect the canonical inclusions $\mathrm{Gl}_n(R) \subset \mathrm{Gl}_{n+1}(R)$, and therefore give a homomorphism $\phi\colon \mathrm{Gl}(R) \longrightarrow K_1(R)$. A result of Bass asserts that ϕ induces an isomorphism

$$\mathrm{Gl}(R)/[\mathrm{Gl}(R), \mathrm{Gl}(R)] \cong K_1(R).$$

The proof of this result uses the Whitehead theorem:

THEOREM 2 (WHITEHEAD). *Let $E(R)$ be the sugbroup of $\mathrm{Gl}(R)$ which is generated by elementary transformations. Then $E(R)$ coincides with the commutator subgroup of* $\mathrm{Gl}(R)$.

$K_1(R)$ is often called the *Whitehead group*. One also says that there is an isomorphism

$$K_1(R) \cong H_1\big(\mathrm{Gl}(R), \mathbb{Z}\big).$$

In the cases where $E(R)$ coincides with the special linear group $\mathrm{Sl}(R)$ (*e.g.* R is a local ring, a field, or R has a Euclidean algorithm), one sees an isomorphism

$$K_1(R) \cong R^*, \tag{1}$$

where R^* is the group of units in R.

The Whitehead theorem is proved by observing that every elementary transformation matrix is a commutator, since there are relations

$$[e_{i,j}(a), e_{j,k}(b)] = e_{i,k}(ab) \quad \text{if } i,j,k \text{ are distinct.}$$

In particular, $E(R)$ is a perfect subgroup of $\mathrm{Gl}(R)$. On the other hand, every commutator is a product of elementary transformations, by the relation

$$ABA^{-1}B^{-1} = \begin{bmatrix} A & 0 \\ 0 & A^{-1} \end{bmatrix} \cdot \begin{bmatrix} B & 0 \\ 0 & B^{-1} \end{bmatrix} \cdot \begin{bmatrix} (BA)^{-1} & 0 \\ 0 & BA \end{bmatrix}$$

and the Chevalley group relations

$$\begin{cases} \begin{bmatrix} 0 & A \\ -A^{-1} & 0 \end{bmatrix} = \begin{bmatrix} 1 & A \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ -A^{-1} & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & A \\ 0 & 1 \end{bmatrix} \\ \begin{bmatrix} A & 0 \\ 0 & A^{-1} \end{bmatrix} = \begin{bmatrix} 0 & A \\ -A^{-1} & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \end{cases}$$

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The group $K_2(R)$ was defined by Milnor [17] to be the Schur multiplier

$$H_2\big(E(R),\mathbb{Z}\big)$$

of the group $E(R)$. Presumably, the original motivation for the definition was to extend the six term exact sequence

$$K_1(I) \longrightarrow K_1(R) \longrightarrow K_1(R/I) \longrightarrow K_0(I) \longrightarrow K_0(R) \longrightarrow K_0(R/I)$$

which is associated to an ideal of R. In the case where R is a field k, the Schur multiplier $H_2\big(E(R),\mathbb{Z}\big)$ was characterized by Matsumoto:

THEOREM 3 (MATSUMOTO). *There is an isomorphism*

$$K_2(k) \cong k^* \otimes k^* / \langle x \otimes (1-x)\rangle.$$

This group $K_2(k)$ is an interesting and important arithmetic invariant. Suppose that ℓ is a prime which is distinct from the characteristic of k. Then the composite

$$k^* \otimes k^* \longrightarrow k^*/(k^*)^\ell \otimes k^*/(k^*)^\ell \xrightarrow{\cup} H^2(k,\mu_\ell^{\otimes 2})$$

induces a map

$$K_2(k)/\ell K_2(k) \xrightarrow{nr} H^2(k,\mu_\ell^{\otimes 2})$$

taking values in degree 2 Galois cohomology, called the *norm residue homomorphism*. The following is the celebrated theorem of Merkurjev and Suslin [15]:

THEOREM 4 (MERKURJEV, SUSLIN). *Suppose that k is any field and that ℓ is a prime which is distinct from the characteristic of k. Then the norm residue homomorphism*

$$K_2(k)/\ell K_2(k) \xrightarrow{nr} H^2(k,\mu_\ell^{\otimes 2})$$

is an isomorphism.

The notation $H^2(k,\mu_\ell^{\otimes 2})$ means the second Galois (or étale) cohomology group of k with coefficients in the 2-fold Tate twisted sheaf $\mu_\ell \otimes \mu_\ell$, which is constructed by tensoring the sheaf μ_ℓ of ℓ-th roots of unity with itself. If k contains a primitive ℓ-th root of unity, then μ_ℓ and all of its Tate twists are isomorphic to the cyclic group $\mathbb{Z}/\ell\mathbb{Z}$, and then $H^2(k,\mathbb{Z}/\ell\mathbb{Z})$ can be identified with the ℓ-torsion subgroup ${}_\ell\operatorname{Br}(k)$ of the Brauer group of the field k.

