ON A GEOMETRIC EXTREMUM PROBLEM

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(received April 2, 1964)

The following problem was brought to our attention by L. Moser: Locate eight points in the closed unit square so that the minimum of the distances between any two of the points should be as large as possible.

L. Moser conjectured the following: Let $P_1, P_2, \ldots, P_8$, be any eight points in the closed unit square. Let $d(P_i, P_j)$ denote the distance between P_i and P_j. Then

$$(1) \qquad \min_{1 \le i < j \le 8} d(P_i, P_j) \le m = \frac{1}{2} \sec \frac{\pi}{12} = \sqrt{2 - \sqrt{3}}$$

and equality holds in (1) only for the configuration indicated in Fig. 1 (where $\alpha = \pi/12$).

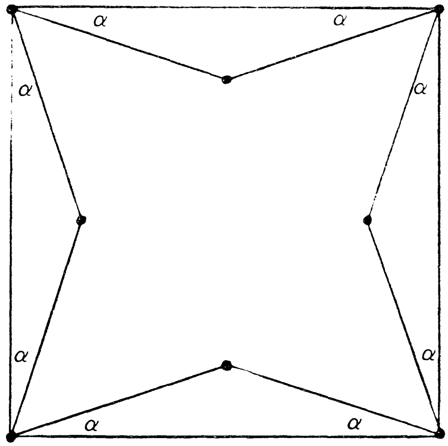


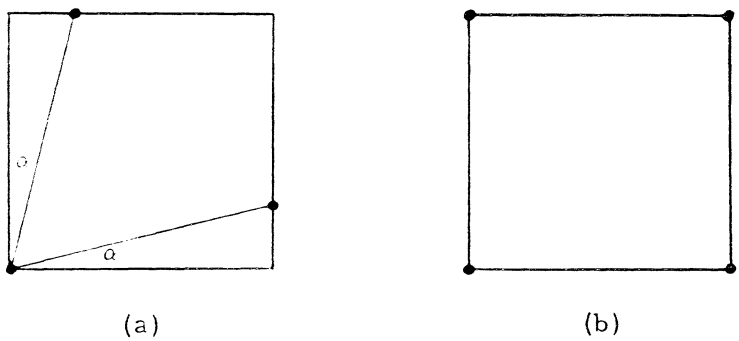
Fig. 1

Canad. Math. Bull. vol. 8, no. 1, February 1965

https://doi.org/10.4153/CMB-1965-004-x Published online by Cambridge University Press

The corresponding "best-location" problems for 2, 3, 4 or 5 points in the unit square are easily solved. For 6 points R. L. Graham obtained the solution recently. We succeeded in solving the problem for 7 and 9 points too, but the proofs will not be included here.

Figure 2 indicates the configurations for 3 and 4 points. These results are used in our proof.



$\alpha = \pi/12$

Fig. 2

We now proceed to prove Moser's conjecture. During the proof we shall use the auxiliary points indicated in Fig. 3.

The points A_1, A_2, A_3, A_4 are the vertices of the unit square, B_1, B_2, B_3, B_4 are the midpoints of the edges, and C is the centre of the square. The points D_i, E_i, F_i, G_i $(1 \leq i \leq 4)$ on the edges are defined by

$$(2) \qquad d(D_i, B_i) = d(E_i, B_{i+1}) = d(F_i, A_i) = d(G_i, A_i)$$

$$= \frac{1}{2} \tan \frac{\pi}{12} = \frac{1}{2}(2 - \sqrt{3}), \qquad 1 \leq i \leq 4 .$$