EXAMPLE 5 ($K_2(\mathbb{Q})$). An explicit calculation (see [17] again) using generators in the Steinberg group $\operatorname{St}(\mathbb{Z})$ shows that the group $K_2(\mathbb{Z})$ is isomorphic to the cyclic group $\mathbb{Z}/2$, with generator given by the "symbol" $\{-1,-1\}$. This symbol is a type of cocycle which measures the failure of the elements in $\operatorname{St}(\mathbb{Z})$ corresponding to the diagonal of $\operatorname{Gl}(\mathbb{Z})$ to be multiplicative in units of $\mathbb{Z}$. Furthermore, this symbol maps to the generator

$$[-1] \cup [-1] \in H^2(\mathbb{R},\mathbb{Z}/2) \cong \mathbb{Z}/2$$

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under the composite

$$K_2(\mathbb{Z}) \longrightarrow K_2(\mathbb{R}) \xrightarrow{nr} H^2(\mathbb{R}, \mathbb{Z}/2).$$

It follows that, in the "localization sequence",

$$K_2(\mathbb{Z}) \xrightarrow{i^*} K_2(\mathbb{Q}) \xrightarrow{\partial} \bigoplus_{p \geq 3,\, p \text{ prime}} K_1(\mathbb{F}_p) \longrightarrow 0,$$

the map i^* which is induced by the inclusion $\mathbb{Z} \subset \mathbb{Q}$ is a split monomorphism, and so there is an isomorphism

$$K_2(\mathbb{Q}) \cong \mathbb{Z}/2 \oplus \bigoplus_{p \geq 3,\, p \text{ prime}} \mathbb{F}_p^*.$$

OTHER EXAMPLES 6. It is known from several points of view that K_2 vanishes on finite fields. It is also known that $K_2(F)$ is uncountable if F is uncountable, and is uniquely divisible if F is algebraically closed. In particular $K_2(\mathbb{C})$ of the complex numbers $\mathbb{C}$ is a very large uniquely divisible group that nobody plans on computing.

3. **Higher K-groups.** It is a basic contribution of Quillen's [19] that the groups $K_0(R)$, $K_1(R)$ and $K_2(R)$ are homotopy groups of a space $BQP(R)$ which is canonically associated to a certain category $QP(R)$, which everybody calls the Q-construction of the category $P(R)$ of finitely generated projective R-modules. The category $QP(R)$ is constructed from exact sequences of projective modules: its objects are the projective modules themselves, and a morphism from P_1 to P_2 is an equivalence class of pictures

$$P_1 \overset{p}{\twoheadleftarrow} Q \overset{m}{\rightarrowtail} P_2,$$

where p is an epimorphism appearing in some exact sequence of $P(R)$, and m is a monomorphism appearing in an exact sequence of $P(R)$ (meaning that m is split). The equivalence relation on such pictures is determined by isomorphisms $Q \cong Q'$ making the obvious diagram commute, and composition is defined by pullback. The associated space $BQP(R)$ has a simplicial structure (in fact, is more properly thought of as a simplicial set), with vertices corresponding to the objects of $QP(R)$, 1-simplices corresponding to its morphisms, 2-simplices given by commutative triangles, and so on. The space $BQP(R)$ is connected; its fundamental groupoid is the free groupoid on the category $QP(R)$ (the free groupoid construction makes every morphism invertible in the most economical way), and this can be used to show that the fundamental group $\pi_1 BQP(R)$ is isomorphic to $K_0(R)$.

The "$Q = +$" theorem asserts that the loop space $\Omega BQP(R)$ has the homotopy type of the space

$$K_0(R) \times \mathrm{BGl}(R)^+,$$

where $K_0(R)$ is discrete, and $\mathrm{BGl}(R)^+$ is an H-space having the homology of the classifying space $\mathrm{BGl}(R)$ of the general linear group $\mathrm{Gl}(R)$. In particular,

$$\pi_1\, \mathrm{BGl}(R)^+ \cong H_1\big(\mathrm{Gl}(R), \mathbb{Z}\big) \cong K_1(R).$$

The "+" notation reflects one of the ways that $\mathrm{BGl}(R)^+$ can be constructed, which involves attaching 2-cells and 3-cells to $\mathrm{BGl}(R)$ in such a way that integral homology is preserved at the same time as the perfect subgroup $E(R)$ is killed in the fundamental group. One can apply the same procedure to the classifying space $BE(R)$ of the group of elementary transformations, and the space $BE(R)^+$ is the universal cover of $\mathrm{BGl}(R)^+$. It follows that there are isomorphisms

$$\pi_2\,\mathrm{BGl}(R)^+ \cong \pi_2 BE(R)^+ \cong H_2\big(E(R), \mathbb{Z}\big) \cong K_2(R),$$

at the level of the second homotopy group.

The algebraic K-groups $K_i(R)$, $i \geq 0$, are defined to be the homotopy groups of the loop space $\Omega BQP(R)$ of the "Q-construction" $BQP(R)$. I emphasized the fact that the category $QP(R)$ is constructed from exact sequences, because an analogous construction can be made for any additive category having a suitable calculus of exact sequences. Such categories are called *exact categories*, and the category of vector bundles $P(X)$ on a scheme X is an example: the (higher) algebraic K-groups $K_i(X)$ of a scheme are defined to be the homotopy groups $\pi_i\Omega BQP(X)$, $i \geq 0$. This theory is contravariant in maps of schemes; the definition of the induced map uses the observation that vector bundles are preserved by pullback. Similar considerations apply to coherent sheaves (*aka.* finitely generated modules) on Noetherian schemes, and the resulting theory is either called *G-theory* or *K'-theory*, depending on what you read.

There hasn't been much in the way of successful characterizations of the higher K-groups. It is known [6] that $K_3(R)$ is isomorphic to the homology group

$$H_3\big(\mathrm{St}(R), \mathbb{Z}\big)$$

of the Steinberg group, but it's hard to know what to do with this. The most striking early calculational success was Quillen's computation of the K-theory of finite fields [18].

THEOREM 7 (QUILLEN). *Suppose that $\mathbb{F}_q$ is a finite field with q elements. Then there are isomorphisms*

$$K_n(\mathbb{F}_q) \cong \begin{cases} 0 & \text{if } n = 2k,\ k \geq 1, \text{ and} \\ \mathbb{Z}/(q^k - 1) & \text{if } n = 2k-1,\ k \geq 1. \end{cases}$$

This result was proved by identifying the space $\mathrm{BGl}(\mathbb{F}_q)$ up to homotopy with the homotopy fixed points of the Adams operation $\Psi^q\colon BU \longrightarrow BU$, for then the map $\Psi^q - 1$ induces multiplication by the integer $q^k - 1$ in the homotopy group $\pi_{2k}BU$. The method of proof was a homology calculation that actually led to the introduction of the plus construction and the higher algebraic K-groups.

4. **The Lichtenbaum-Quillen conjecture.** We shall assume henceforth that k is a field containing a primitive ℓ-th root of unity ζ_ℓ, and that ℓ is a prime which is distinct from the characteristic of k. We shall also assume that $\ell > 3$, because of a homotopy theoretic gotcha concerning Moore spectra. Finally, suppose that k has finite Galois cohomological dimension with respect to ℓ-torsion sheaves. Since we've chosen to avoid $\ell = 2$, this last assumption is not a big deal from a number theoretic or algebraic geometric point of view.

On account of phenomena like the bad experience with $K_2(\mathbb{C})$ that was referred to above, all modern attempts at calculating algebraic K-groups have concentrated on determining their torsion subgroups. The main tool for investigating this torsion is mod n algebraic K-theory, as introduced by Browder [2] some years back. On the space level, the mod n algebraic K-groups $K_i(R, \mathbb{Z}/n)$ can be defined as homotopy classes of maps

$$K_i(R,\mathbb{Z}/n) = \begin{cases} [Y^n, \Omega^{i-1}BQP(R)] & \text{if } i \geq 1, \text{ and} \\ K_0(R) \otimes \mathbb{Z}/n & \text{if } i = 0. \end{cases}$$

The space Y^n is the Moore space, which is defined to be the cofibre of the multiplication by n map

$$S^1 \xrightarrow{\times n} S^1$$

on the circle. The resulting cofibre sequence gives rise to a long exact sequence in homotopy groups, in which one finds short exact sequences of the form

$$0 \longrightarrow K_i(R) \otimes \mathbb{Z}/n \longrightarrow K_i(R, \mathbb{Z}/n) \longrightarrow \operatorname{Tor}\big(\mathbb{Z}/n, K_{i-1}(R)\big) \longrightarrow 0.$$

In particular $K_i(R, \mathbb{Z}/n)$ maps onto the n-torsion in $K_{i-1}(R)$, for all i.

If you think that the space level definition of mod n K-theory looks ad hoc in degree 0, you're right. It makes much more sense to define the invariant on the spectrum level, once you know that the space $\Omega BQP(R)$ is the 0-th space of a connective spectrum $K(R)$ which arises from the symmetric monoidal structure on the category $QP(R)$ which is given by direct sum. This last sentence means that the stable homotopy groups of the spectrum $K(R)$ naturally have the form

$$\pi_j K(R) = \begin{cases} K_j(R) & \text{if } j \geq 0, \text{ and} \\ 0 & \text{if } j < 0. \end{cases}$$

You can think of a spectrum Z as a generalized space, if you like, with homotopy groups $\pi_i Z$ indexed by $i \in \mathbb{Z}$ instead of just the non-negative integers. Spectra have a notion of addition up to homotopy, so one can form a cofibre sequence

$$K(R) \xrightarrow{\times n} K(R) \longrightarrow K/n(R)$$

which defines the mod n K-theory spectrum $K/n(R)$, and the mod n K-groups $K_i(R, \mathbb{Z}/n)$ are its stable homotopy groups.