The points H_i, K_i, N_i, O_i $(1 \leq i \leq 4)$ are intersection points of

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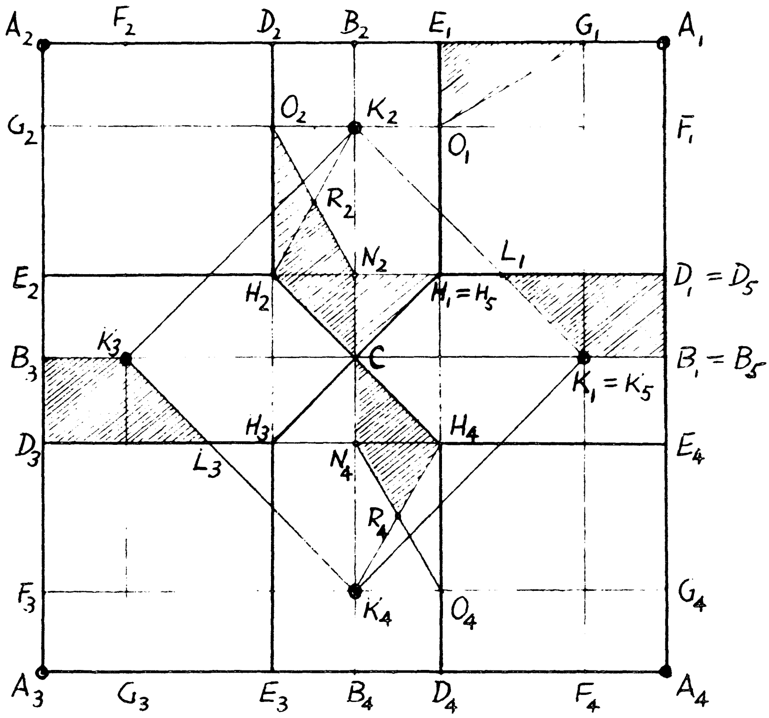

Fig. 3

side-parallels passing through the above introduced points. Note that

$$(3) \qquad d(A_i, K_i) = d(B_i, G_i) = d(F_i, B_{i+1}) = d(A_i, K_{i+1})$$

$$= d(A_i, H_i) = d(D_i, E_i) = d(C, E_i)$$

$$= d(C, D_i) = d(K_i, K_{i+1}) = m, \qquad (1 \le i \le 4).$$

We shall show that if $P_1, P_2, \ldots, P_8$ is any set $\mathcal{S}$ of eight points in the closed unit square and

$$(4) \qquad \min_{1 \le i < j \le 8} d(P_i, P_j) \ge m$$

then the location-figure of the points is that of Fig. 1, and in (4) the equality sign holds.

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PROPOSITION 1. If (4) holds, then $C \notin \mathcal{S}$.

Proof: Assume $C \in \mathcal{S}$. Obviously (see Figure 2(b)) no four points of $\mathcal{S}$ can lie in a closed square of side-length 1/2. Thus, in each of the closed squares $A_i B_{i+1} C B_i$ $(1 \leq i \leq 4)$ there can lie at most two points of $\mathcal{S}$ different from C. Therefore in three of these squares there must lie exactly two further points of $\mathcal{S}$, and so there are two adjacent squares, say $A_1B_2CB_1$ and $A_2B_3CB_2$, in each of which lie two points different from C. Because of (4), the only possibility is that these points are located at D_1, E_1, D_2, E_2 (see Fig 2(a)). But by (2)

$$d(E_1, D_2) = 2 - \sqrt{3} < \sqrt{2 - \sqrt{3}} = m ,$$

yielding a contradiction to (4).

In the following we shall use the obvious

LEMMA. If the distance between any two vertices of a closed polygon is less than or equal to m, then the distance between two points lying in or on the polygon can be equal to m only when the points are two vertices.

PROPOSITION 2. If (4) holds, then in each of the point-sets $\sigma_1, \sigma_2, \ldots, \sigma_8$ listed below there must lie exactly one point of $\mathcal{S}$.

σ_{2i-1}: The square $A_iE_iH_iD_i$ excluding the closed segment $\overline{D_iH_i}$ $(1 \leq i \leq 4)$.

σ_{2i}: The pentagon $CH_iE_iD_{i+1}H_{i+1}C$ excluding C $(1 \leq i \leq 4)$.

Proof: It is easy to see, by (3) and the lemma, that no σ_i $(1 \leq i \leq 8)$ can contain two points of $\mathcal{S}$. Since the union of all these sets is the closed unit square excluding C, by Proposition 1 each must contain at least one point, and thus

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exactly one point, of $\mathcal{S}$. In the sequel we shall denote the point belonging to the point set σ_i by P_i $(1 \leq i \leq 8)$.

The following proposition is the main argument of our paper.

PROPOSITION 3. If (4) holds, then in the closed square $K_1K_2K_3K_4$ there must be at least four points of $\mathcal{S}$.