Now let's take the field k and the prime number ℓ as above, and analyze the lower dimensional mod ℓ K-groups for k. Remember in particular that k contains a primitive ℓ-th root of unity. First of all, there are isomorphisms

$$K_0(k, \mathbb{Z}/\ell) \cong \mathbb{Z}/\ell \cong H^0(k, \mathbb{Z}/\ell),$$

since k contains a primitive ℓ-th root of unity. Secondly, in the exact sequence,

$$0 \longrightarrow K_1(k) \otimes \mathbb{Z}/\ell \longrightarrow K_1(k, \mathbb{Z}/\ell) \longrightarrow \mathrm{Tor}\big(\mathbb{Z}/\ell, K_0(k)\big) \longrightarrow 0,$$

one sees that the Tor term is 0, because $K_0(k) \cong \mathbb{Z}$, so that there are isomorphisms

$$\begin{aligned} K_1(k, \mathbb{Z}/\ell) &\cong K_1(k) \otimes \mathbb{Z}/\ell \\ &\cong k^*/(k^*)^{\times \ell} \\ &\cong H^1(k, \mathbb{Z}/\ell). \end{aligned}$$

Next, the Merkurjev-Suslin theorem says that the defining exact sequence for $K_2(k, \mathbb{Z}/\ell)$ can be written in the form

$$0 \longrightarrow H^2(k, \mathbb{Z}/\ell) \longrightarrow K_2(k, \mathbb{Z}/\ell) \longrightarrow H^0(k, \mathbb{Z}/\ell) \longrightarrow 0.$$

Finally, a more recent theorem of, separately, Levine [14] and Merkurjev-Suslin [16] says that, with respect to a ring structure on $K_*(k, \mathbb{Z}/\ell)$ that is somehow determined by tensor product, there is an isomorphism

$$K_3(k, \mathbb{Z}/\ell)_{\mathrm{ind}} \cong H^1(k, \mathbb{Z}/\ell)$$

which relates the indecomposable part of $K_3(k, \mathbb{Z}/\ell)$ (*i.e.*, quotient by the image of Milnor K-theory) to mod ℓ Galois cohomology.

Therefore, in all cases that anybody has been able to compute, the mod ℓ K-theory of a field k is controlled by its Galois cohomology. The intuition behind the Lichtenbaum-Quillen conjecture is that this should be true for all of the mod ℓ K-theory of k.

The standard method for turning this intuition into a conjecture is to compare the mod ℓ K-groups $K_*(k, \mathbb{Z}/\ell)$ to another list of invariants $K_*^{\mathrm{et}}(k, \mathbb{Z}/\ell)$, collectively called the mod ℓ *étale K-theory* of k, which is determined by the Galois cohomology of k.

The étale K-groups do not have an elementary construction. The original method for defining them (due to Friedlander [4]; see also [3]) is to take the ordinary mod ℓ non-connective complex K-theory spectrum KU/ℓ, and let it represent a cohomology theory on the étale homotopy type of k, the étale homotopy type itself being constructed from path components of hypercovers for k with respect to the étale topology.

The construction that will be used here starts with the assertion that the assignment

$$U \xrightarrow{\phi} \mathrm{Sp}(k) \longmapsto K/\ell(U) \tag{2}$$

defines a contravariant functor on any category of k-schemes after invoking a suitable theory of categorical coherence, and the resulting presheaf of spectra K/ℓ represents

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a generalized étale (or Galois) cohomology theory on the field k, whose cohomology groups coincide with the Dwyer-Friedlander étale K-groups of k in degrees -1 or more.

In slightly more detail, the assignment (2) restricts to a definition of the mod ℓ K-theory presheaf of spectra K/ℓ on an étale site for k (of whatever size). There is a homotopy theory for such objects which arises from a Quillen closed model structure, which structure has weak equivalences computed stalkwise. This homotopy theory is a descendant of Joyal's closed model structure for simplicial sheaves [11], later bootstrapped to simplicial presheaves and presheaves of spectra [9], [10]. The presheaves of spectra which are fibrant for this homotopy theory are said to be *globally fibrant*. Then, from this point of view, the mod ℓ étale (or Galois) K-group $K_i^{\mathrm{et}}(k, \mathbb{Z}/\ell)$ is defined to be the i-th stable homotopy group $\pi_i GK/\ell(k)$ of the global sections spectrum of a globally fibrant model $\alpha\colon K/\ell \longrightarrow GK/\ell$ for the mod ℓ K-theory presheaf of spectra.

Explicitly, when I say that the map α is a globally fibrant model, it means that α is a stalkwise stable equivalence of presheaves of spectra, and that GK/ℓ is globally fibrant. The stable homotopy groups of its global sections spectrum are independent of the choice of the stalkwise weak equivalence α. Then the comparison map

$$\alpha_*\colon K_i(k, \mathbb{Z}/\ell) \longrightarrow K_i^{\mathrm{et}}(k, \mathbb{Z}/\ell)$$

is induced by the global sections map $\alpha(k)\colon K/\ell(k) \longrightarrow GK/\ell(k)$.

The map of spectra $\alpha(k)$ is not a stable equivalence in general—see Example 13 below. The globally fibrant model GK/ℓ should rather be thought of as a generalized injective resolution for the presheaf of spectra K/ℓ, with sufficiently good formal properties that the stable homotopy groups of the global sections spectrum $GK/\ell(k)$ can be recovered from Galois cohomology. There is also technical sense to the assertion that the G in GK/ℓ should stand for "Godement", in that the Godement resolution for K/ℓ (suitably defined as a homotopy inverse limit of some cosimplicial gadget) is a globally fibrant model for K/ℓ under standard hypotheses (see [22], [9]).

In general, presheaves of spectra represent generalized étale cohomology theories. There are essentially two reasons for this:

(1) Let F be a presheaf of spectra, and let $\alpha\colon F \longrightarrow GF$ be a globally fibrant model for F. Then there are isomorphisms

$$[\Sigma^n \Gamma^* S, F] \overset{\alpha_*}{\cong} [\Sigma^n \Gamma^* S, GF] \cong \pi_n GF(k),$$

for $n \in \mathbb{Z}$, where S is the sphere spectrum, and $\Gamma^* S$ is it associated constant presheaf of spectra. The square brackets means morphisms in the homotopy category associated to presheaves of spectra. The stable homotopy groups $\pi_n GF(k)$ are therefore analogous to the cohomology of a point in ordinary stable homotopy theory. The "point" is $\mathrm{Sp}(k)$ from a topos theoretic point of view, and so these groups can naturally be interpreted as the étale cohomology of k with coefficients in F. In particular, the way I've defined it, mod ℓ étale K-theory is the theory represented by the K-theory presheaf of spectra K/ℓ.

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(2) This is a true generalization of sheaf cohomology: the Eilenberg-Mac Lane object $H(A)$ associated to a sheaf of abelian groups gives rise to ordinary étale cohomology, in the sense that there is an isomorphism

$$\pi_i GH(A)(k) \cong H^{-i}(k, A), \qquad \text{for each } i \in \mathbb{Z}.$$

Spectra have Postnikov towers, which are towers of fibrations with Eilenberg-Mac Lane objects in the fibres, and so it's not much of a step within the ambient homotopical machinery to see that the Postnikov tower for a presheaf of spectra F gives rise to a spectral sequence, with

$$E_2^{p,q} = H^p\big(k, \tilde{\pi}_q(F)\big) \Rightarrow \pi_{q-p} GF(k). \tag{3}$$

The notation $\tilde{\pi}_q(F)$ stands for the sheaf associated to the presheaf $\pi_q(F)$ of stable homotopy groups of F. This spectral sequence converges if F has ℓ-torsion presheaves of stable homotopy groups, and k has finite Galois cohomological dimension with respect to ℓ-torsion sheaves. This is the étale, or Galois cohomological descent spectral sequence for the generalized étale cohomology theory that is represented by F.

Now here's what I mean by the Lichtenbaum-Quillen conjecture for fields:

CONJECTURE 8 (LICHTENBAUM-QUILLEN). *Suppose that k is a field, and that ℓ is a prime which is distinct from the characteristic of k, as above (in particular, k contains a primitive ℓ-th root of unity). Suppose that k has finite Galois cohomological dimension d with respect to ℓ-torsion sheaves. Let $\alpha\colon K/\ell \longrightarrow GK/\ell$ be a globally fibrant model for the K-theory presheaf K/ℓ. Then the global sections map induces isomorphisms*

$$K_i(k, \mathbb{Z}/\ell) \xrightarrow[\cong]{\alpha_*} K_i^{\text{et}}(k, \mathbb{Z}/\ell) \cong \pi_i GK/\ell(k)$$

for $i \geq d-1$.

I want to explain a bit about what the conjecture means in terms of the descent spectral sequence, and address the question of why the bound $d-1$ appears in the statement.

The first step is to talk about the Gabber rigidity theorem [5]; it's one of the most important algebraic K-theory results of the past ten years. The following is really only a special case:

THEOREM 9 (GABBER). *Suppose that O is a Henselian local ring containing $1/\ell$, and let K denote the residue field. Then the residue map $O \longrightarrow K$ induces isomorphisms*

$$K_i(O, \mathbb{Z}/\ell) \xrightarrow{\cong} K_i(K, \mathbb{Z}/\ell)$$

in mod ℓ *K-theory for $i \geq 0$.*

The consequences of this result are legion. Perhaps the most striking application to date is Suslin's calculation of the mod ℓ K-theory of algebraically closed fields [20], [21].

THEOREM 10 (SUSLIN). *Suppose that k is an algebraically closed field of characteristic prime to ℓ. Then there are isomorphisms*

$$K_j(k, \mathbb{Z}/\ell) \cong \begin{cases} \mathbb{Z}/\ell & \text{if } j = 2n,\ n \geq 0, \text{ and} \\ 0 & \text{if } j = 2n+1,\ n \geq 0. \end{cases}$$

The proof of this result can be encapsulated [8] as follows. The Gabber rigidity theorem implies that the canonical comparison map

$$\epsilon \colon \Gamma^* \operatorname{BGl}(k) \longrightarrow \operatorname{BGl}$$

induces an isomorphism of mod ℓ homology sheaves, on the big étale site of smooth k-schemes. In effect, if x is a closed point of some smooth k-scheme X, then the map in homology sheaves associated to ϵ at the stalk corresponding to x specializes to the map

$$i_* \colon H_*\big(\operatorname{BGl}(k), \mathbb{Z}/\ell\big) \longrightarrow H_*\big(\operatorname{BGl}(\mathcal{O}_x^{\mathrm{sh}}), \mathbb{Z}/\ell\big),$$

induced by the k-structure map $i \colon k \longrightarrow \mathcal{O}_x^{\mathrm{sh}}$, where $\mathcal{O}_x^{\mathrm{sh}}$ is the strict Henselization of the local ring $\mathcal{O}_x$ for x, and this map is an isomorphism by Gabber rigidity. A universal coefficients spectral sequence argument then implies that ϵ induces an isomorphism

$$H^*_{\mathrm{et}}\big(\operatorname{BGl}, \mathbb{Z}/\ell\big) \cong H^*_{\mathrm{et}}\big(\Gamma^* \operatorname{BGl}(k), \mathbb{Z}/\ell\big).$$