Proof: Suppose the proposition is false. Then (see Fig. 3)

(i) At least one of the points P_{2i} $(1 \leq i \leq 4)$ lies outside the square $K_1K_2K_3K_4$. Without loss of generality we may assume that this point is P_8. By symmetry we may assume that P_8 lies in the closed trapezium $B_1D_1L_1K_1$, excluding the point K_1 (in fact the whole segment K_1L_1, where L_1 is the intersection point of D_1E_2 with K_1K_2).

(ii) The point P_1 lies in the closed triangle $G_1E_1O_1$ since, by the Lemma, the polygon $B_1A_1G_1O_1H_1L_1K_1B_1$, excluding the points K_1, H_1, G_1 and O_1, cannot contain two points of $\mathcal{S}$ and by (i) P_8 is already there.

(iii) The point P_2 lies in the closed polygon $CN_2R_2H_2C$ excluding C and R_2 (where R_2 is the intersection point of O_2N_2 and K_2H_2). By the Lemma, the closed polygon $G_1D_2O_2N_2H_1O_1G_1$, excluding the points O_2 and N_2, cannot contain two points of $\mathcal{S}$ and by (ii) P_1 is already there. So P_2 must lie in the closed polygon $CH_1N_2O_2H_2C$. Now, by the Lemma, the closed polygon $A_2E_2H_2R_2O_2D_2A_2$, excluding the points D_2 and H_2, cannot contain two points of $\mathcal{S}$ and by Proposition 2 the point P_3 is already there. So P_2 must lie

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in the closed polygon $CH_1N_2R_2H_2C$ excluding R_2. Again by the Lemma, the closed rectangle $B_1D_1N_2C$, excluding the points N_2 and C, cannot contain two points of $\mathcal{S}$ and by (i) P_8 is already there. Thus P_2 lies in the closed polygon $CN_2R_2H_2C$ excluding R_2. That P_2 cannot lie at C was proved by Proposition 1.

(iv) The point P_4 lies in the closed trapezium $B_3D_3L_3K_3$, excluding the point K_3 (L_3 is the intersection point of K_3K_4 with D_3E_4). For, by easy computation, the distances of any two of the vertices of the closed polygon $E_2B_3K_3L_3H_3CN_2R_2H_2E_2$, excluding the points B_3 and C, are less than m. By the Lemma no two points of $\mathcal{S}$ can lie in this polygon, but by (iii) the point P_2 is already there.

(v) The point P_6 lies in the closed polygon $CN_4R_4H_4C$ excluding R_4 (and C). This is a consequence of the fact that the closed polygon $B_3D_3L_3K_3$ (excluding K_3) in which, by (iv), the point P_4 lies, is central-symmetric to the polygon $B_1D_1L_1K_1$ (excluding K_1) in which, by (i), P_8 lies. Thus, the same arguments which yielded, by (iv), the location of P_2 in $CN_2R_2H_2C$ excluding R_2 (and C), will yield the central-symmetric location for P_6 in the polygon $CN_4R_4H_4C$ excluding R_4 (and C).

The proof of Proposition 4 follows now by observing that (iii) and (v) contradict the assumption (4). For, the distance between any two of the vertices of the polygon $R_2H_2CN_4R_4H_4CN_2R_2$, excluding R_2 and R_4, is <u>less</u> than m, and since by (iii) and (v) two points of $\mathcal{S}$ are located in it, (4) cannot hold.

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PROPOSITION 4. If (4) holds, then Moser's conjecture is true.

Proof: By Proposition 3, at least four points of $\mathcal{S}$ are located in the closed square $K_1K_2K_3K_4$. Since $d(K_i, K_{i+1}) = m$, by Fig. 2(b) it follows that (4) can only be satisfied if there are exactly four points located there and they coincide with the vertices K_1, K_2, K_3, K_4. In other words, necessarily $P_2 \equiv K_2$, $P_4 \equiv K_3$, $P_6 \equiv K_4$, $P_8 \equiv K_1$. But then, by Fig. 2(a), in any of the closed squares $CB_iA_iB_{i+1}C$ $(1 \le i \le 4)$ the only possibility for P_{2i-1} is to coincide with A_i. This yields the conjectured configuration.

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