The simplicial sheaf associated to the constant functor $\Gamma^* \operatorname{BGl}(k)$ is represented by a discrete simplicial k-scheme consisting of a disjoint union of copies of $\operatorname{Sp}(k)$ in each simplicial degree. The algebraically closed field k is acyclic in the eyes of étale cohomology, and so there is a canonical isomorphism

$$H^*_{\mathrm{et}}\big(\Gamma^* \operatorname{BGl}(k), \mathbb{Z}/\ell\big) \cong H^*\big(\operatorname{BGl}(k), \mathbb{Z}/\ell\big).$$

Finally, composing these maps gives a canonical isomorphism

$$H^*_{\mathrm{et}}(\operatorname{BGl}, \mathbb{Z}/\ell) \cong H^*\big(\operatorname{BGl}(k), \mathbb{Z}/\ell\big), \tag{4}$$

relating the étale cohomology of BGl to the cohomology of the simplicial set $\operatorname{BGl}(k)$.

In particular, by comparing dimensions of vector spaces, one sees that the comparison map

$$\operatorname{BGl}(\mathbb{C}) \longrightarrow BU$$

is a mod ℓ cohomology isomorphism, giving Suslin's result for $k = \mathbb{C}$. But the cohomological isomorphism (4) also implies a "Lefschetz principle" for $K_*(k, \mathbb{Z}/\ell)$, and the Gabber result itself (applied to extensions of Witt rings) implies that $K_*(\overline{\mathbb{F}}_p, \mathbb{Z}/\ell)$ is isomorphic to $K_*(\mathbb{C}, \mathbb{Z}/\ell)$.

The Suslin theorem and Gabber rigidity together determine the mod ℓ étale K-theory sheaves for fields which are not algebraically closed:

COROLLARY 11. *Suppose that the field k contains a primitive ℓ-th root of unity, where ℓ is prime to the characteristic of k. Then the* mod ℓ *étale K-theory sheaves of k have the form*

$$\tilde{\pi}_i K/\ell = \begin{cases} \mathbb{Z}/\ell & \text{if } i = 2n,\ n \geq 0, \text{ and} \\ 0 & \text{if } i = 2n+1,\ n \geq 0. \end{cases}$$

We are then able to verify the Lichtenbaum-Quillen conjecture for fields of Galois cohomological dimension 0:

COROLLARY 12. *Suppose that k contains a primitive ℓ-th root of unity, where ℓ is prime to the characteristic of k. Suppose further that k has Galois cohomological dimension* 0 *with respect to ℓ-torsion sheaves. Then the Lichtenbaum-Quillen conjecture holds for k.*

PROOF. The Bott element $\beta \in K_2(k, \mathbb{Z}/\ell)$ is the element mapping to the primitive ℓ-th root of unity ζ_ℓ under the isomorphism

$$K_2(k, \mathbb{Z}/\ell) \xrightarrow{\cong} \mathrm{Tor}(\mathbb{Z}/\ell, k^*).$$

The mod ℓ K-theory of the algebraic closure $\bar{k}$ of k has a ring structure of the form

$$K_*(\bar{k}, \mathbb{Z}/\ell) \cong \mathbb{Z}/\ell[\beta],$$

by comparison with mod ℓ homotopy groups of BU. All finite algebraic extensions of k have index prime to ℓ by the assumption on the cohomological dimension of k, so that the inclusion $j : k \subset \bar{k}$ induces a monomorphism

$$K_*(k, \mathbb{Z}/\ell) \longrightarrow K_*(\bar{k}, \mathbb{Z}/\ell),$$

by a transfer argument. But $K_2(k, \mathbb{Z}/\ell)$ contains β, and j_* is a ring homomorphism, so j_* is surjective as well. The descent spectral sequence implies that $K_i^{\text{et}}(k, \mathbb{Z}/\ell)$ vanishes in negative degrees, as well as in all odd degrees. Finally, the calculation of $K_*(k, \mathbb{Z}/\ell)$ implies that there are commutative diagrams in even positive degrees of the form

$$\begin{tikzcd} K_{2j}(k, \mathbb{Z}/\ell) \arrow[r] & \pi_{2j} GK/\ell(k) \arrow[d, "\cong"] \\ \mathbb{Z}/\ell \arrow[u, "\cong"] \arrow[r, "\cong"'] & H^0(k, \mathbb{Z}/\ell) \end{tikzcd}$$ ∎

The cohomological descent spectral sequence is so sparse that it collapses in cases of low Galois cohomological dimension. Here is how this works in a very specific example:

EXAMPLE 13 (mod 5 ÉTALE K-THEORY FOR $\mathbb{Q}(\zeta_5)$). The following is a picture of the E_2-term of the descent spectral sequence for $K_*^{\text{et}}\big(\mathbb{Q}(\zeta_5), \mathbb{Z}/5\big)$:

$$
\begin{array}{c|cccc}
\uparrow & & & & \\
q & & & & \\
& & & & \\
3 & 0 & 0 & 0 & 0 \\
2 & \mathbb{Z}/5 & \mathbb{Q}(\zeta_5)^*/\big(\mathbb{Q}(\zeta_5)^*\big)^{\times 5} & {}_5\operatorname{Br}\big(\mathbb{Q}(\zeta_5)\big) & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & \mathbb{Z}/5 & \mathbb{Q}(\zeta_5)^*/\big(\mathbb{Q}(\zeta_5)^*\big)^{\times 5} & {}_5\operatorname{Br}\big(\mathbb{Q}(\zeta_5)\big) & 0 \\
\hline
& 0 & 1 & 2 & 3 \quad p \longrightarrow
\end{array}
$$

$$E_2^{p,q} = H^p\big(\mathbb{Q}(\zeta_5), \tilde{\pi}_q K/5\big)$$

The field $\mathbb{Q}(\zeta_5)$ has Galois cohomological dimension 2 with respect to 5-torsion sheaves, so all groups to the right of those indicated in the above table are trivial. The differentials of the descent spectral sequence have the form

$$d_r\colon E_r^{p,q} \longrightarrow E_r^{p+r,q+r-1},$$

and the E_∞ groups which calculate the group $K_n^{\mathrm{et}}\big(\mathbb{Q}(\zeta_5), \mathbb{Z}/5\big)$ lie along the line $n = q-p$. The numbers dictate that all differentials for this spectral sequence are trivial, and so the spectral sequence collapses. In particular, there are isomorphisms

$$K_{-2}^{\mathrm{et}}\big(\mathbb{Q}(\zeta_5), \mathbb{Z}/5\big) \cong {}_5\operatorname{Br}\big(\mathbb{Q}(\zeta_5)\big),$$
$$K_{-1}^{\mathrm{et}}\big(\mathbb{Q}(\zeta_5), \mathbb{Z}/5\big) \cong \mathbb{Q}(\zeta_5)^*/\big(\mathbb{Q}(\zeta_5)^*\big)^{\times 5},$$

and there is a short exact sequence

$$0 \longrightarrow {}_5\operatorname{Br}\big(\mathbb{Q}(\zeta_5)\big) \longrightarrow K_0^{\mathrm{et}}\big(\mathbb{Q}(\zeta_5), \mathbb{Z}/5\big) \longrightarrow \mathbb{Z}/5 \longrightarrow 0.$$

Furthermore, the groups $K_i^{\mathrm{et}}\big(\mathbb{Q}(\zeta_5), \mathbb{Z}/5\big)$ are periodic of period 2 in degrees greater than or equal to -1, and are trivial in degrees strictly less than -2, so we've completely computed the mod 5 étale K-theory of this field.

It is exactly this sort of calculation which makes one wish that the Lichtenbaum-Quillen conjecture were true.

Note that the ordinary K-group $K_0\big(\mathbb{Q}(\zeta_5), \mathbb{Z}/5\big)$ is a copy of $\mathbb{Z}/5$, whereas the 5-torsion subgroup ${}_5\operatorname{Br}\big(\mathbb{Q}(\zeta_5)\big)$ of the Brauer group $\operatorname{Br}\big(\mathbb{Q}(\zeta_5)\big)$ sits in an exact sequence

$$0 \longrightarrow {}_5\operatorname{Br}\big(\mathbb{Q}(\zeta_5)\big) \longrightarrow \bigoplus_{P \text{ finite}} \mathbb{Z}/5 \xrightarrow{\nabla} \mathbb{Z}/5 \longrightarrow 0,$$

by classfield theory, and is therefore very big. It follows that the comparison map

$$K_0\big(\mathbb{Q}(\zeta_5), \mathbb{Z}/5\big) \xrightarrow{\alpha_*} K_0^{\mathrm{et}}\big(\mathbb{Q}(\zeta_5), \mathbb{Z}/5\big)$$

is *not* an isomorphism.

More generally, Bruno Kahn has recently shown [13] that the descent spectral sequence for $K_*^{\mathrm{et}}(k, \mathbb{Z}/\ell)$ collapses at the E_2-level in the cases where one can verify Kato's conjecture that the cohomology ring $H^*(k, \mathbb{Z}/\ell)$ is multiplicatively generated by

$H^1(k, \mathbb{Z}/\ell)$. The multiplicative structure of the descent spectra sequence is used to prove this: in the case at hand (k and ℓ as above), the groups $E_2^{0,2}$ and $E_2^{1,2}$ consist of permanent cycles (see [12]), essentially because they are images of ordinary K-groups, and the Kato conjecture implies that they generate the entire E_2-term as an algebra over $\mathbb{Z}/\ell = E_2^{0,0}$.

The comparison map

$$K_j(k, \mathbb{Z}/\ell) \xrightarrow{\alpha_*} K_j^{\mathrm{et}}(k, \mathbb{Z}/\ell)$$

is an isomorphism in degrees $j = 1, 2$ if the cohomological dimension of k is 2, and is an isomorphism in degrees $j = 0, 1, 2$ if k has cohomological dimension 1 (see Kahn's paper [12]). Note as well that

$$K_{-1}^{\mathrm{et}}(k, \mathbb{Z}/\ell) \cong k^*/(k^*)^{\times \ell}$$

in the cohomological dimension 1 case. Thus, the lower bound $d - 1$ on degrees for the isomorphisms in the Lichtenbaum-Quillen conjecture is the best possible, as well as being consistent with all calculations that we know about.

I want to finish by analyzing the relative cohomological dimension 1 case. Suppose that we have a diagram of field inclusions of the form

$$\begin{array}{ccccc} & & KL & & \\ & \nearrow & & \nwarrow & \\ k_{\mathrm{sep}} = K & & & & L \\ & \nwarrow & & \nearrow_{j} & \\ & & k, & & \end{array} \tag{5}$$

where k has cohomological dimension 1 and the compositum KL has cohomological dimension d. Suppose further that the Lichtenbaum-Quillen conjecture holds for KL. Then L has cohomological dimension $d + 1$, and we want to derive a condition for the Lichtenbaum-Quillen conjecture to hold for L.

Choose a globally fibrant model $\alpha\colon K/\ell \longrightarrow GK/\ell$ for K/ℓ over L. We want to show that the induced global sections map

$$K/\ell(L) \xrightarrow{\alpha} GK/\ell(L)$$

of spectra induces an isomorphism in stable homotopy groups in degrees greater than or equal to d.

Consider the induced map

$$j_*K/\ell \xrightarrow{j_*\alpha} j_*GK/\ell$$

of direct images on the level of presheaves of spectra over k, and let the presheaf of spectra X be the homotopy fibre of $j_*\alpha$. Then evaluating stalkwise, one sees a fibre sequences of ordinary spectra of the form

$$X(K) \longrightarrow K/\ell(KL) \longrightarrow GK/\ell(KL),$$

and so the inductive assumption implies that $\pi_s X(K) = 0$ if $s \geq d-1$. But then formal nonsense implies that $\pi_s GX(k) = 0$ if $s \geq d-1$, so that the global sections map

$$Gj_*K/\ell(k) \xrightarrow{Gj_*\alpha} Gj_*GK/\ell(k)$$

induces an isomorphism in stable homotopy groups π_s for $s \geq d$. The map $Gj_*\alpha$ is the "globally fibrant model" for the direct image map $j_*\alpha$, and there is a commutative diagram of maps of spectra of the form

$$\begin{array}{ccc} j_*K/\ell(k) & \xrightarrow{j_*\alpha} & j_*GK/\ell(k) \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle \cong} \\ Gj_*K/\ell(k) & \xrightarrow[G_{j_*\alpha}]{} & Gj_*GK/\ell(k). \end{array}$$

The vertical maps are induced by choices of globally fibrant models for the spectra j_*K/ℓ and j_*GK/ℓ, respectively. But j_* preserves global fibrations, so that j_*GK/ℓ is globally fibrant, and any globally fibrant model of it induces a stable equivalence on (global) sections.

The Lichtenbaum-Quillen conjecture will therefore hold for L if we can show that the map

$$m: j_*K/\ell(k) \longrightarrow Gj_*K/\ell(k)$$

of spectra induces an isomorphism in stable homotopy groups π_r for $r \geq d$. In other words, we want the K-theory groups $K_r(L, \mathbb{Z}/\ell)$ to be recoverable in a suitable range of degrees from Galois cohomological descent over the cohomological dimension 1 subfield k.

I am willing to believe that the condition $r \geq d$ may have to be altered a little bit—perhaps d can be replaced by a smaller number. The alert reader will also notice that the method outlined above does not prove the Lichtenbaum-Quillen conjecture for number fields. To get at the cyclotomic extension case, and hence number fields, one would have to replace the extension k_{sep}/k in the diagram (5) by a more general Galois extension K/k of cohomological dimension 1.

Note finally that this technique yields, and was derived from, the corresponding inductive step in Thomason's argument for his descent theorem for Bott periodic K-theory [22]. The special case of it that is most relevant here can be stated as follows:

THEOREM 14 (THOMASON). *Suppose that k and ℓ are as above, and for the presheaf of spectra $K/\ell(1/\beta)$ by formally inverting the Bott element $\beta \in K_2(k, \mathbb{Z}/\ell)$. Then any globally fibrant model*

$$K/\ell(1/\beta) \longrightarrow GK/\ell(1/\beta)$$

induces a stable equivalence

$$K/\ell(1/\beta)(k) \xrightarrow{\simeq} GK/\ell(1/\beta)(k)$$

on the level of global sections.

In particular, the spectrum $K/\ell(1/\beta)(k)$ is a model for étale K-theory, and has a Galois cohomological descent spectral sequence.

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Mathematics Department
University of Western Ontario
London, Ontario
N6A 5B7